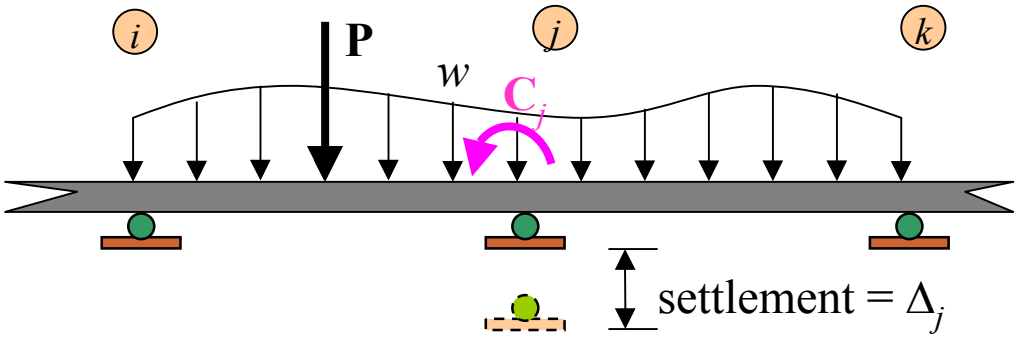
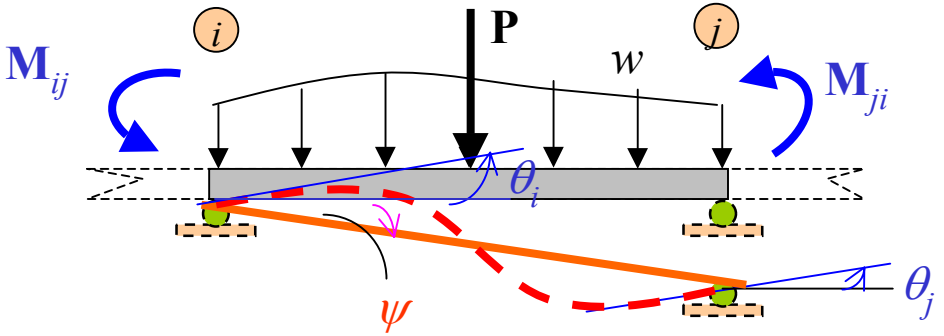
BEAM ANALYSIS USING THE STIFFNESS METHOD

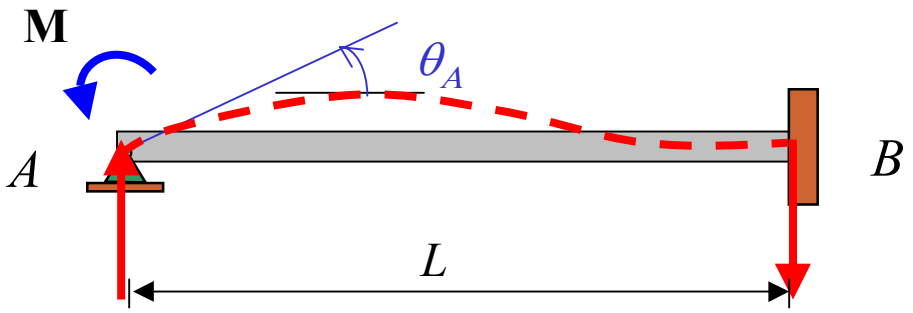
- Development: The Slope-Deflection Equations
- Stiffness Matrix
- General Procedures
- Internal Hinges
- Temperature Effects
- Force & Displacement Transformation
- Skew Roller Support

