

Section 5.7
Numerical Integration

Math 1a
Introduction to Calculus

April 25, 2008

Announcements

- Midterm II is Wednesday 4/30 in class (covers §4.9–5.6)
- Friday 5/2 is Movie Day!
- Final (tentative) 5/23 9:15am
- Problem Sessions Sunday, Thursday, 7pm, SC 310
- Office hours Tues, Weds, 2–4pm SC 323

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Outline

Last Time: Integration by Parts

The Trapezoidal Rule
Trapezoids instead of rectangles

The Midpoint Rule

Simpson's Rule

Error estimates for the Trapezoidal and Midpoint Rules

Error estimates for Simpson's Rule

Theorem (Integration by Parts)

Let u and v be differentiable functions. Then

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx.$$

Succinctly,

$$\int u\,dv = uv - \int v\,du.$$

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Theorem (Integration by Parts, definite form)

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

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- Trapezoids instead of rectangles

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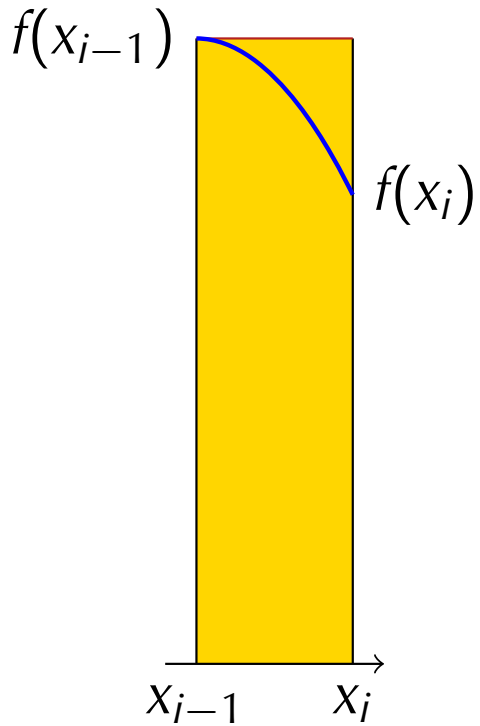
Error estimates for the Trapezoidal and Midpoint Rules

Error estimates for Simpson's Rule

Why estimate an integral when we have the FTC?

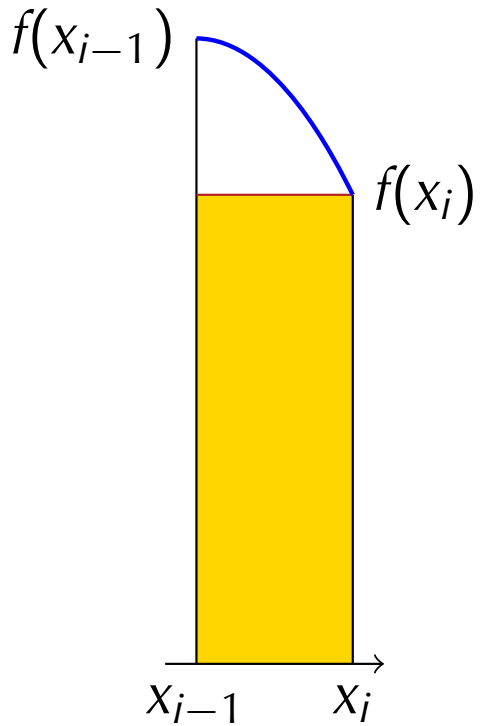
- Antidifferentiation is "hard"
- Sometimes antidifferentiation is *impossible*
- Sometimes all we need is an approximation
- These methods actually work pretty well!

Trapezoids instead of rectangles



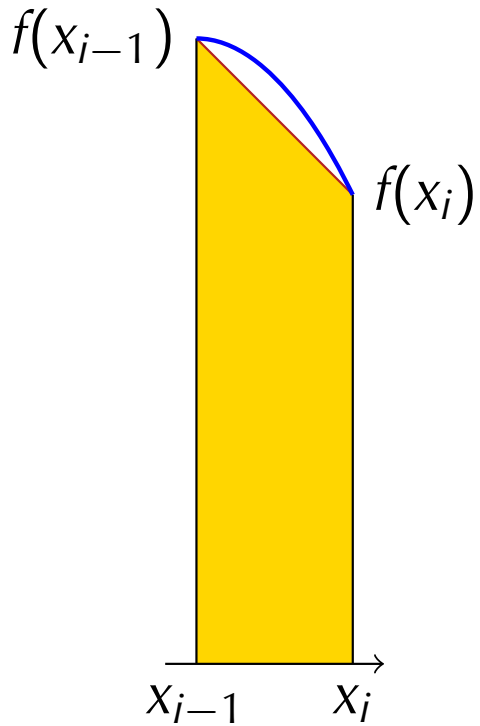
- In the case of a simple decreasing function like this one, clearly L_n is too much and R_n is not enough.

