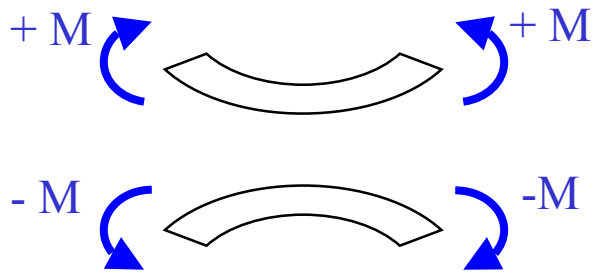
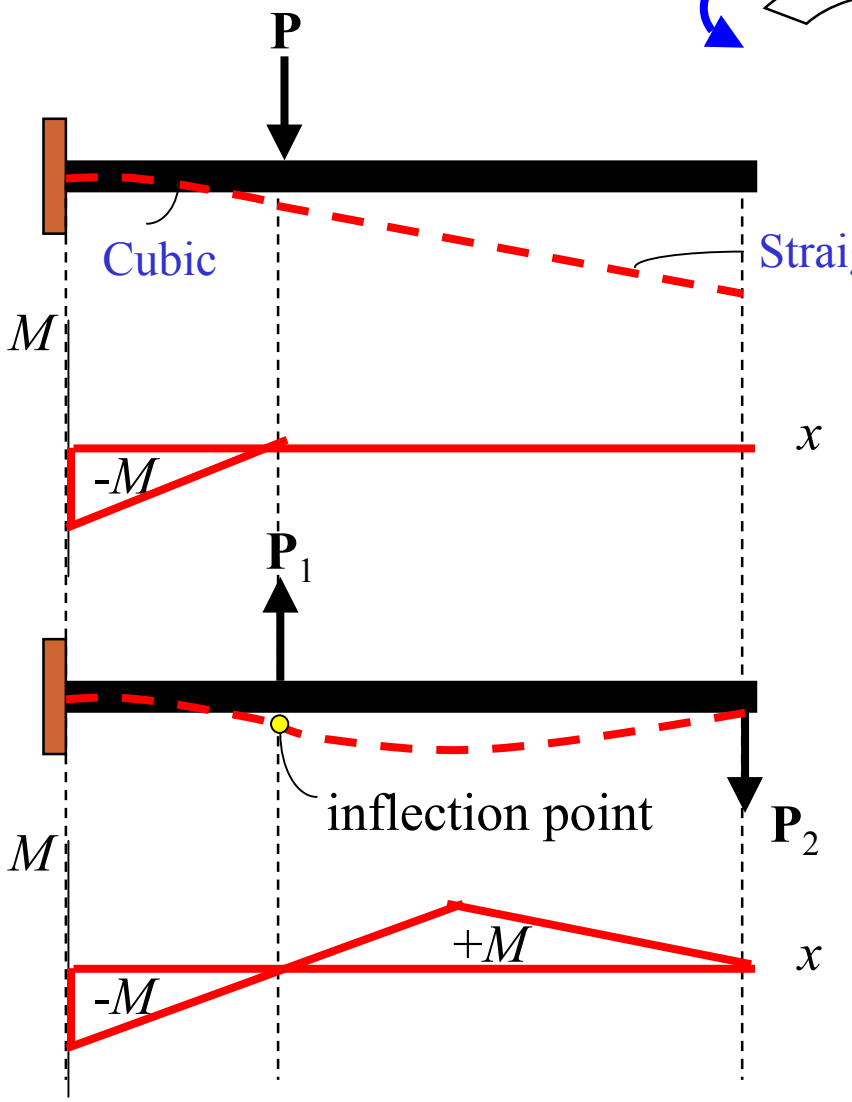
DEFLECTIONS OF BEAMS AND SHAFTS

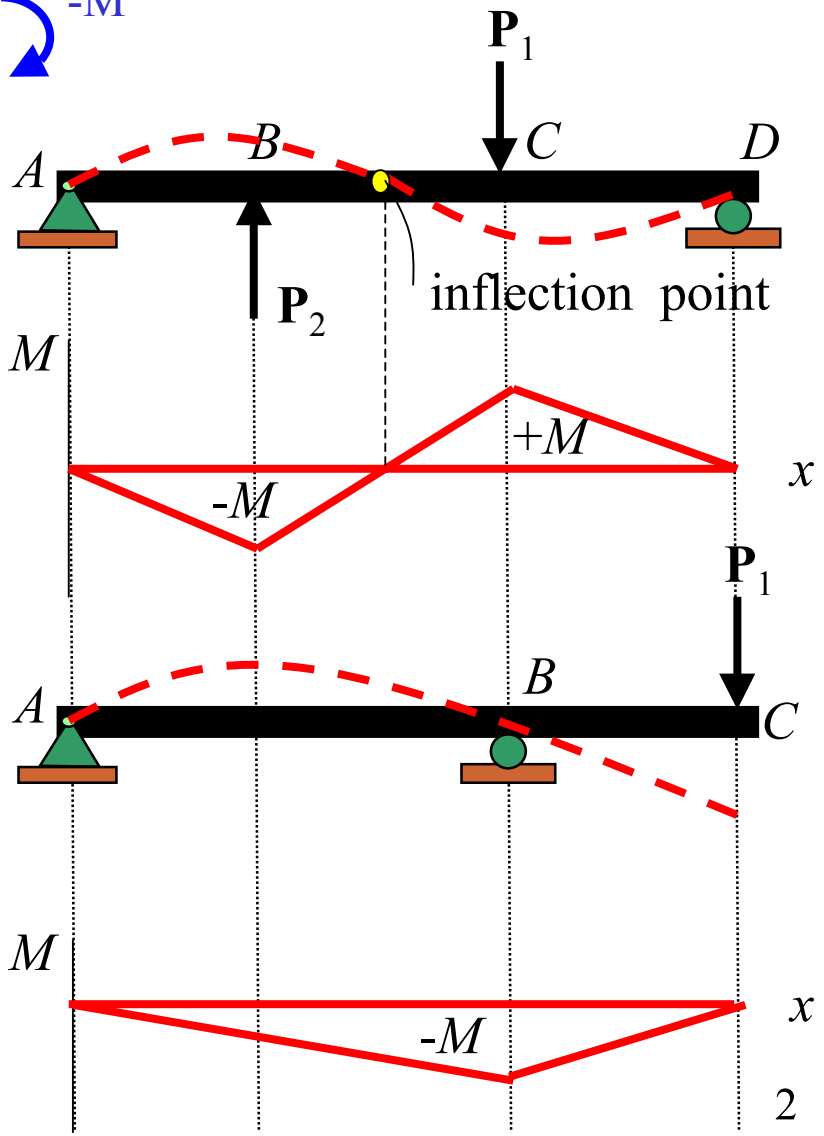
- The Elastic Curve
- Slope and Displacement by Integration
- Method of Superposition