Slope – Deflection Equations

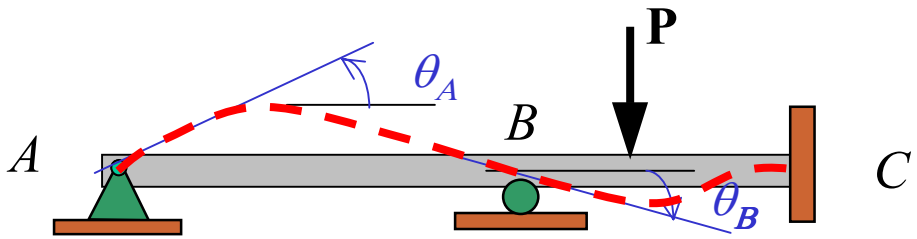




• **Degrees of Freedom**

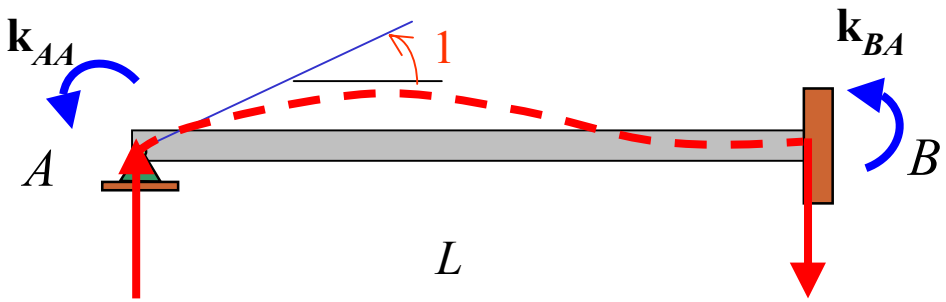


1 DOF: θ_A



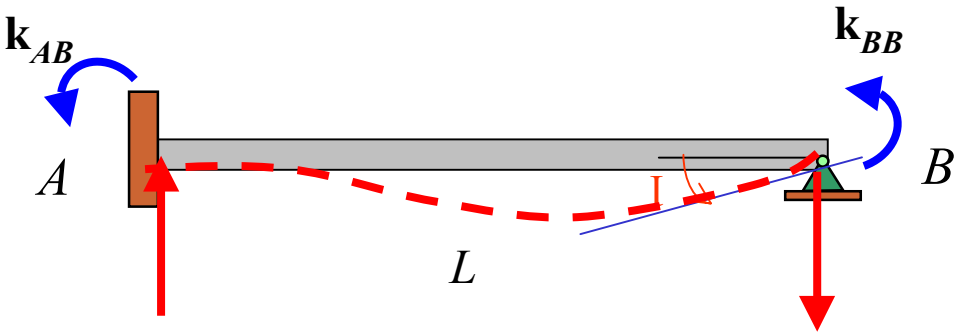