Trapezoids instead of rectangles



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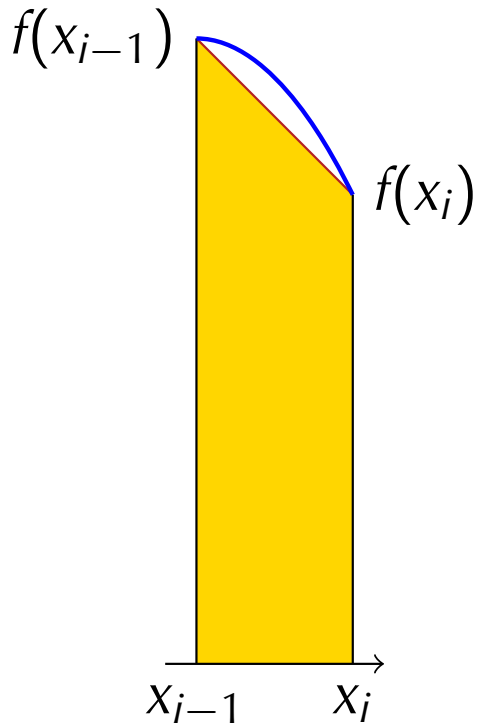
Trapezoids instead of rectangles



- In the case of a simple decreasing function like this one, clearly L_n is too much and R_n is not enough.
- Why not average them out and make a trapezoid?

$$\Delta A =$$

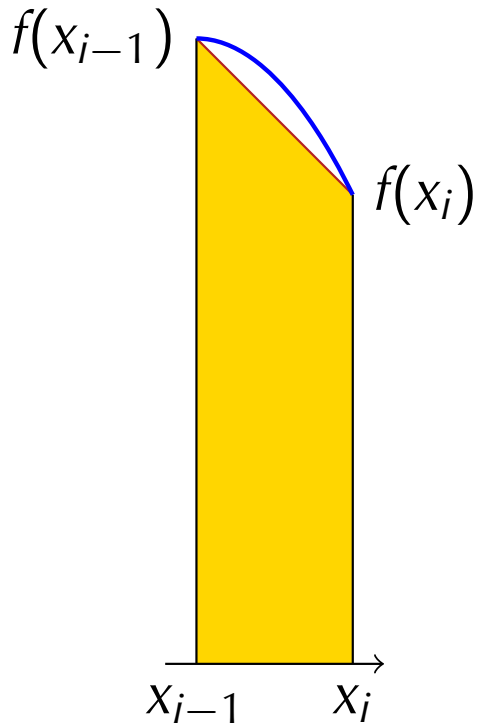
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$$\Delta A = \frac{f(x_{i-1}) + f(x_i)}{2} \Delta x$$

Trapezoids instead of rectangles



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$$\Delta A = \frac{f(x_{i-1}) + f(x_i)}{2} \Delta x$$

- The smaller the intervals, the better the approximation

The Trapezoidal Rule

Definition

Divide the interval $[a, b]$ up into n pieces. Let $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, and $y_i = f(x_i)$ for each i from 1 to n.

The $(n+1)$-point **Trapezoidal Rule** approximation to $\displaystyle\int_a^b f(x)\,dx$ is given by

$$\begin{aligned} T_n(f) &= \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}\Delta x \\ &= \Delta x \left[\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \cdots + \frac{y_{n-1} + y_n}{2}\right] \\ &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right) \end{aligned}$$

Example

Example

Estimate $\int_0^1 x^2\,dx$ with the Trapezoidal Rule and $n = 4$.

Example

Example

Estimate $\int_0^1 x^2\,dx$ with the Trapezoidal Rule and $n = 4$.

Solution

The set of values is

$$\{y_0, y_1, y_2, y_3, y_4\} = \left\{0, \left(\tfrac{1}{4}\right)^2, \left(\tfrac{2}{4}\right)^2, \left(\tfrac{3}{4}\right)^2, \left(\tfrac{4}{4}\right)^2\right\}$$

So

$$\begin{aligned} T_n(f) &= \frac{1}{2\cdot 4}\left(0 + 2\cdot\frac{1}{16} + 2\cdot\frac{4}{16} + 2\cdot\frac{9}{16} + \frac{16}{16}\right) \\ &= \frac{1}{8}\left(\frac{2+8+18+16}{16}\right) \\ &= \frac{44}{128} = \frac{11}{32} \end{aligned}$$

Your Turn

Example

Estimate

$$\ln 2 = \int_1^2 \frac{1}{x}\, dx$$

with the Trapezoidal Rule and $n = 8$.