The elastic curve

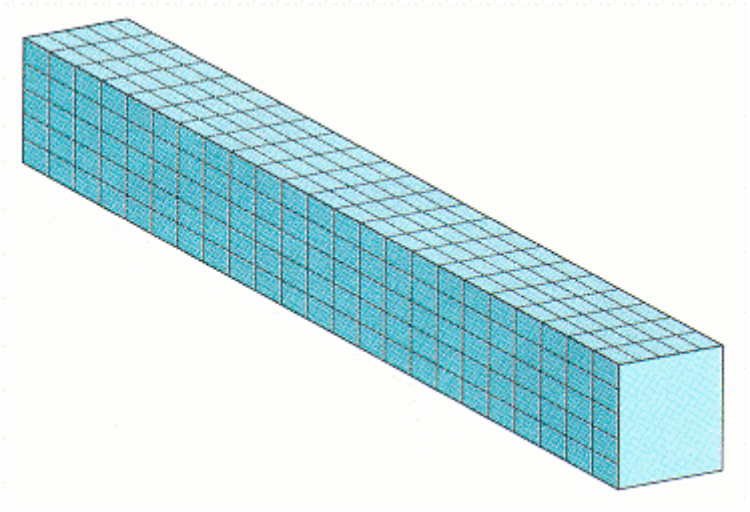
- **Elastic Curve**



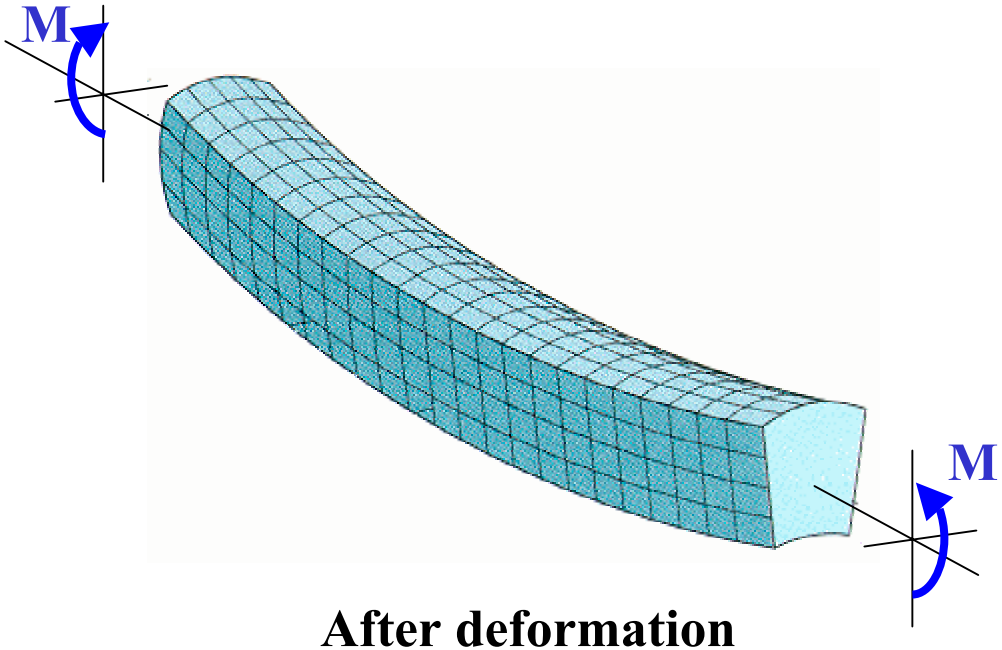




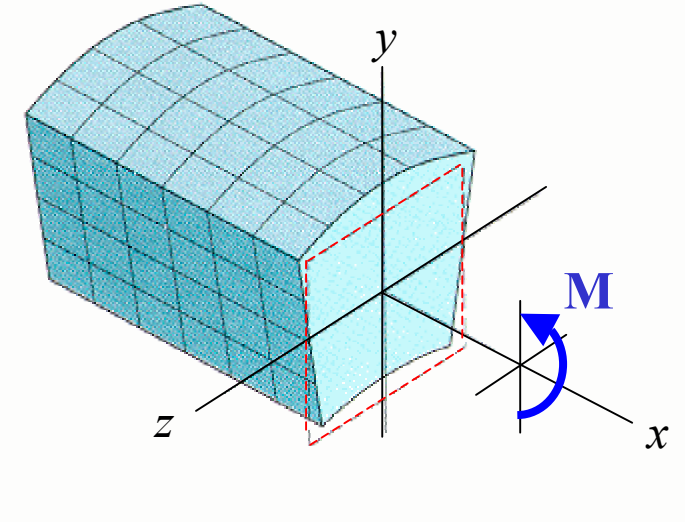
- **Elastic-Beam Theory**



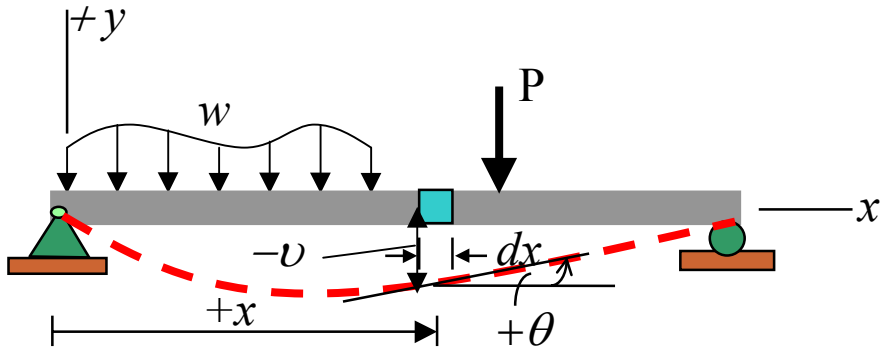
Before deformation

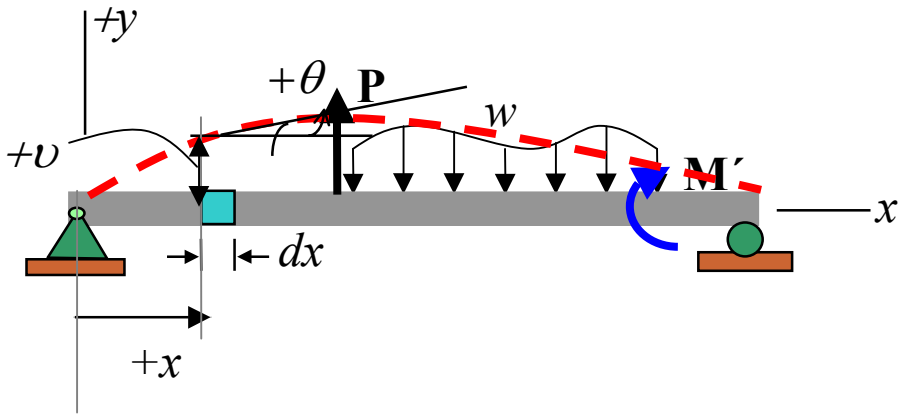


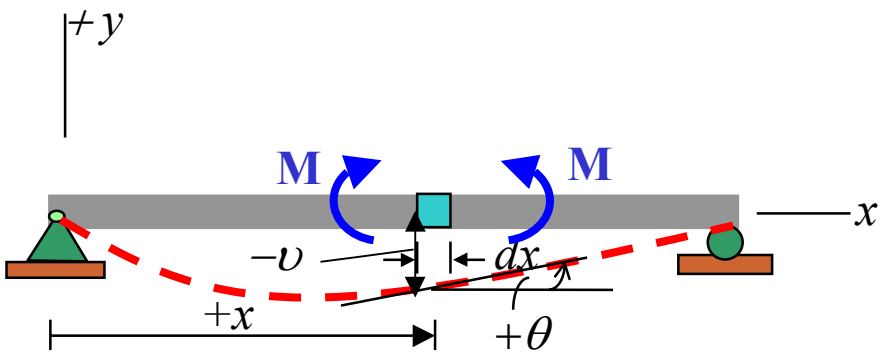
After deformation



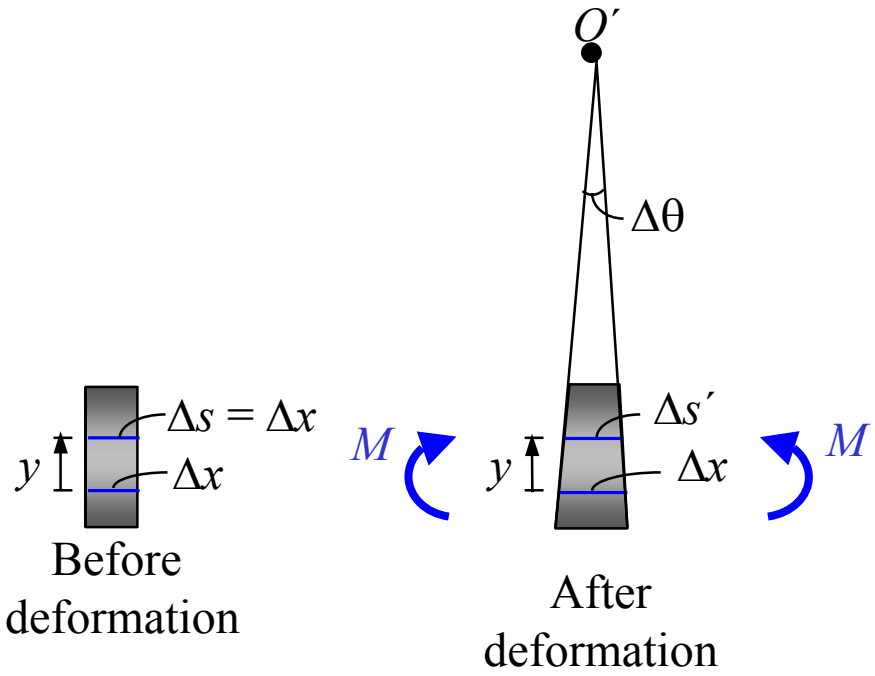
- **Sign Convention**



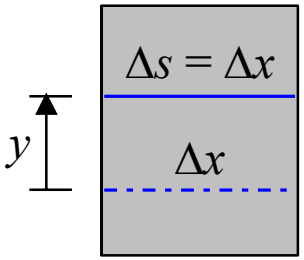




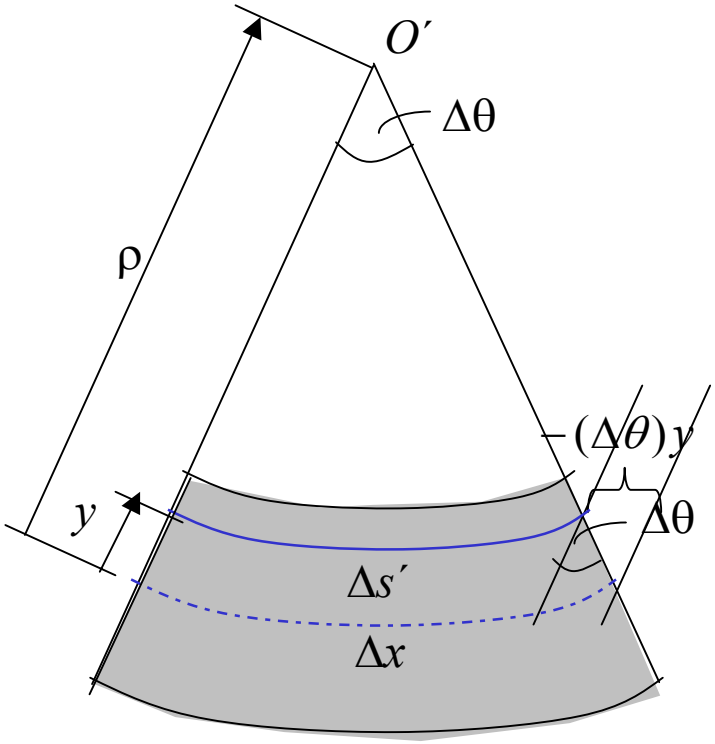
- **Pure Bending**



- **Moment-curvature relationship**



Before deformation



After deformation

$$\varepsilon = \frac{-(\Delta\theta)y}{(\Delta\theta)\rho} = -\frac{y}{\rho}$$

Then $\varepsilon = \sigma/E$ and $\sigma = -My/I$

$$\varepsilon = \frac{\sigma}{E} = -\frac{y}{\rho}$$

$$\frac{1}{\rho} = -\frac{\sigma}{yE} = -(-\frac{My}{I})(\frac{1}{yE}) = \frac{M}{EI}$$

$$\frac{M}{EI} = \frac{1}{\rho}$$

Curvature

Radius of Curvature

If we chose the υ axis positive upward, and if we can express the curvature ($1/\rho$) in terms of x and , we can then determine the elastic curve for the beam.

$$\frac{1}{\rho} = \frac{d^2\upsilon / dx^2}{[1+(d\upsilon / dx)^2]^{3/2}}$$

Therefore,

$$\frac{M}{EI} = \frac{d^2\upsilon / dx^2}{[1+(d\upsilon / dx)^2]^{3/2}}$$

$$(\frac{d\upsilon}{dx})^2 \approx 0$$

$$\boxed{\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}}$$

- **Assumptions**

  - **Linearly elastic behavior applies**
  - **Small displacement theory applies**
  - **Displacement due to shear is neglected**

Slope and displacement by integration

- **Moment-curvature**
  - $\frac{M}{EI} = \frac{1}{\rho}$

$$\frac{M}{EI} = \frac{\frac{d^2\upsilon}{dx^2}}{(1+(\frac{d\upsilon}{dx})^2)^{3/2}}$$

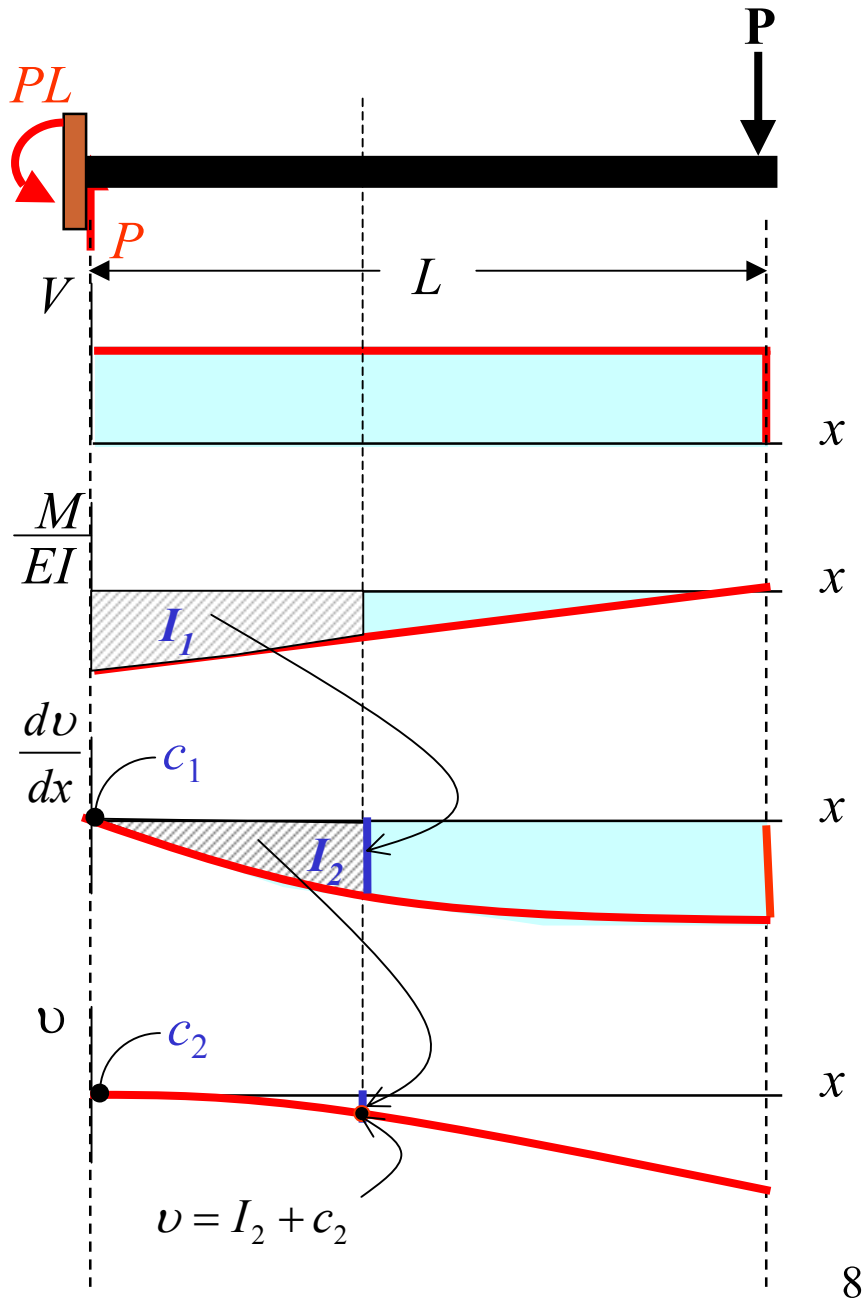
$$\frac{M}{EI} \approx \frac{d^2\upsilon}{dx^2}$$

Note : ≈ 99 % Small Deflection Theory

- **Double Integration Method**
  - $\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}$

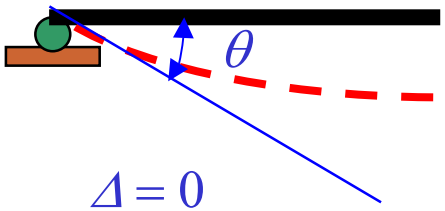
Slope (θ) $\frac{d\upsilon}{dx} = \int \frac{M}{EI}dx = (I_1) + c_1$

Deflection (Δ) $\upsilon = \int (I_1 + c_1)dx = I_2 + c_2$



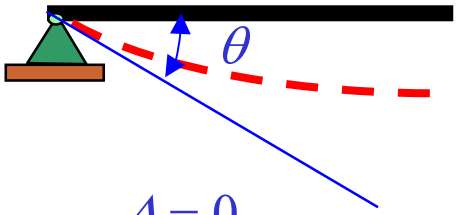
- **Boundary condition**

roller or rocker



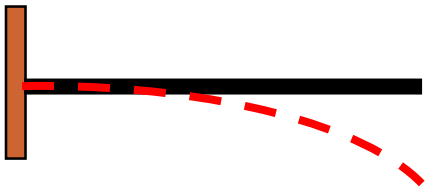
$\Delta = 0$

pin



$\Delta = 0$

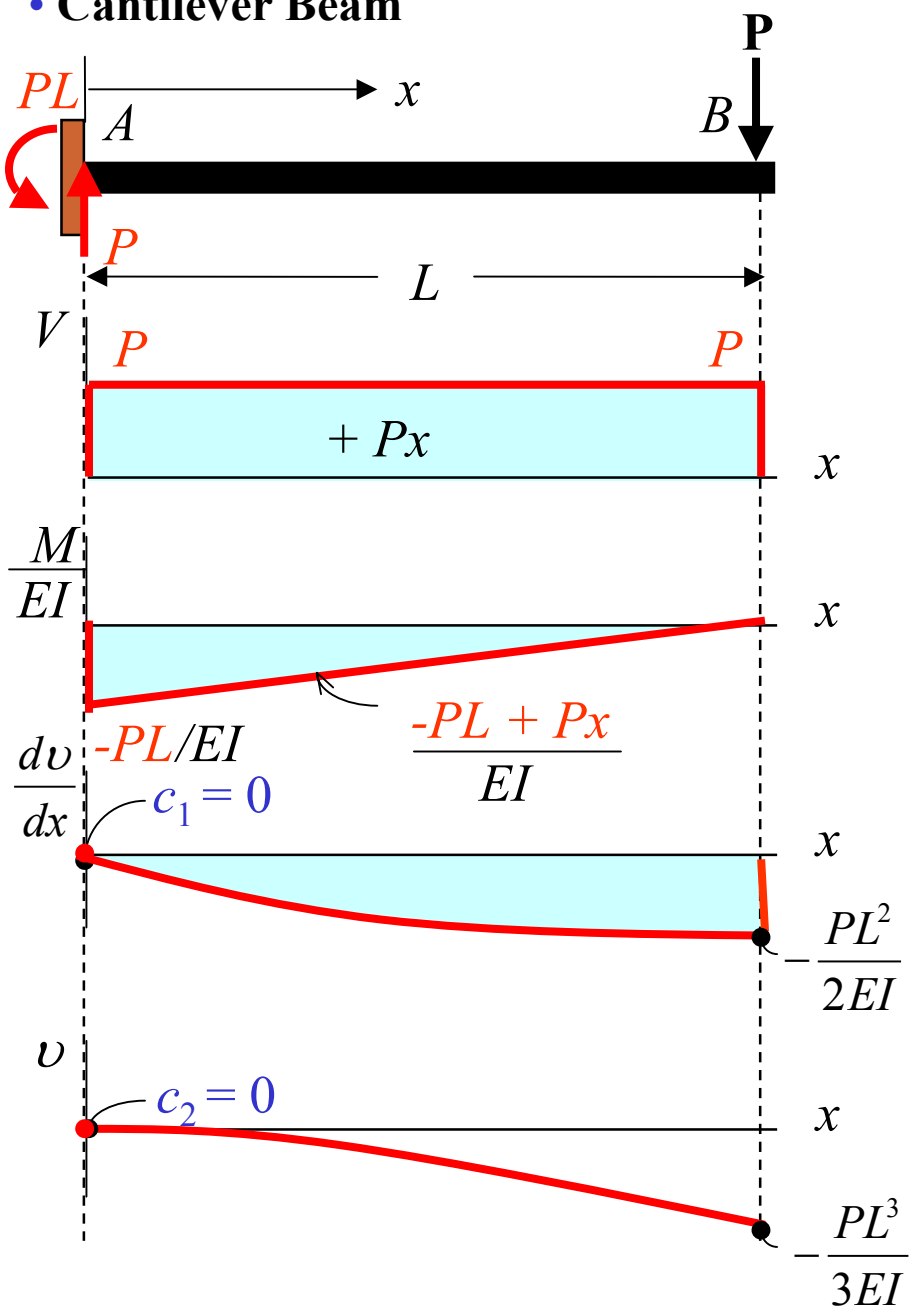
fixed support



$\Delta = 0$

$\theta = 0$

• **Cantilever Beam**



Moment-curvature relationship

$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}$$

$$\frac{d^2\upsilon}{dx^2} = \frac{1}{EI}(-PL + Px)$$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}(-PLx + \frac{Px^2}{2}) + C_1 \qquad ---(1)$$