2 DOF: θ_A, θ_B

- **Stiffness Definition**



$$k_{AA} = \frac{4EI}{L}$$

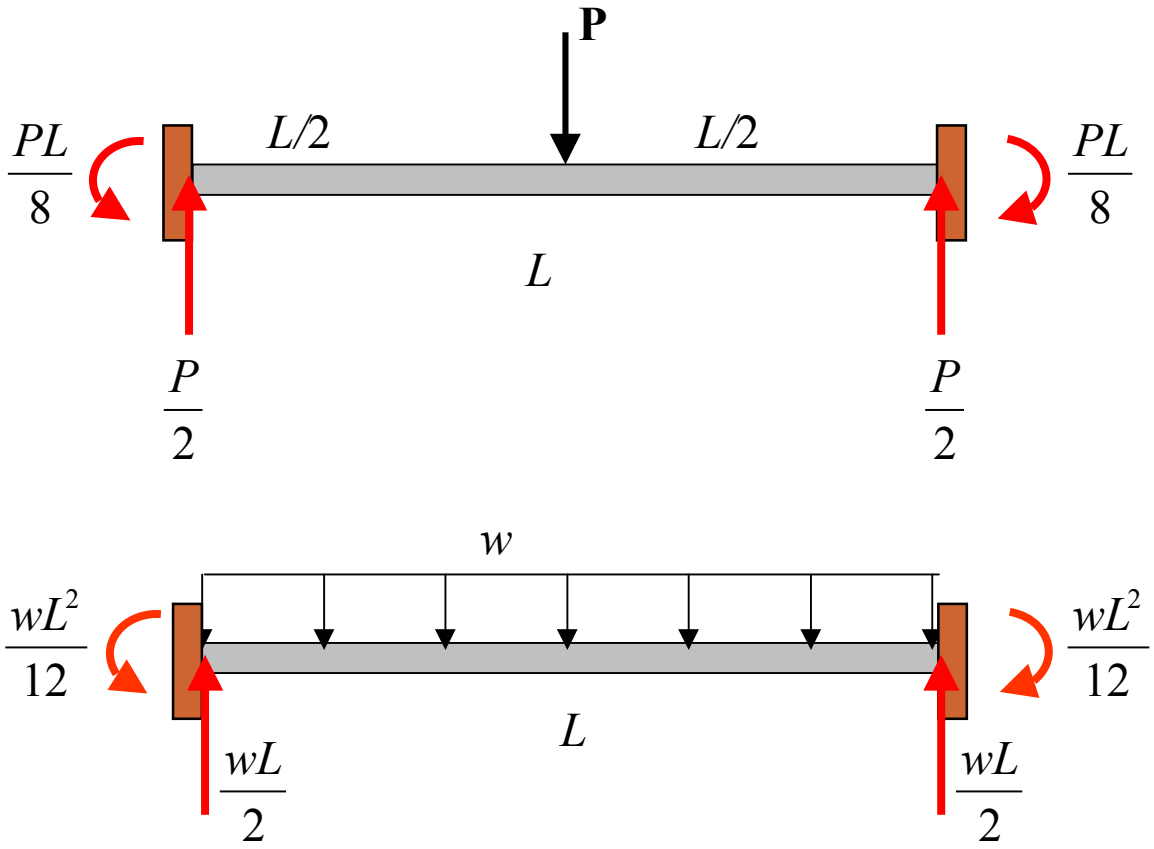
$$k_{BA} = \frac{2EI}{L}$$



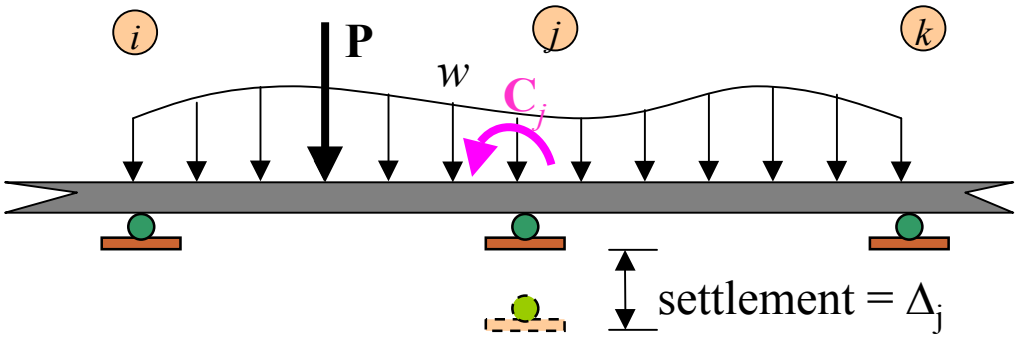