Your Turn

Example

Estimate

$$\ln 2 = \int_1^2 \frac{1}{x}\, dx$$

with the Trapezoidal Rule and $n = 8$.

Solution

0.694122

Outline

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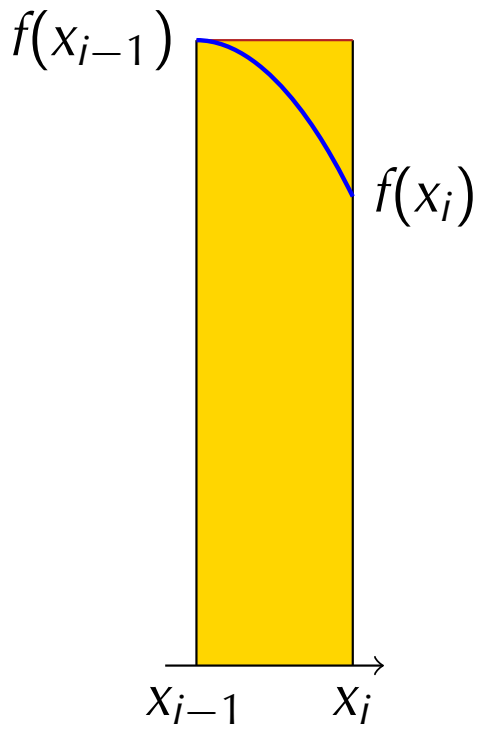
The Midpoint Rule

Simpson's Rule

Error estimates for the Trapezoidal and Midpoint Rules

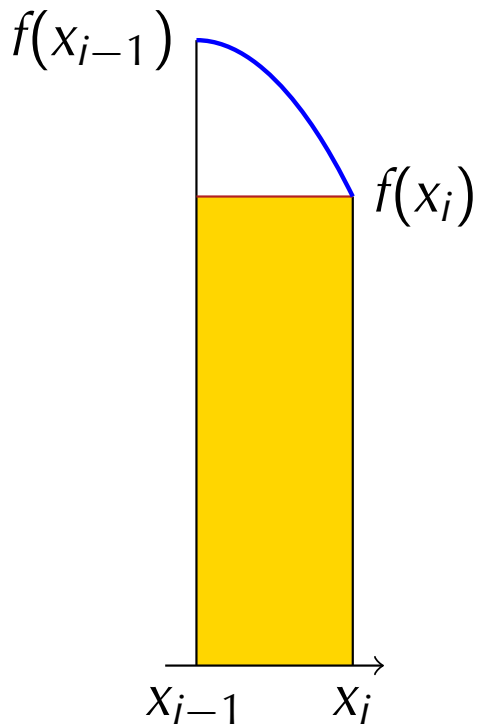
Error estimates for Simpson's Rule

Midpoints to cancel out the slop



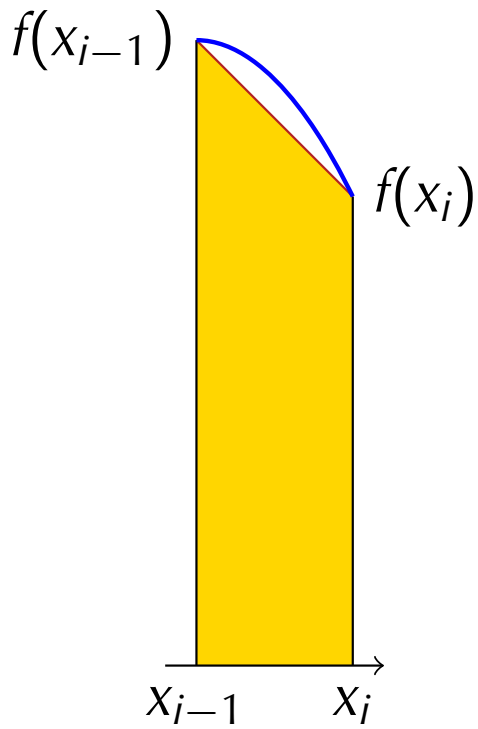
- The Trapezoidal Rule averages the values