$$\upsilon = \frac{1}{EI}(-\frac{PLx^2}{2} + \frac{Px^3}{6}) + C_1x + C_2 \qquad ---(2)$$

Boundary Condition

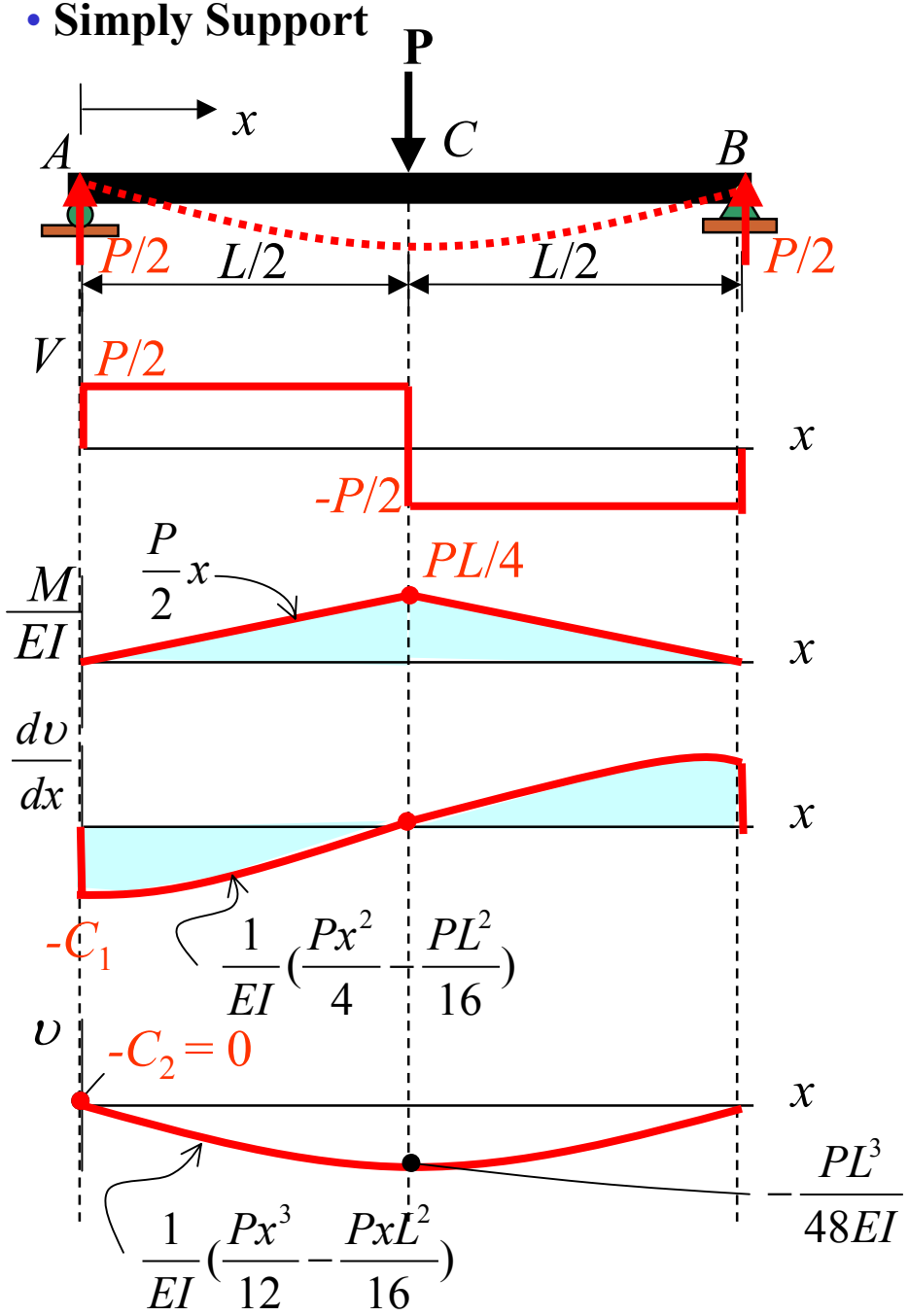
- $(\frac{d\upsilon}{dx})\Big|_{x=0} = 0$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}(-PLx + \frac{Px^2}{2}) + C_1 \quad ---> C_1 = 0$$

- $(\upsilon)\Big|_{x=0} = 0$

$$\upsilon = \frac{1}{EI}(-\frac{PLx^2}{2} + \frac{Px^3}{6}) + C_1x + C_2 \quad -> C_2 = 0$$

- **Simply Support**



Moment-curvature relationship

$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}$$

$$A \to C : \frac{d^2\upsilon}{dx^2} = \frac{1}{EI}\left(\frac{Px}{2}\right)$$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}\left(\frac{Px^2}{4}\right) + C_1 \qquad ---(1)$$

$$\upsilon = \frac{1}{EI}\left(\frac{Px^3}{12}\right) + C_1 x + C_2 \qquad ---(2)$$

Boundary Condition

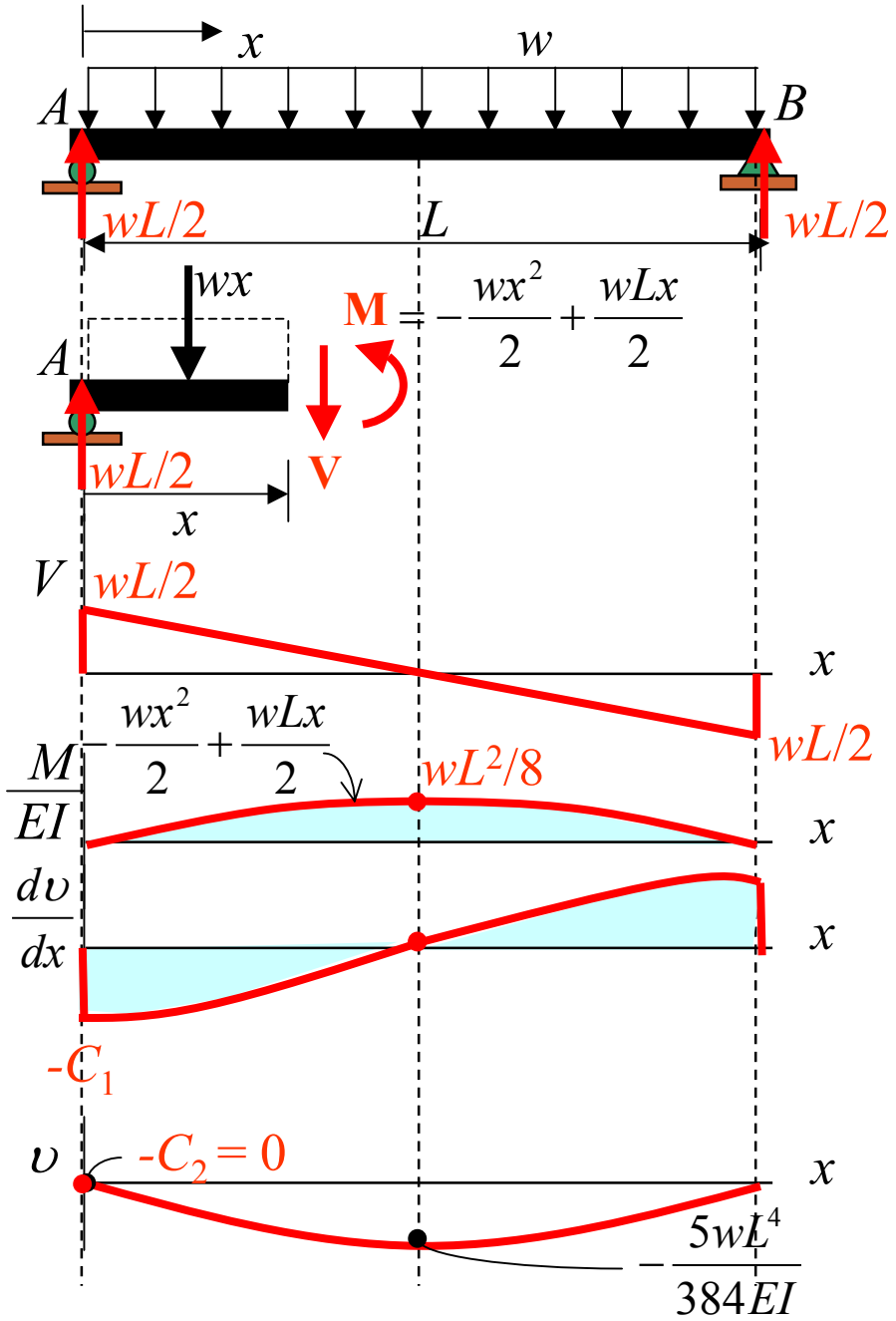
- $(\upsilon)\big|_{x=0} = 0$

$$\upsilon = \frac{1}{EI}\left(\frac{Px^3}{12}\right) + C_1 x + C_2 \dashrightarrow C_2 = 0$$

- $\left(\frac{d\upsilon}{dx}\right)\Big|_{x=L/2} = 0$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}\left(\frac{Px^2}{4}\right) + C_1 \dashrightarrow C_1 = -\frac{PL^2}{16EI}$$

- **Simply Support**



Moment-curvature relationship

$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}$$

$$A \to B : \frac{d^2\upsilon}{dx^2} = -\frac{1}{EI}\left(\frac{wx^2}{2} - \frac{wLx}{2}\right)$$

$$\frac{d\upsilon}{dx} = -\frac{1}{EI}\left(\frac{wx^3}{6} - \frac{wLx^2}{4}\right) + C_1 \qquad ---(1)$$

$$\upsilon = -\frac{1}{EI}\left(\frac{wx^4}{24} - \frac{wLx^3}{12}\right) + C_1 x + C_2 \qquad ---(2)$$

Boundary Condition

- $(\upsilon)\big|_{x=0} = 0$

$$\upsilon = -\frac{1}{EI}\left(\frac{wx^4}{24} - \frac{wLx^3}{12}\right) + C_1 x + C_2 \dashrightarrow C_2 = 0$$

(0, 0, 0, 0)

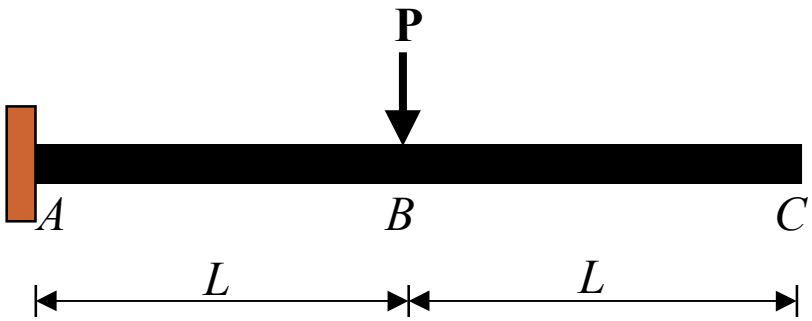
- $\left(\frac{d\upsilon}{dx}\right)\Big|_{x=L/2} = 0$