$$k_{BB} = \frac{4EI}{L}$$

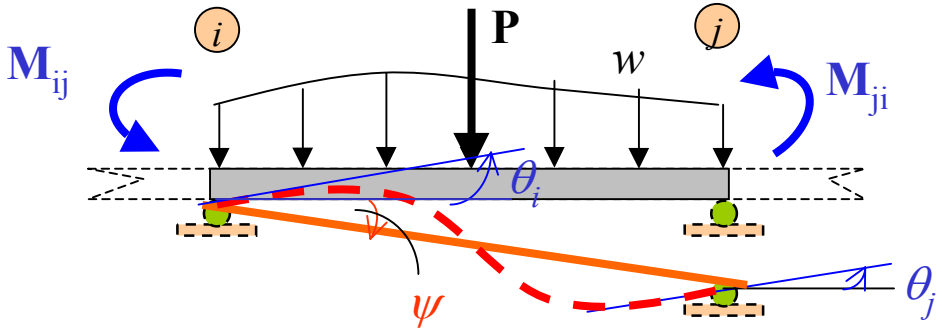
$$k_{AB} = \frac{2EI}{L}$$

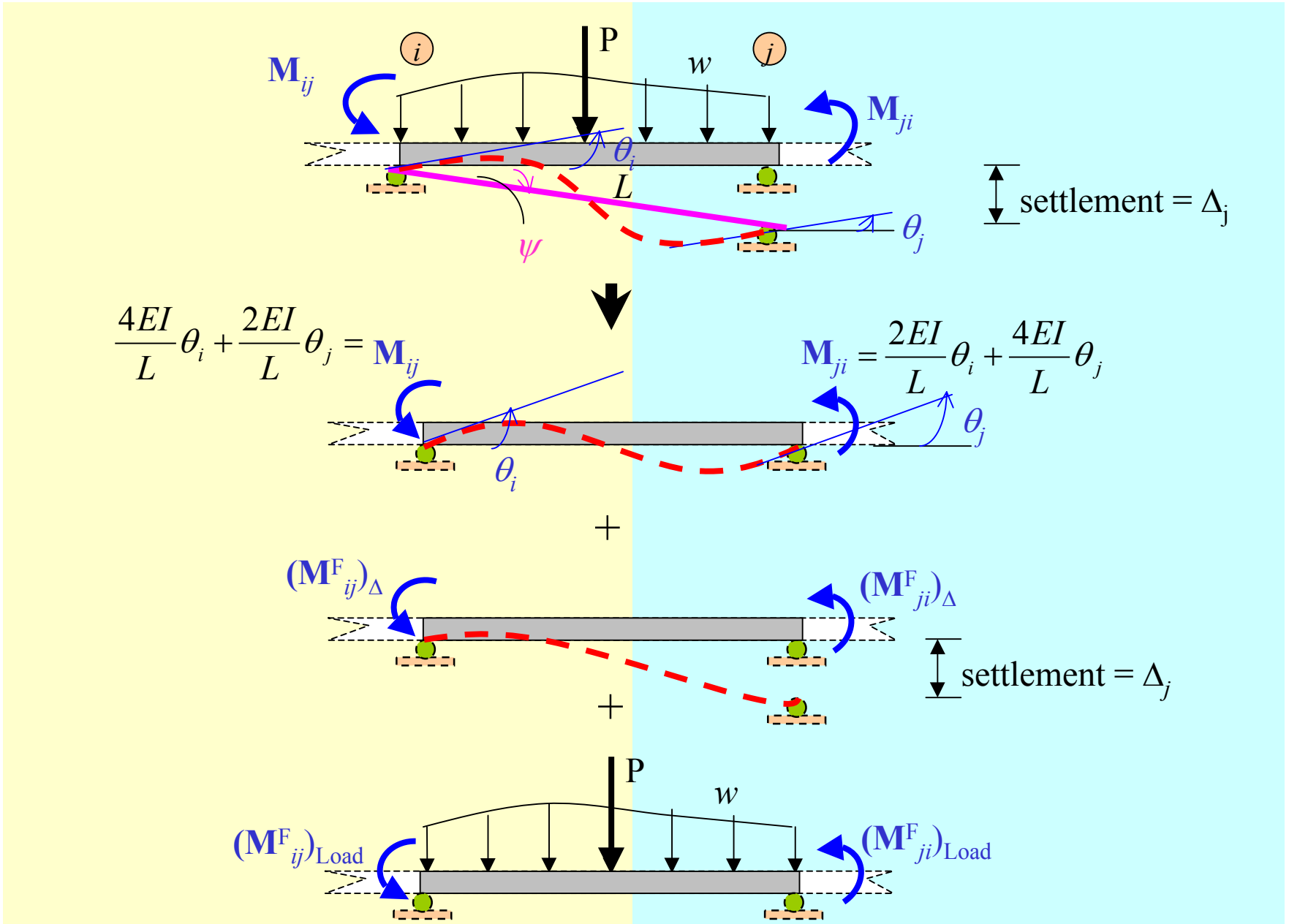
- **Fixed-End Forces**
  - **Fixed-End Forces: Loads**