Midpoints to cancel out the slop



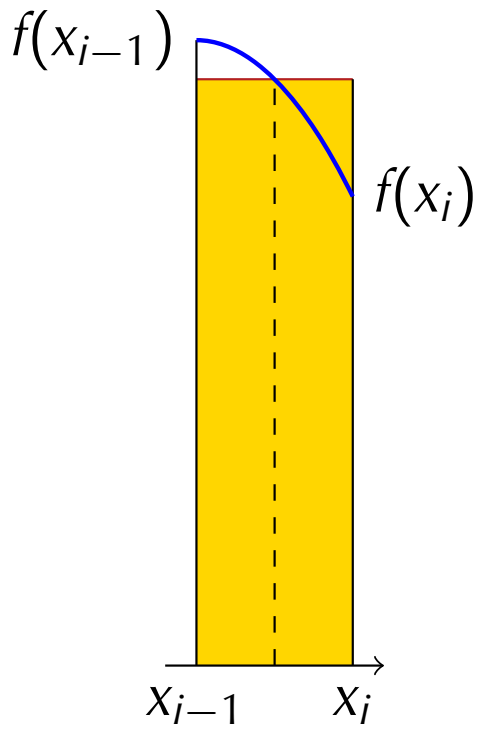
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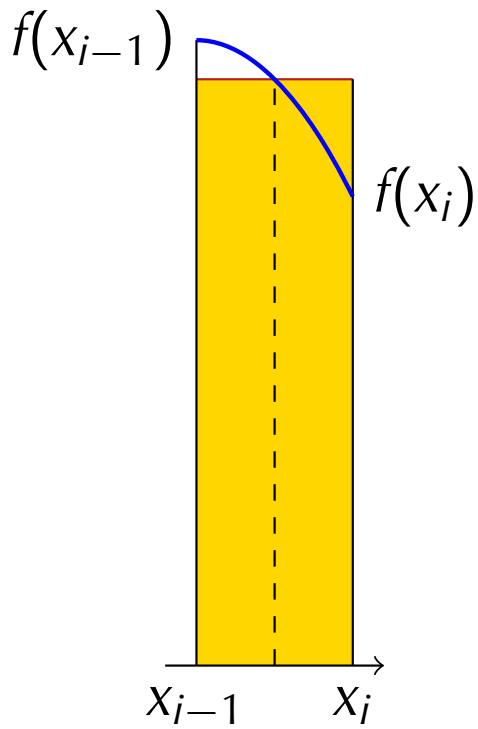
Midpoints to cancel out the slop



- The Trapezoidal Rule averages the values
- Another possibility would be to average the *points*.

$$\Delta A =$$

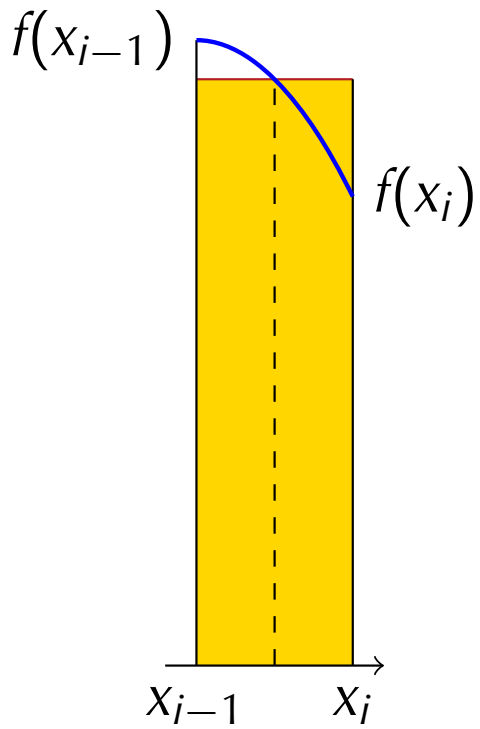
Midpoints to cancel out the slop



- The Trapezoidal Rule averages the values
- Another possibility would be to average the *points*.

$$\Delta A = f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x$$

Midpoints to cancel out the slop



- The Trapezoidal Rule averages the values
- Another possibility would be to average the *points*.

$$\Delta A = f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x$$

- Again, the smaller the intervals, the better the approximation

The Midpoint Rule

Definition

Divide the interval $[a, b]$ up into n pieces. Let $\Delta x = \dfrac{b-a}{n}$, and $x_i = a + i\Delta x$ for each i from 1 to n.