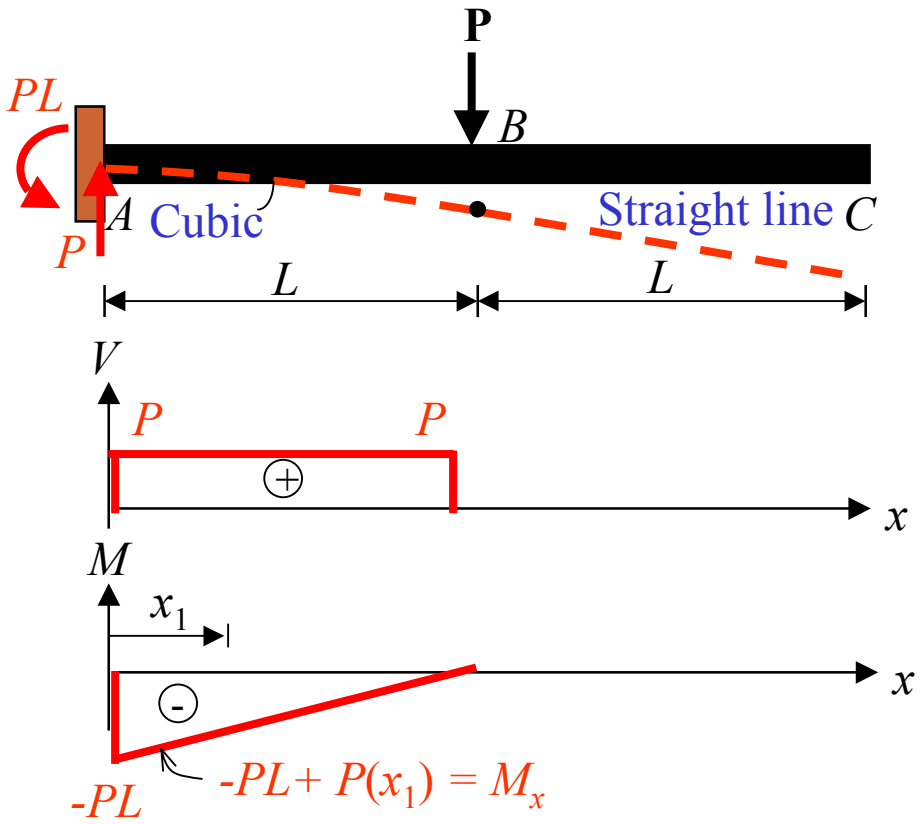
$$\frac{d\upsilon}{dx} = -\frac{1}{EI}\left(\frac{wx^3}{6} - \frac{wLx^2}{4}\right) + C_1 \dashrightarrow C_1 = -\frac{wL^3}{24EI}$$

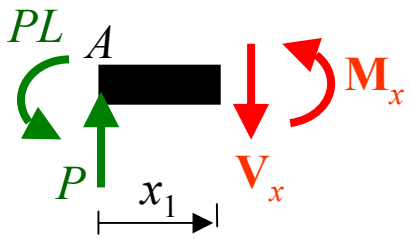
(0, L/2, L/2)

Example 1

The cantilevered beam shown is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. *EI* is constant.







- **Moment-curvature relationship**

$$\frac{d^2\upsilon}{dx^2}=\frac{M}{EI}$$

$$A\to B:\frac{d^2\upsilon}{dx^2}=\frac{1}{EI}(-PL+Px)$$

$$\frac{d\upsilon}{dx}=\frac{1}{EI}(-PLx+\frac{Px^2}{2})+C_1$$

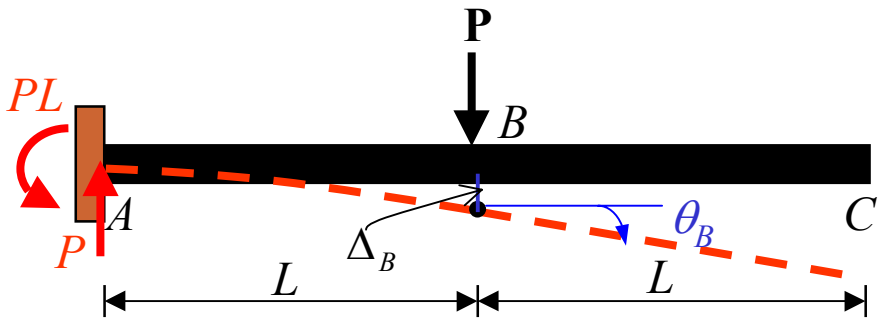
$$\upsilon=\frac{1}{EI}(-\frac{PLx^2}{2}+\frac{Px^3}{6})+C_1x+C_2$$

Boundary condition :

- $(\frac{d\upsilon}{dx})\Big|_{x=0}=0 \quad --->C_1=0$
- $(\upsilon)\big|_{x=0}=0 \quad --->C_2=0$

$$\frac{d\upsilon}{dx}=\frac{1}{EI}(-PLx+\frac{Px^2}{2}) \quad ---(1)$$

$$\upsilon=\frac{1}{EI}(-\frac{PLx^2}{2}+\frac{Px^3}{6}) \quad ---(2)$$



$$\frac{d\upsilon}{dx} = \frac{1}{EI}(-PLx + \frac{Px^2}{2}) \quad ---(1)$$

$$\upsilon = \frac{1}{EI}(-\frac{PLx^2}{2} + \frac{Px^3}{6}) \quad ---(2)$$

- **Elastic Curve Equation ; x : 0 - L**

$$(\upsilon)_{x=L} = \frac{1}{EI}(-\frac{PL^3}{2} + \frac{PL^3}{6}) = -\frac{PL^3}{3EI} \quad ,\downarrow \Delta_B$$

$$(\frac{d\upsilon}{dx})_{x=L} = \frac{1}{EI}(-PL^2 + \frac{PL^2}{2}) = -\frac{PL^2}{2EI}$$ θ_B

- **Elastic Curve Equation ; x : L - $2L$**

$$\upsilon_{BC} = \upsilon_B + \theta_B(x - L)$$

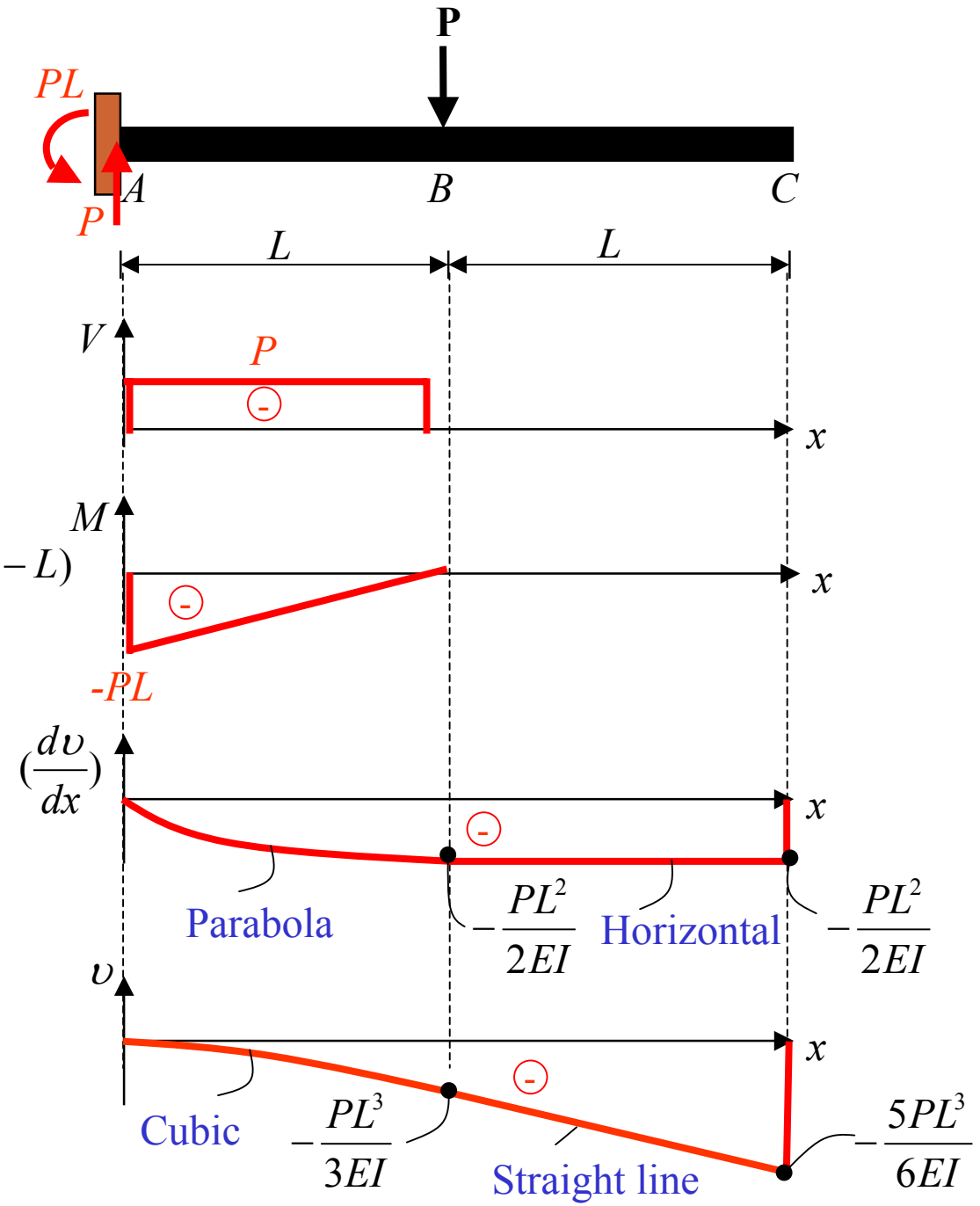
$$= (-\frac{PL^3}{3EI}) + (\frac{PL^2}{2EI})(x - L)$$

- **Conclusion**

$$(\frac{d\upsilon}{dx})_{x=L} = -\frac{PL^2}{2EI}$$

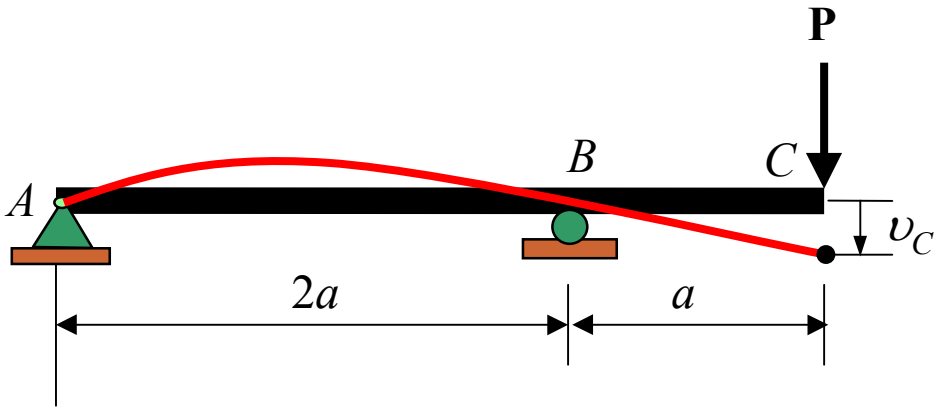
$$(\upsilon)_{x=L} = -\frac{PL^3}{3EI}$$

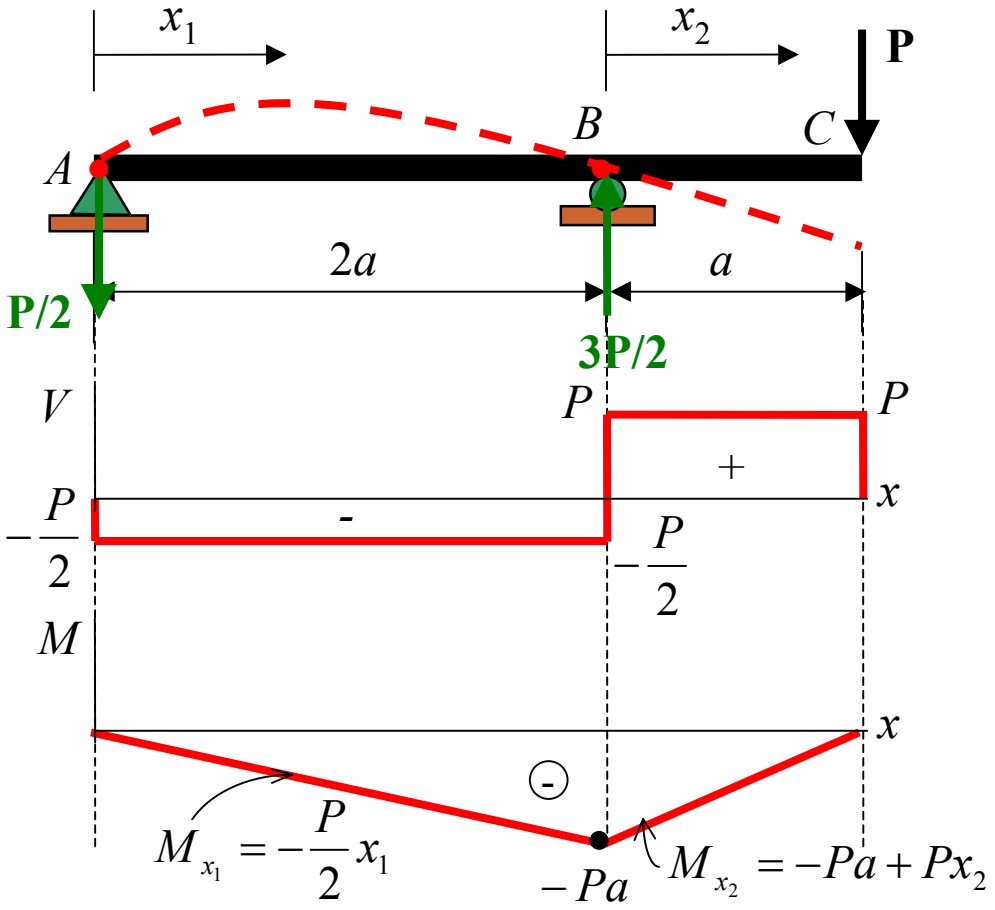
$$(\upsilon)_{x:L-2L} = (-\frac{PL^3}{3EI}) + (\frac{PL^2}{2EI})(x-L)$$



Example 2

The beam shown is subjected to a load **P** at its end. Determine the equation of the elastic curve and the slope and displacement at *C*. *EI* is constant.