• **General Case**

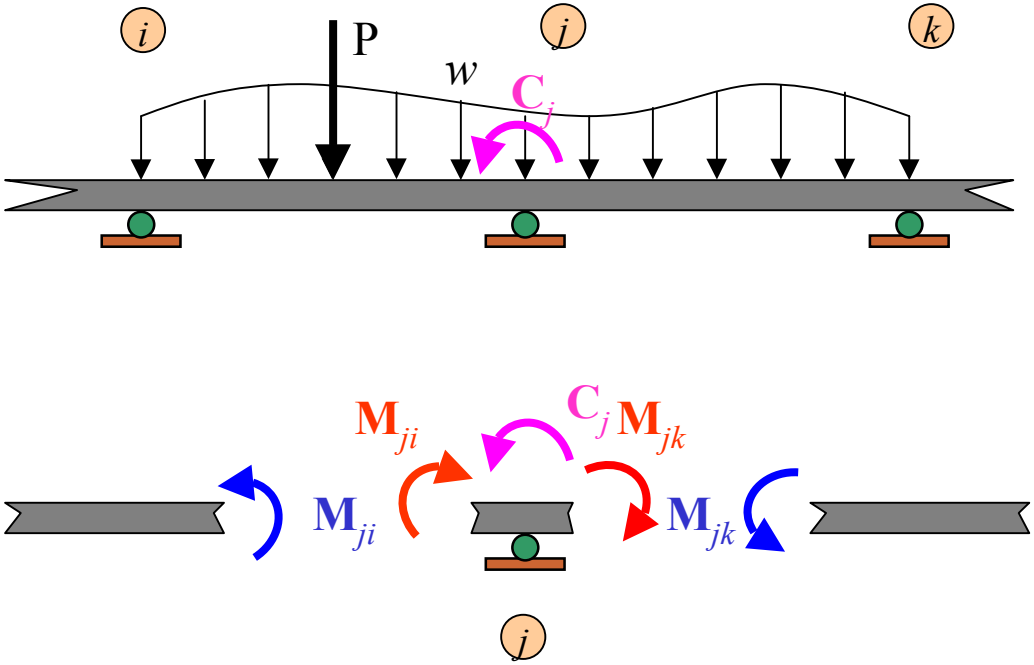






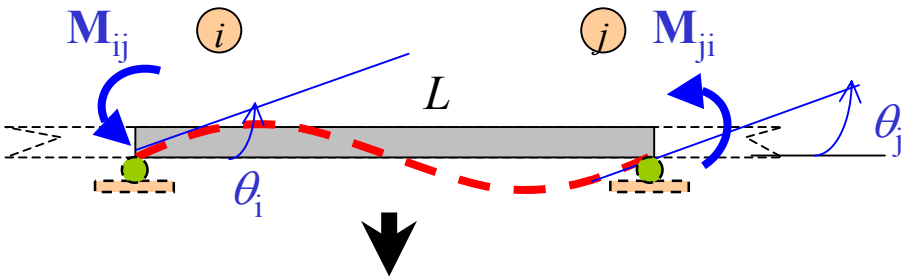
$$M_{ij} = (\frac{4EI}{L})\theta_i + (\frac{2EI}{L})\theta_j + (M^F{}_{ij})_\Delta + (M^F{}_{ij})_{Load}\ ,\quad M_{ji} = (\frac{2EI}{L})\theta_i + (\frac{4EI}{L})\theta_j + (M^F{}_{ji})_\Delta + (M^F{}_{ji})_{Load}$$