The $(n+1)$-point **Midpoint Rule** approximation to $\displaystyle\int_a^b f(x)\,dx$ is given by

$$M_n(f) = \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)\Delta x$$

$$= \Delta x\left(f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right)\right)$$

Example

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Estimate $\int_0^1 x^2\,dx$ with the Midpoint Rule and $n = 4$.

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Estimate $\int_0^1 x^2\,dx$ with the Midpoint Rule and $n = 4$.

Solution

The set of midpoints is

$$\left\{0, \tfrac{1}{8}, \tfrac{3}{8}, \tfrac{5}{8}, \tfrac{7}{8}\right\}$$

So

$$\begin{aligned} M_n(f) &= \frac{1}{4}\left(0 + \left(\tfrac{1}{8}\right)^2 + \left(\tfrac{3}{8}\right)^2 + \left(\tfrac{5}{8}\right)^2 + \left(\tfrac{7}{8}\right)^2\right) \\ &= \frac{1}{4}\left(\frac{1+9+25+49}{64}\right) \\ &= \frac{84}{256} = \frac{21}{64}. \end{aligned}$$

Your Turn

Example

Estimate

$$\ln 2 = \int_1^2 \frac{1}{x}\, dx$$

with the Midpoint Rule and $n = 8$.

Your Turn

Example

Estimate

$$\ln 2 = \int_1^2 \frac{1}{x}\, dx$$

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Solution

0.692661

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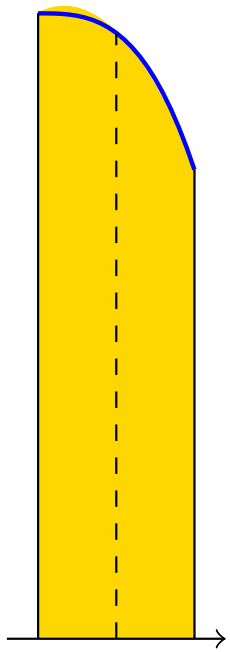
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Simpson's Rule

Error estimates for the Trapezoidal and Midpoint Rules

Error estimates for Simpson's Rule

Parabolas instead of lines



- Trapezoidal Rule approximates the function with a line
- Why not use a parabola?
- A section of a parabola passing through equally spaced points is

$$\frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

- need an even number of points

Simpson's Rule

Definition

Divide the interval $[a, b]$ up into n pieces, where n is even. Let $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$, and $y_i = f(x_i)$ for each i from 1 to n.

The $(n+1)$-point **Simpson's Rule** approximation to $\displaystyle\int_a^b f(x)\,dx$ is given by

$$\begin{aligned} S_n(f) &= \sum_{i=1}^{n/2} \frac{\Delta x}{3}\left(y_{2i} + 4y_{2i+1} + y_{2i+2}\right) \\ &= \frac{b-a}{3n}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right) \end{aligned}$$

Meet the Simpsons



Thomas Simpson
(1710-1761)



Homer Simpson

Examples

Example

Use Simpson's rule with $n = 4$ to estimate $\int_0^1 x^2\,dx$

Examples

Example

Use Simpson's rule with $n = 4$ to estimate $\int_0^1 x^2\,dx$

Solution

$$\{y_0, y_1, y_2, y_3, y_4\} = \left\{0, \left(\tfrac{1}{4}\right)^2, \left(\tfrac{2}{4}\right)^2, \left(\tfrac{3}{4}\right)^2, \left(\tfrac{4}{4}\right)^2\right\}$$

So

$$\begin{aligned} S_4(f) &= \frac{1}{3\cdot 4}\left(0 + 4\cdot\frac{1}{16} + 2\cdot\frac{4}{16} + 4\cdot\frac{9}{16} + \frac{16}{16}\right) \\ &= \frac{1}{12}\left(\frac{4+8+36+16}{16}\right) = \frac{1}{12}\cdot\frac{64}{16} = \frac{1}{3} \end{aligned}$$

Examples

Example

Use Simpson's rule with $n = 4$ to estimate $\int_0^1 x^2\,dx$

Solution

$$\{y_0, y_1, y_2, y_3, y_4\} = \left\{0, \left(\tfrac{1}{4}\right)^2, \left(\tfrac{2}{4}\right)^2, \left(\tfrac{3}{4}\right)^2, \left(\tfrac{4}{4}\right)^2\right\}$$

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No surprise we get the exact value; approximating a parabola with a parabola should be exact!

Your turn

Example

Use Simpson's Rule with $n = 8$ to estimate $\ln 2$.

Your turn

Example

Use Simpson's Rule with $n = 8$ to estimate $\ln 2$.

Solution

0.693155