- **Double Integration of x_1**

$$\frac{d^2\upsilon_1}{dx_1^2} = \frac{M_{x_1}}{EI} = -\frac{Px_1}{2EI}$$

$$\frac{d\upsilon_1}{dx_1} = \int \left(\frac{M_{x_1}}{EI}\right)dx \quad = -\frac{Px_1^2}{4EI} + C_1 \quad ---(1)$$

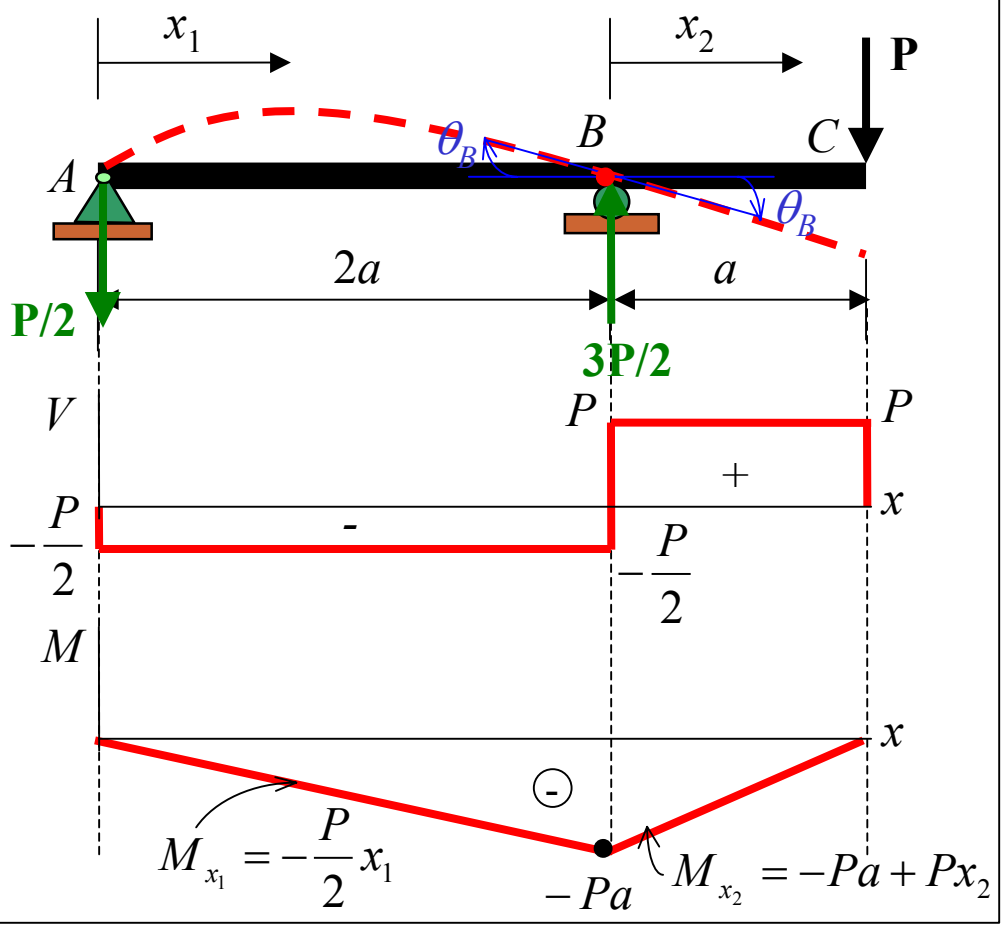
$$\upsilon_1 = \int \left(\int \frac{M_{x_1}}{EI}\right)dx)dx$$

$$\upsilon_1 = -\frac{Px_1^3}{12EI} + C_1x_1 + C_2 \quad ---(2)$$

Boundary conditions

$$\upsilon_1\Big|_{x_1=0} = 0, \qquad C_2 = 0$$

$$\upsilon_1\Big|_{x_1=2a} = 0, \qquad C_1 = \frac{Pa^2}{3EI}$$



- **Double Integration of x_2**

$$\frac{d^2\upsilon_2}{dx_2^2} = \frac{M_{x2}}{EI} = -\frac{Pa}{EI} + \frac{Px_2}{EI}$$

$$\frac{d\upsilon_2}{dx_2} = \int (\frac{M_{x_2}}{EI})dx = -\frac{Pax_2}{EI} + \frac{Px_2^2}{2EI} + C_3 \quad --(3)$$

$$\upsilon_2 = \int (\int \frac{M_{x_2}}{EI})dx)dx$$

$$\upsilon_2 = -\frac{Pax_2^2}{2EI} + \frac{Px_2^3}{6EI} + C_3x_2 + C_4 \quad ---(4)$$

Boundary conditions

$$\upsilon_2\Big|_{x_2=0} = 0, \qquad C_4 = 0$$

$$\frac{d\upsilon_1}{dx_1}\Big|_{x_1=2a} = \frac{d\upsilon_2}{dx_2}\Big|_{x_2=0}, \qquad C_3 = -\frac{2Pa^2}{3EI}$$

- **Deflection and Slope at *C***

$$\upsilon_C = \upsilon_2\Big|_{x_2=a} = -\frac{Pa^3}{2EI} + \frac{Pa^3}{6EI} - \frac{2Pa^3}{3EI} = \frac{-Pa^3}{EI} \Leftarrow$$

$$\theta_C = \frac{d\upsilon_2}{dx_2}\Big|_{x_2=a} = -\frac{Pa^2}{EI} + \frac{Pa^2}{2EI} - \frac{2Pa^2}{3EI} = -\frac{7Pa^2}{6EI} \Leftarrow$$

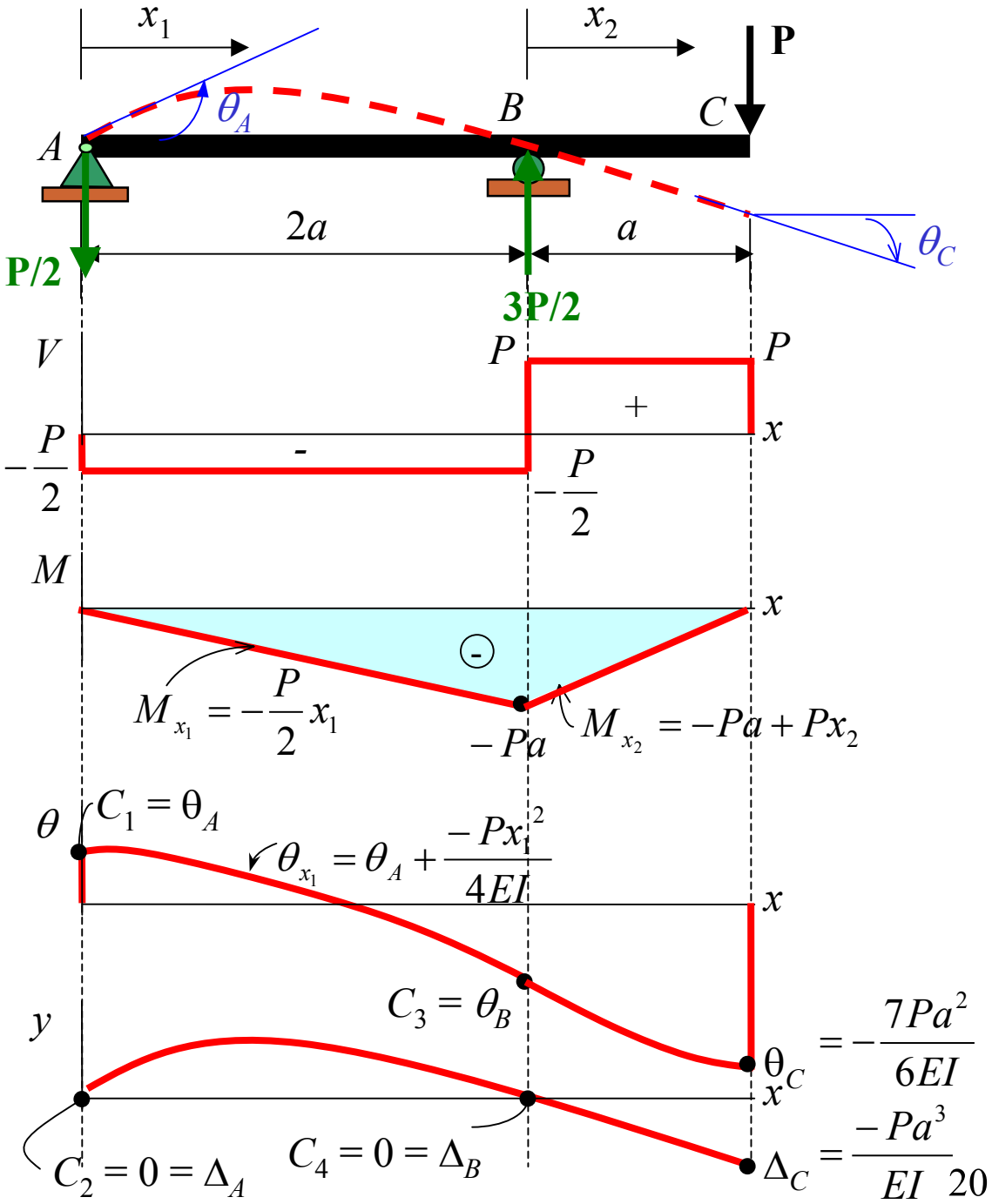
- **Conclusion**

$$\frac{d\upsilon_1}{dx_1} = -\frac{Px_1^{\,2}}{4EI} + C_1 \qquad C_1 = \frac{Pa^2}{3EI}$$

$$\upsilon_1 = -\frac{Px_1^{\,3}}{12EI} + C_1 x_1 + C_2 \qquad C_2 = 0$$

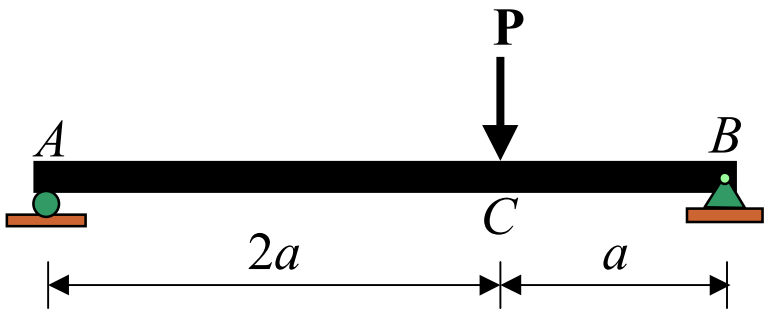
$$\frac{d\upsilon_2}{dx_2} = -\frac{Pax_2}{EI} + \frac{Px_2^{\,2}}{2EI} + C_3 \qquad C_3 = -\frac{2Pa^2}{3EI}$$

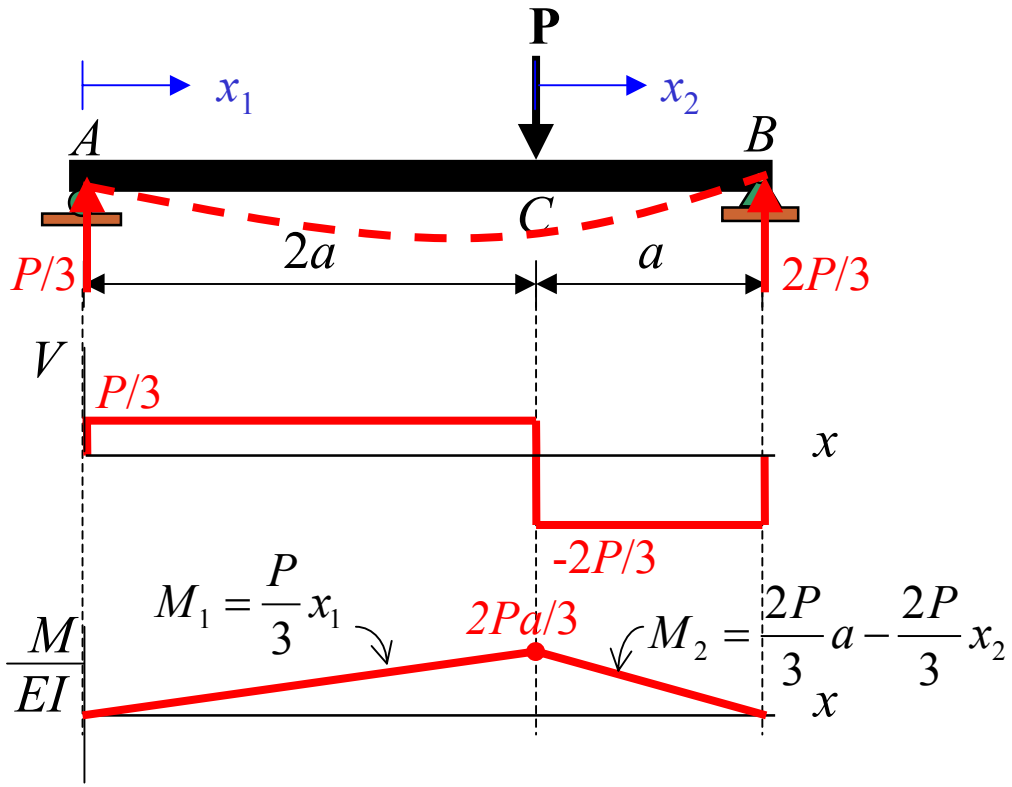
$$\upsilon_2 = -\frac{Pax_2^{\,2}}{2EI} + \frac{Px_2^{\,3}}{6EI} + C_3 x_2 + C_4 \qquad C_4 = 0$$



Example 3

The simply supported beam shown is subjected to the concentrated force *P*. Determine the maximum deflection of the beam. *EI* is constant.