• **Equilibrium Equations**

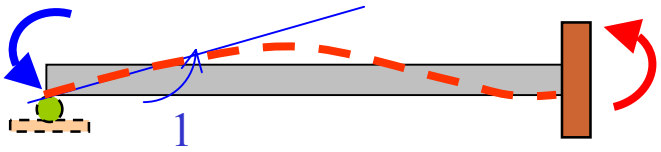


$$+\circlearrowleft \Sigma M_j = 0: \quad -M_{ji} - M_{jk} + C_j = 0$$

- **Stiffness Coefficients**



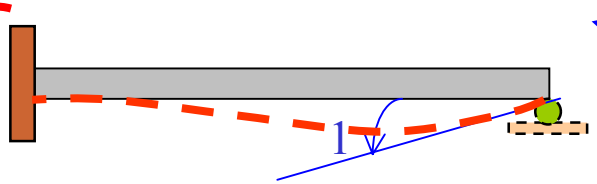
$$k_{ii} = \frac{4EI}{L}$$



$$k_{ji} = \frac{2EI}{L} \qquad \times \theta_i$$

$+$

$$k_{ij} = \frac{2EI}{L}$$



$$k_{jj} = \frac{4EI}{L} \qquad \times \theta_j$$

- **Matrix Formulation**

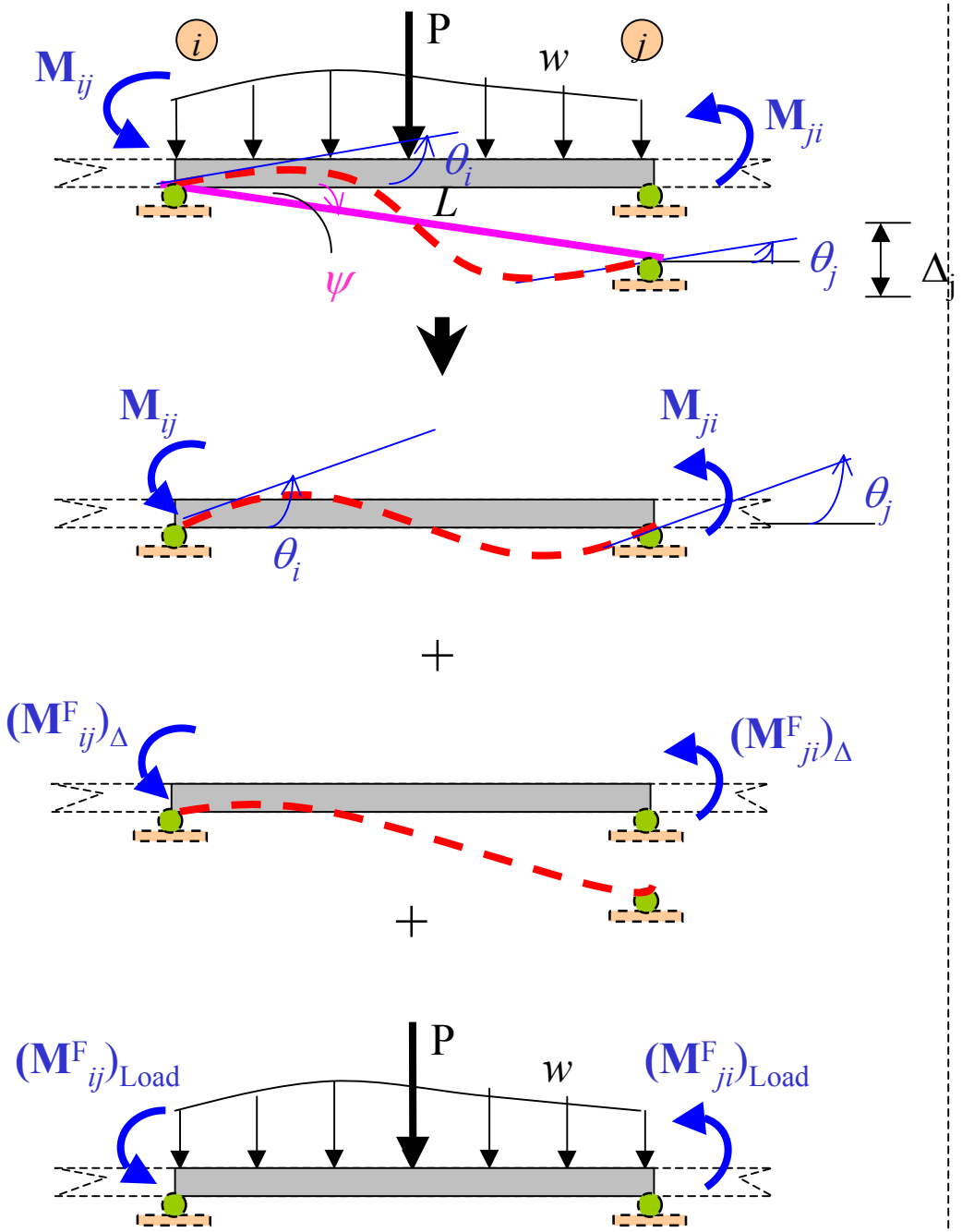
$$M_{ij} = (\frac{4EI}{L})\theta_i + (\frac{2EI}{L})\theta_j + (M^F{}_{ij})$$