- **Segment *AC***

$$\frac{d^2\upsilon_1}{dx_1^2} = \frac{M_1}{EI} = \frac{1}{EI}\left(\frac{P}{3}x_1\right)$$

$$\frac{d\upsilon_1}{dx_1} = \frac{1}{EI}\left(\frac{P}{6}x_1^2\right) + C_1 \qquad ---(1)$$

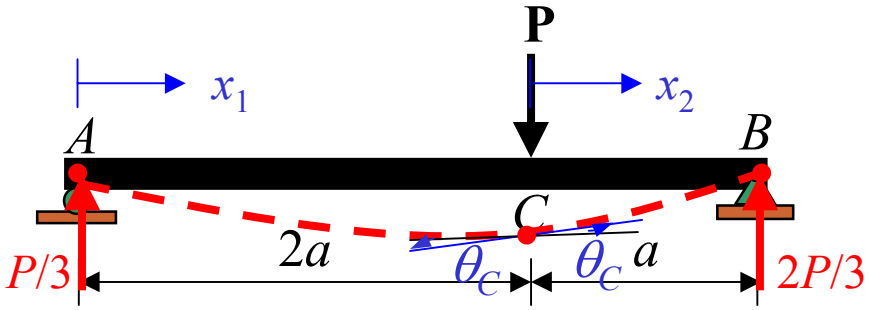
$$\upsilon_1 = \frac{1}{EI}\left(\frac{P}{18}x_1^3\right) + C_1x_1 + C_2 \qquad ---(2)$$

- **Segment *CB***

$$\frac{d^2\upsilon_2}{dx_2^2} = \frac{M_2}{EI} = \frac{1}{EI}\left(\frac{2Pa}{3} - \frac{2P}{3}x_2\right)$$

$$\frac{d\upsilon_2}{dx_2} = \frac{1}{EI}\left(\frac{2Pa}{3}x_2 - \frac{2P}{6}x_2^2\right) + C_3 \qquad ---(3)$$

$$\upsilon_2 = \frac{1}{EI}\left(\frac{2Pa}{6}x_2^2 - \frac{2P}{18}x_2^3\right) + C_3x_2 + C_4 \qquad ---(4)$$



$$\frac{d\upsilon_1}{dx_1} = \frac{1}{EI}\left(\frac{P}{6}x_1^{\,2}\right) + C_1 \quad ---(1)$$

$$\upsilon_1 = \frac{1}{EI}\left(\frac{P}{18}x_1^{\,3}\right) + C_1x_1 + C_2 \quad ---(2)$$

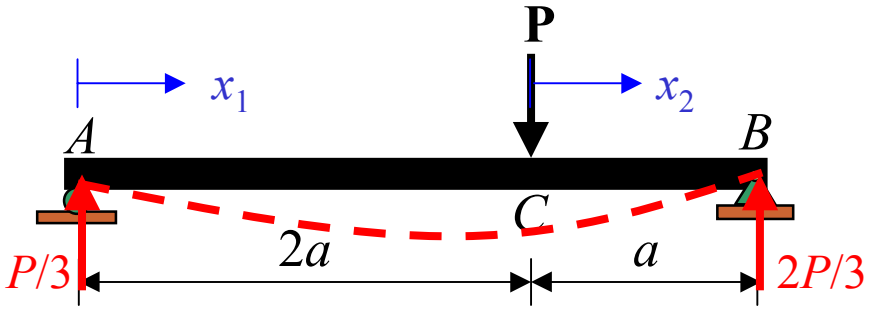
$$\frac{d\upsilon_2}{dx_2} = \frac{1}{EI}\left(\frac{2P}{3}ax_2 - \frac{2P}{6}x_2^{\,2}\right) + C_3 \quad ---(3)$$

$$\upsilon_2 = \frac{1}{EI}\left(\frac{2P}{6}ax_2^{\,2} - \frac{2P}{18}x_2^{\,3}\right) + C_3x_2 + C_4 \quad ---(4)$$

- **Boundary Conditions**
  - $(\upsilon_1)\big|_{x_1=0} = 0$, $\quad 0 = 0 + 0 + C_2 \quad ---(1^*)$
  - $(\upsilon_2)\big|_{x_2=a} = 0$, $\quad 0 = \frac{1}{EI}\left(\frac{2P}{6}a^3 - \frac{2P}{18}a^3\right) + C_3a + C_4 \quad ---(2^*)$
  - $(\upsilon_1)\big|_{x_1=2a} = (\upsilon_2)\big|_{x_2=0}$, $\quad \frac{1}{EI}\left(\frac{P}{18}(2a)^3\right) + C_1(2a) + 0 = 0 - 0 + 0 + C_4 \quad ---(3^*)$
  - $\left.\frac{d\upsilon_1}{dx_1}\right|_{x_1=2a} = \left.\frac{d\upsilon_2}{dx_2}\right|_{x_2=0}$, $\quad \frac{1}{EI}\left(\frac{P}{6}(2a)^2\right) + C_1 = 0 - 0 + C_3 \quad ---(4^*)$

Solving for (1*) to (4*) :

$$C_1 = -\frac{4Pa^2}{9EI} \qquad C_2 = 0 \qquad C_3 = \frac{2Pa^2}{9EI} \qquad C_4 = -\frac{4Pa^3}{9EI}$$



- **Elastic Curve Equation**

$$\frac{d\upsilon_1}{dx_1} = \frac{1}{EI}\left(\frac{P}{6}x_1^{\,2} - \frac{4Pa^2}{9}\right)$$

$$\upsilon_1 = \frac{1}{EI}\left(\frac{P}{18}x_1^{\,3} - \frac{4Pa^2}{9}x_1\right)$$

$$\frac{d\upsilon_2}{dx_2} = \frac{1}{EI}\left(\frac{2Pa}{3}x_2 - \frac{2P}{6}x_2^{\,2} + \frac{2Pa^2}{9}\right)$$

$$\upsilon_2 = \frac{1}{EI}\left(\frac{2P}{6}ax_2^{\,2} - \frac{2P}{18}x_2^{\,3} + \frac{2Pa^2}{9}x_2 - \frac{4Pa^3}{9}\right)$$

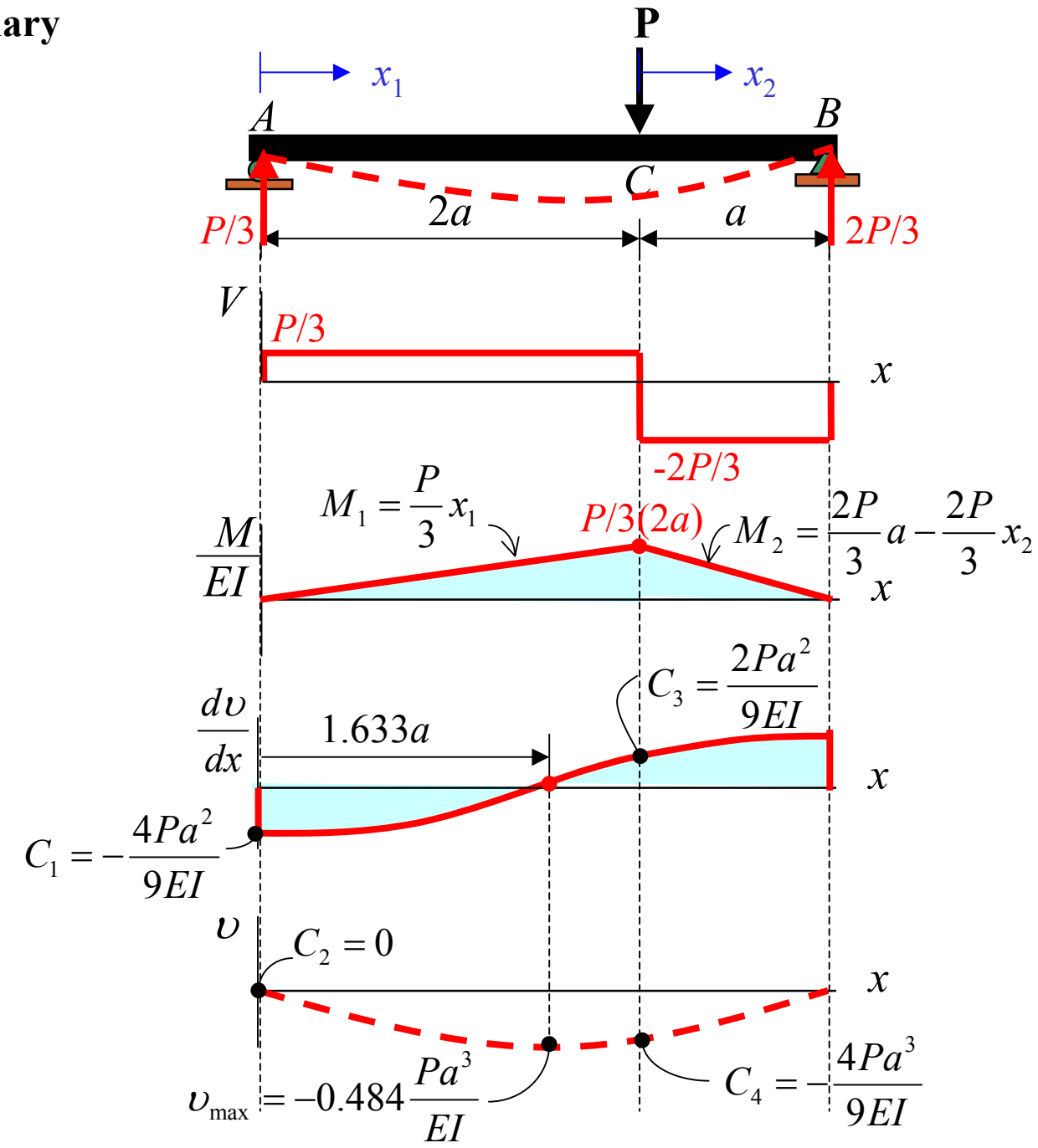
- **Maximum Deflection**

  - $\left.\dfrac{d\upsilon_1}{dx_1}\right|_{\upsilon_{\max}} = 0$; $\dfrac{d\upsilon_1}{dx_1}$ (= 0) $= \dfrac{1}{EI}\left(\dfrac{P}{6}x_1^{\,2} - \dfrac{4Pa^2}{9}\right)$

$$x_1 = 1.633a$$

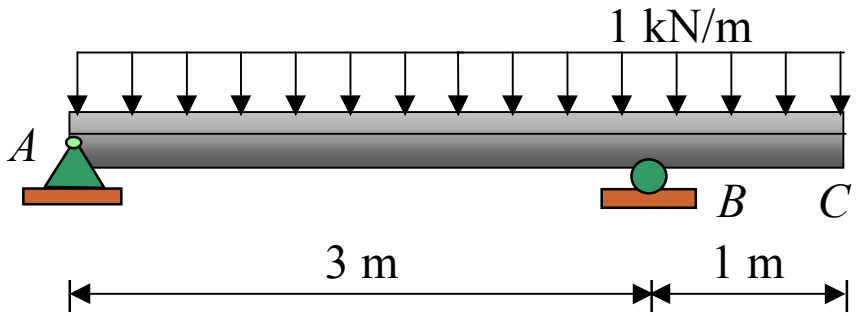
$$\upsilon_{\max}\big|_{x_1=1.633a} = \frac{1}{EI}\left[\frac{P}{18}(1.633a)^3 - \frac{4Pa^2}{9}(1.633a)\right] = -0.484\frac{Pa^3}{EI}$$

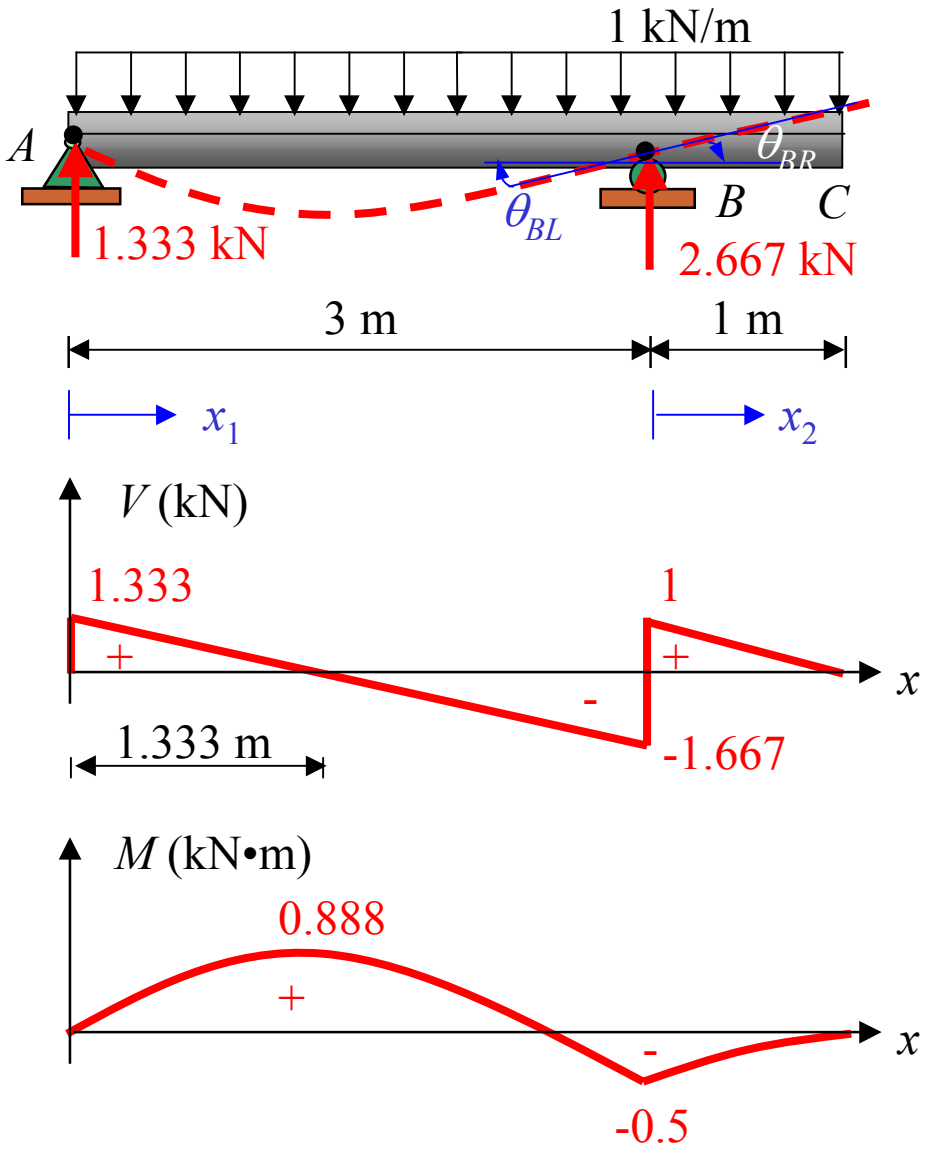
- **Summary**


P
x_1
x_2
A
B
C
2a
a
P/3
2P/3
V
P/3
x
-2P/3
$M_1 = \frac{P}{3}x_1$
P/3(2a)
$M_2 = \frac{2P}{3}a - \frac{2P}{3}x_2$
$\frac{M}{EI}$
x
$C_3 = \frac{2Pa^2}{9EI}$
$\frac{d\upsilon}{dx}$
1.633a
x
$C_1 = -\frac{4Pa^2}{9EI}$
υ
$C_2 = 0$
x
$\upsilon_{max} = -0.484\frac{Pa^3}{EI}$
$C_4 = -\frac{4Pa^3}{9EI}$

Example 4

From the beam shown determine the slope at supports , the maximum deflection of the beam and the deflection at *C*. Take $E = 10$ GPa and $I_{NA} = 15(10^6)$ mm^4.





- **Boundary condition**

1) **Deflection** at support is zero

$$(a)\ \upsilon_1|_{x_1=0} = 0$$

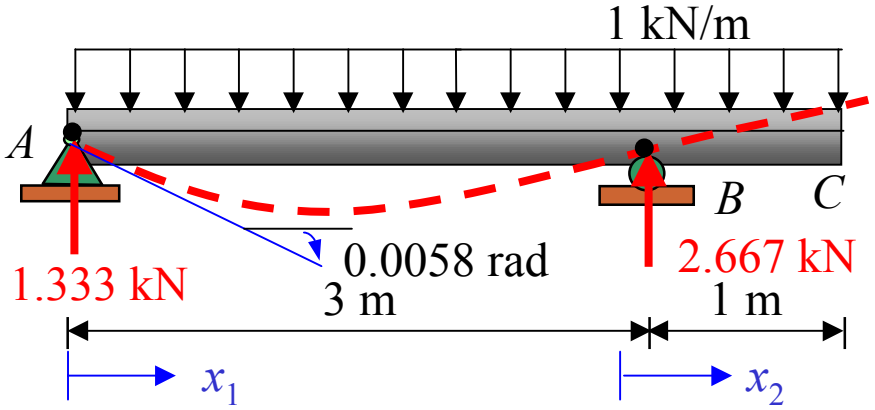
$$(b)\ \upsilon_1|_{x_1=3} = 0$$

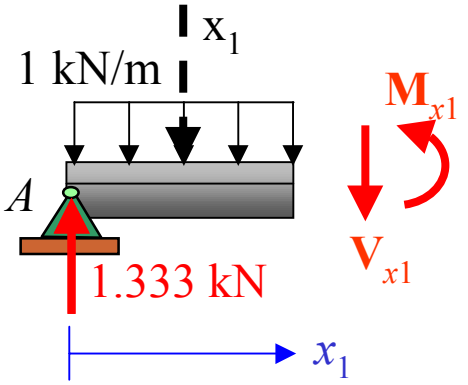
$$(c)\ \upsilon_2|_{x_2=0} = 0$$

2) The left of slope and the right of **slope at *B*** are equivalent

$$(d)\ \left.\frac{d\upsilon_1}{dx_1}\right|_{x_1=3} = \left.\frac{d\upsilon_2}{dx_2}\right|_{x_2=0}$$

Segment *AB* $E = 10$ GPa and $I_{NA} = 15(10^6)$ mm^4.





$+\circlearrowleft \Sigma M_{x1} = 0:$

$$M_{x_1} = \frac{4x_1}{3} - \frac{1x_1^{\,2}}{2}$$

- *Moment Curvature*

$$\frac{d^2\upsilon}{dx_1^{\,2}} = \frac{1}{EI}\left(\frac{4x_1}{3} - \frac{1x_1^{\,2}}{2}\right)$$

$$\frac{d\upsilon_1}{dx_1} = \frac{1}{EI}\left(\frac{2x_1^{\,2}}{3} - \frac{x_1^{\,3}}{6}\right) + C_1 \quad ---(1)$$

$$\upsilon_1 = \frac{1}{EI}\left(\frac{2x_1^{\,3}}{9} - \frac{x_1^{\,4}}{24}\right) + C_1x_1 + C_2 \quad ---(2)$$

- *Boundary Condition*

$(a)\quad \upsilon_1\big|_{x_1=0} = 0 \quad : \quad C_2 = 0$

$(b)\quad \upsilon_1\big|_{x_1=3} = 0 \quad : \quad C_1 = -\frac{7}{8EI}$

- *Slope at A*

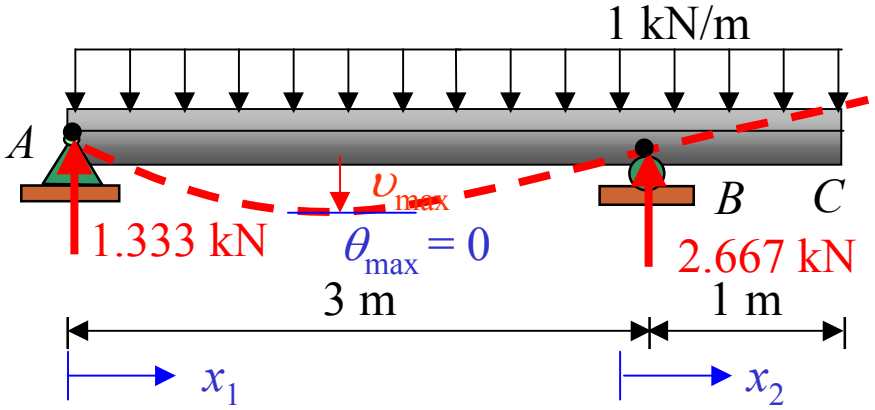
from (1) : *at* $x_1 = 0$,

$$\theta_A = \left(\frac{d\upsilon_1}{dx_1}\right)\Big|_{x_1=0} = \frac{1}{EI}\left(\frac{2\overset{0}{\cancel{x_1}^{\,2}}}{3} - \frac{\overset{0}{\cancel{x_1}^{\,3}}}{6}\right) + C_1$$

$$\theta_A = C_1 = -\frac{7}{8EI}$$

$$\theta_A = -\frac{7}{8(10\times10^6\ kPa)(15\times10^{-6}\ m^4)}$$

$$= -0.005833\ rad \Leftarrow$$



$E = 10$ GPa and $I_{NA} = 15(10^6)$ mm^4.

$$\frac{d\upsilon_1}{dx_1} = \frac{1}{EI}\left(\frac{2x_1^{\,2}}{3} - \frac{x_1^{\,3}}{6}\right) - \frac{7}{8EI} \quad ---(1)$$

$$\upsilon_1 = \frac{1}{EI}\left(\frac{2x_1^{\,3}}{9} - \frac{x_1^{\,4}}{24}\right) - \frac{7}{8EI}x_1 \quad ---(2)$$

- *The Maximum Displacement*

υ_{max} *when* $\dfrac{d\upsilon_1}{dx_1} = 0$ *substute in* (1)

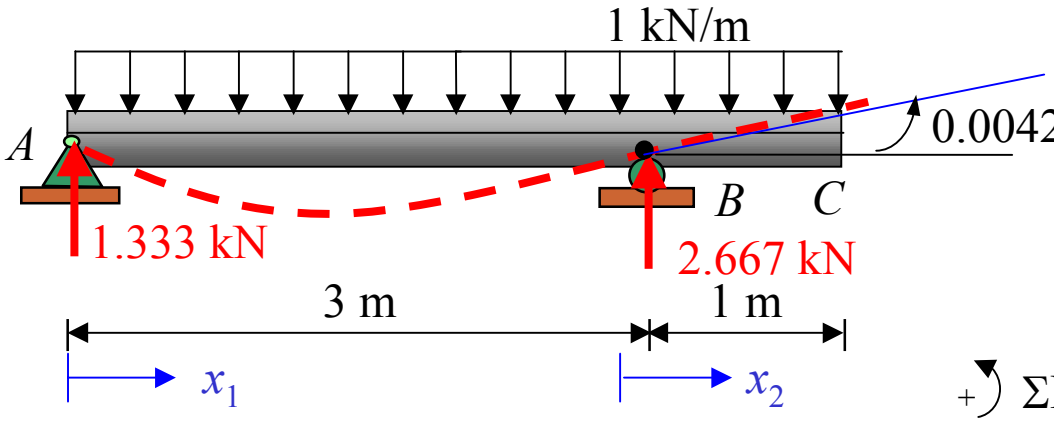
$$0 = \frac{1}{(10\times10^{+6})(15\times10^{-6})}\left(\frac{2x_1^{\,2}}{3} - \frac{x_1^{\,3}}{6}\right) - \frac{7}{8[(10\times10^{+6})(15\times10^{-6})]}$$

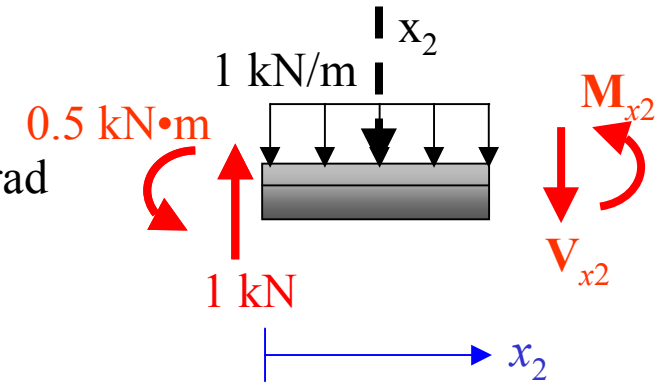
$$x_1 = 1.427 \text{ m}$$

Substitute $x_1 = 1.427$ m, $E = 10\text{x}10^6$ kPa and $I = 15(10^{-6})$ m^4 in (2)

$\upsilon_{max} = -5.171$ mm, $\downarrow$ ⇦

Segment *BC*





$$+\circlearrowleft \Sigma M_{x2} = 0:\ M_{x_2} = \frac{-1}{2} + x_2 - \frac{x_2^{\,2}}{2}$$

- *Moment Curvature*

$$\frac{d^2\upsilon_2}{dx_2^{\,2}} = \frac{1}{EI}\left(\frac{-1}{2} + x_2 - \frac{x_2^{\,2}}{2}\right)$$

$$\frac{d\upsilon_2}{dx_2} = \frac{1}{EI}\left(\frac{-x_2}{2} + \frac{x_2^{\,2}}{2} - \frac{x_2^{\,3}}{6}\right) + C_3$$

$$\upsilon_2 = \frac{1}{EI}\left(\frac{-x_2^{\,2}}{4} + \frac{x_2^{\,3}}{6} - \frac{x_2^{\,4}}{24}\right) + C_3 x_2 + C_4$$

- *Boundary Condition*

$$(c)\quad \upsilon_2\big|_{x_2=0} = 0, \qquad C_4 = 0$$

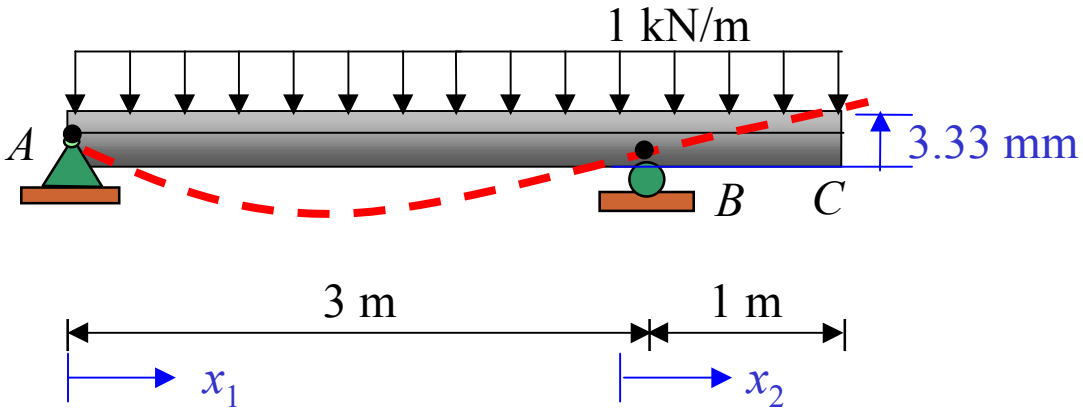
$$(c)\ \left.\frac{d\upsilon_1}{dx_1}\right|_{x_1=3} = \left.\frac{d\upsilon_2}{dx_2}\right|_{x_2=0}$$

$$\frac{1}{EI}\left(\frac{2x_1^{\,2}}{3} - \frac{x_1^{\,3}}{6}\right) - \frac{7}{8EI} = \frac{1}{EI}\left(\frac{-x_2}{2} + \frac{x_2^{\,2}}{2} - \frac{x_2^{\,3}}{6}\right) + C_3$$

(3, 3, 0, 0, 0)

$$C_3 = \frac{5}{8EI}$$

$$\theta_B = \left(\frac{d\upsilon_2}{dx_2}\right)_{x_2=0} = C_3 = \frac{5}{8(10\times 15)} = 0.004167\ rad \Leftarrow$$



$$\upsilon_2 = \frac{1}{EI}\left(\frac{-x_2^{\,2}}{4} + \frac{x_2^{\,3}}{6} - \frac{x_2^{\,4}}{24}\right) + \frac{5}{8EI}x_2$$

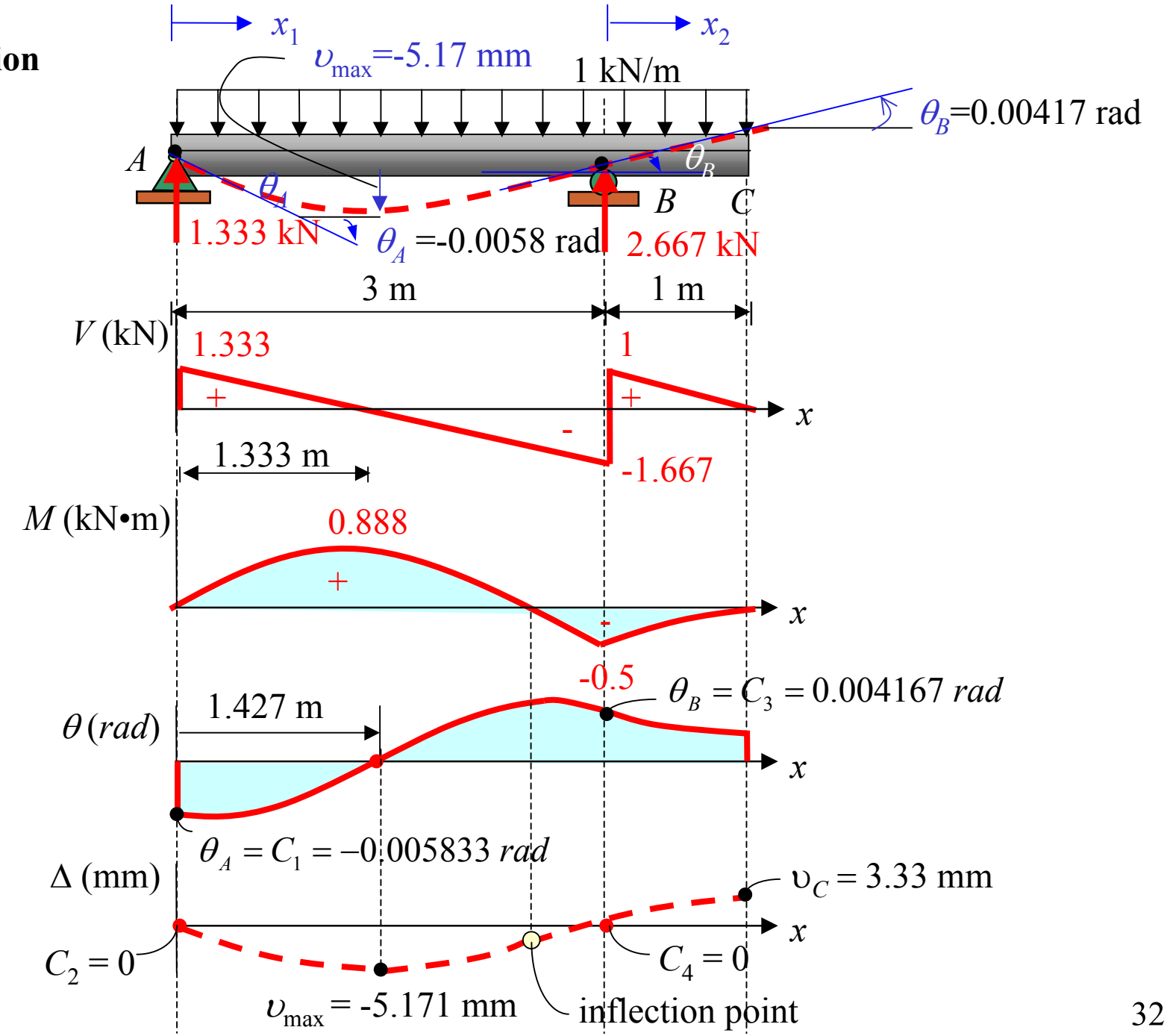
- *Displacement at C*

$$\upsilon_C = \upsilon_2\Big|_{x_2=1} = \frac{1}{EI}\left(\frac{-\overset{1}{\overbrace{x_2^{\,2}}}}{4} + \frac{\overset{1}{\overbrace{x_2^{\,3}}}}{6} - \frac{\overset{1}{\overbrace{x_2^{\,4}}}}{24}\right) + \frac{5}{8EI}\overset{1}{\overbrace{x_2}}$$

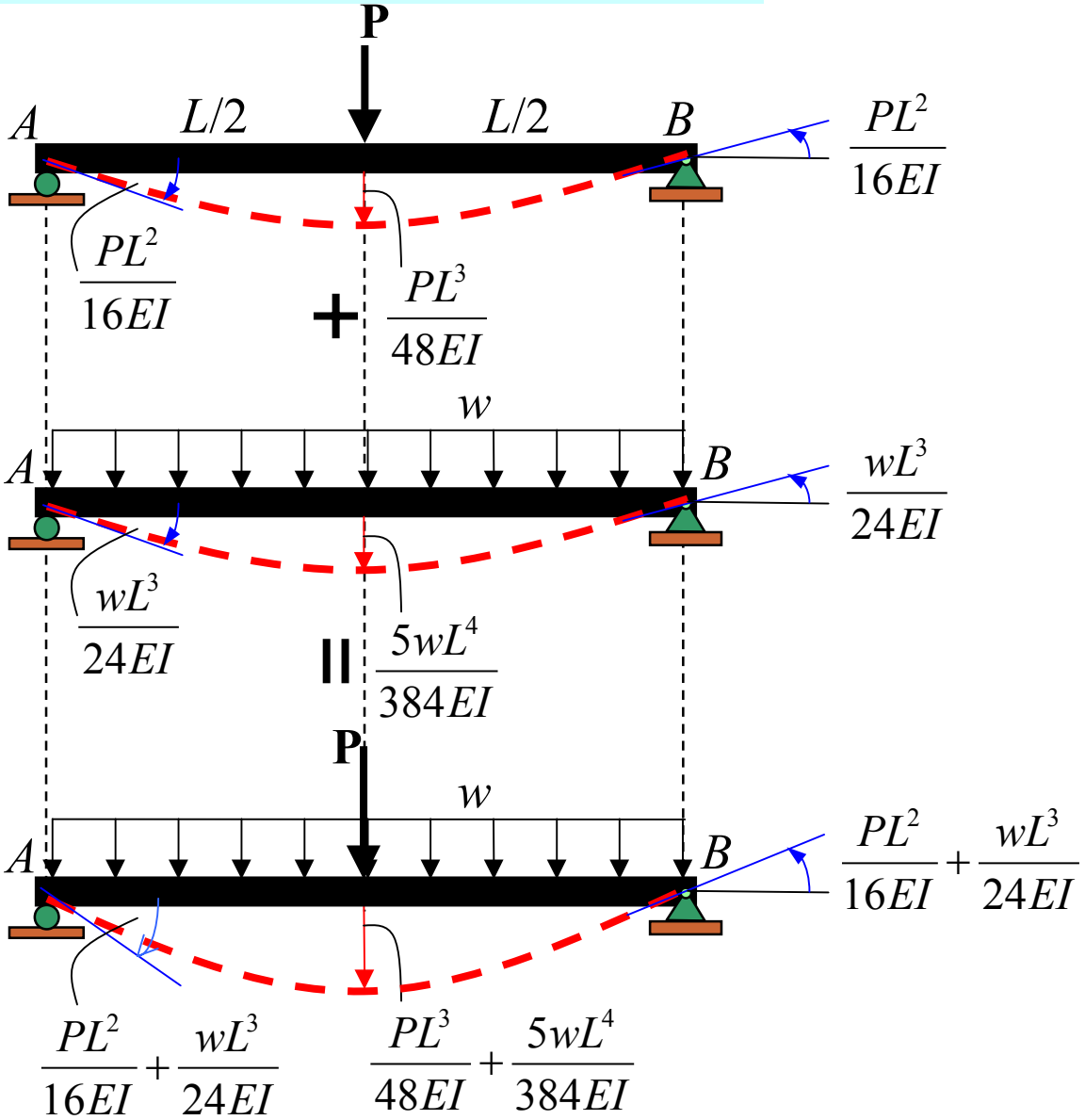
$$= -\frac{3}{24EI} + \frac{5}{8EI} = \frac{12}{24EI}$$

$$\upsilon_C = \frac{12}{24(10\times 15)} = 3.333\ mm\ , \quad \uparrow$$

• **Conclusion**


x_1
x_2
υ_{max}=-5.17 mm
1 kN/m
θ_B=0.00417 rad
A
B
C
θ_B
θ_A
1.333 kN
θ_A =-0.0058 rad
2.667 kN
3 m
1 m
V (kN)
1.333
1
+
+
-
x
-1.667
1.333 m
M (kN•m)
0.888
+
-
x
-0.5
$\theta_B = C_3 = 0.004167\ rad$
1.427 m
θ (rad)
x
$\theta_A = C_1 = -0.005833\ rad$
Δ (mm)
υ_C = 3.33 mm
x
$C_2 = 0$
$C_4 = 0$
υ_{max} = -5.171 mm
inflection point

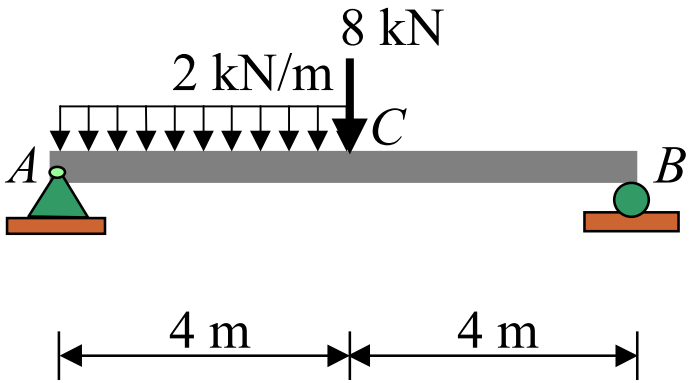
Slope and displacement by superposition method

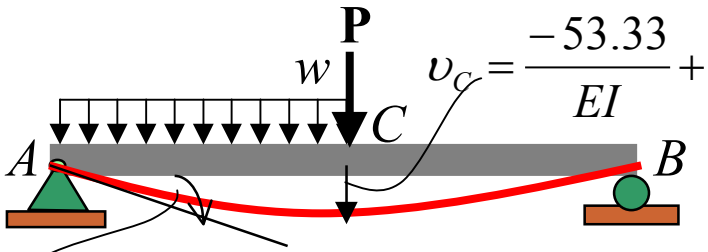


Note: *The displacement and the slope are using the table in Appendix C for each part*

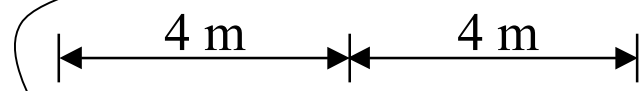
Example 5

Determine the displacement at point *C* and the slope at the support *A* of the beam shown. *EI* is constant.



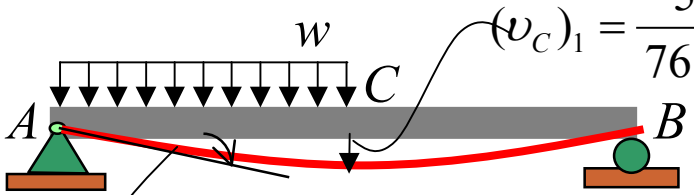


$$\upsilon_C = \frac{-53.33}{EI} + \left(\frac{-85.33}{EI}\right) = \frac{-138.66\ \ kN \bullet m^3}{EI} \downarrow \Leftarrow$$



$$\theta_A = \frac{-24}{EI} + \left(\frac{-32}{EI}\right) = \frac{(-56\ kN \bullet m^2)}{EI} \Leftarrow$$

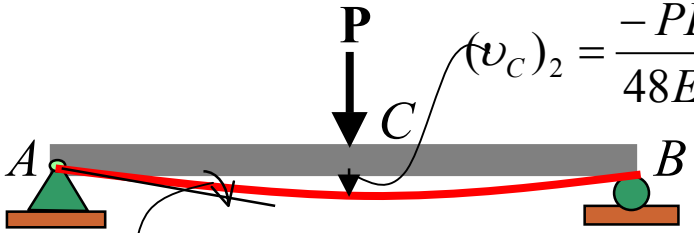
=



$$(\upsilon_C)_1 = \frac{-5wL^4}{768EI} = \frac{-5(2\ \ kN/m)(8\ m)^4}{768EI} = \frac{-53.33\ \ kN \bullet m^3}{EI} \downarrow$$

$$(\theta_A)_1 = \frac{-3wL^3}{128EI} = \frac{-3(2\ \ kN/m)(8\ m)^3}{128EI} = \frac{-24\ \ kN \bullet m^2}{EI}$$

+



$$(\upsilon_C)_2 = \frac{-PL^3}{48EI} = \frac{-8\ \ kN(8\ m)^3}{48EI} = \frac{-85.33\ \ kN \bullet m^3}{EI} \downarrow$$

$$(\theta_A)_2 = \frac{-PL^2}{16EI} = \frac{-8\ \ kN(8\ m)^2}{16EI} = \frac{-32\ \ kN \bullet m^2}{EI}$$