$$M_{ji} = (\frac{2EI}{L})\theta_i + (\frac{4EI}{L})\theta_j + (M^F{}_{ji})$$

$$\begin{bmatrix} M_{ij} \\ M_{ji} \end{bmatrix} = \begin{bmatrix} (4EI/L) & (2EI/L) \\ (2EI/L) & (4EI/L) \end{bmatrix} \begin{bmatrix} \theta_{iI} \\ \theta_j \end{bmatrix} + \begin{bmatrix} M_{ij}{}^F \\ M_{ji}{}^F \end{bmatrix}$$

$$[k] = \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix}$$

Stiffness Matrix



$$[M] = [K][\theta] + [FEM]$$

$$([M] - [FEM]) = [K][\theta]$$

$$[\theta] = [K]^{-1}[M] - [FEM]$$

Stiffness matrix

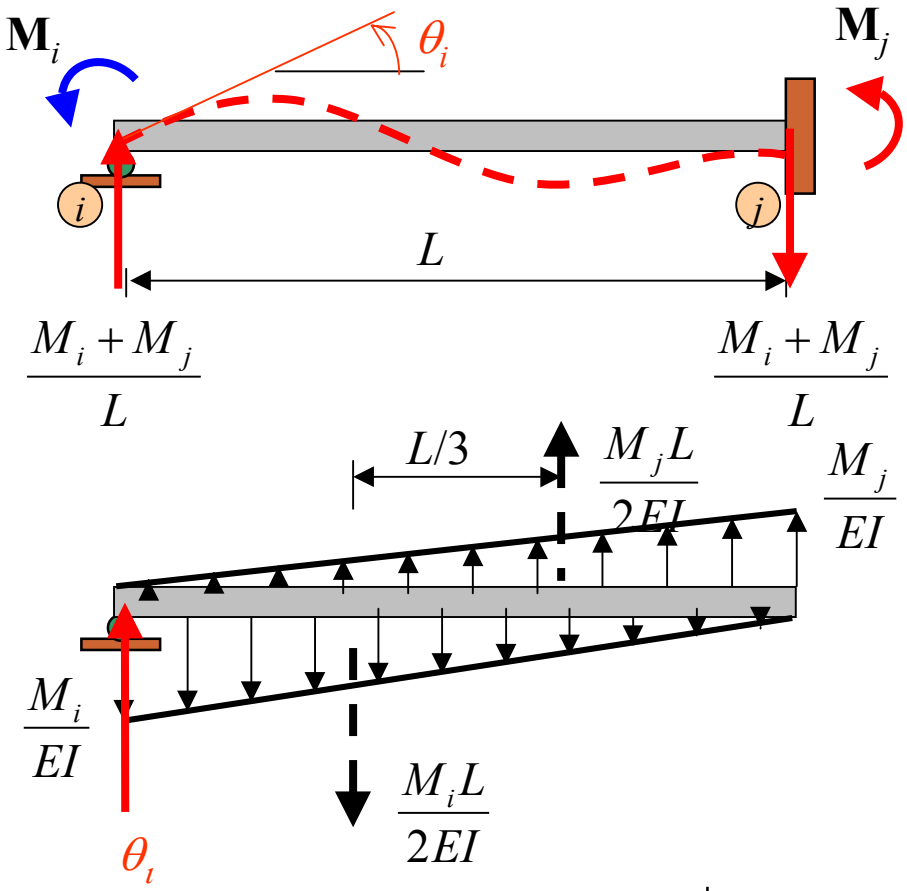
Fixed-end moment matrix

$$[D] = [K]^{-1}([Q] - [\mathrm{FEM}])$$

Displacement matrix

Force matrix

- **Stiffness Coefficients Derivation**



Real beam

Conjugate beam

$$+\circlearrowleft \quad \Sigma M'_i = 0: \quad -(\frac{M_i L}{2EI})(\frac{L}{3}) + (\frac{M_j L}{2EI})(\frac{2L}{3}) = 0$$

$$M_i = 2M_j \quad ---(1)$$

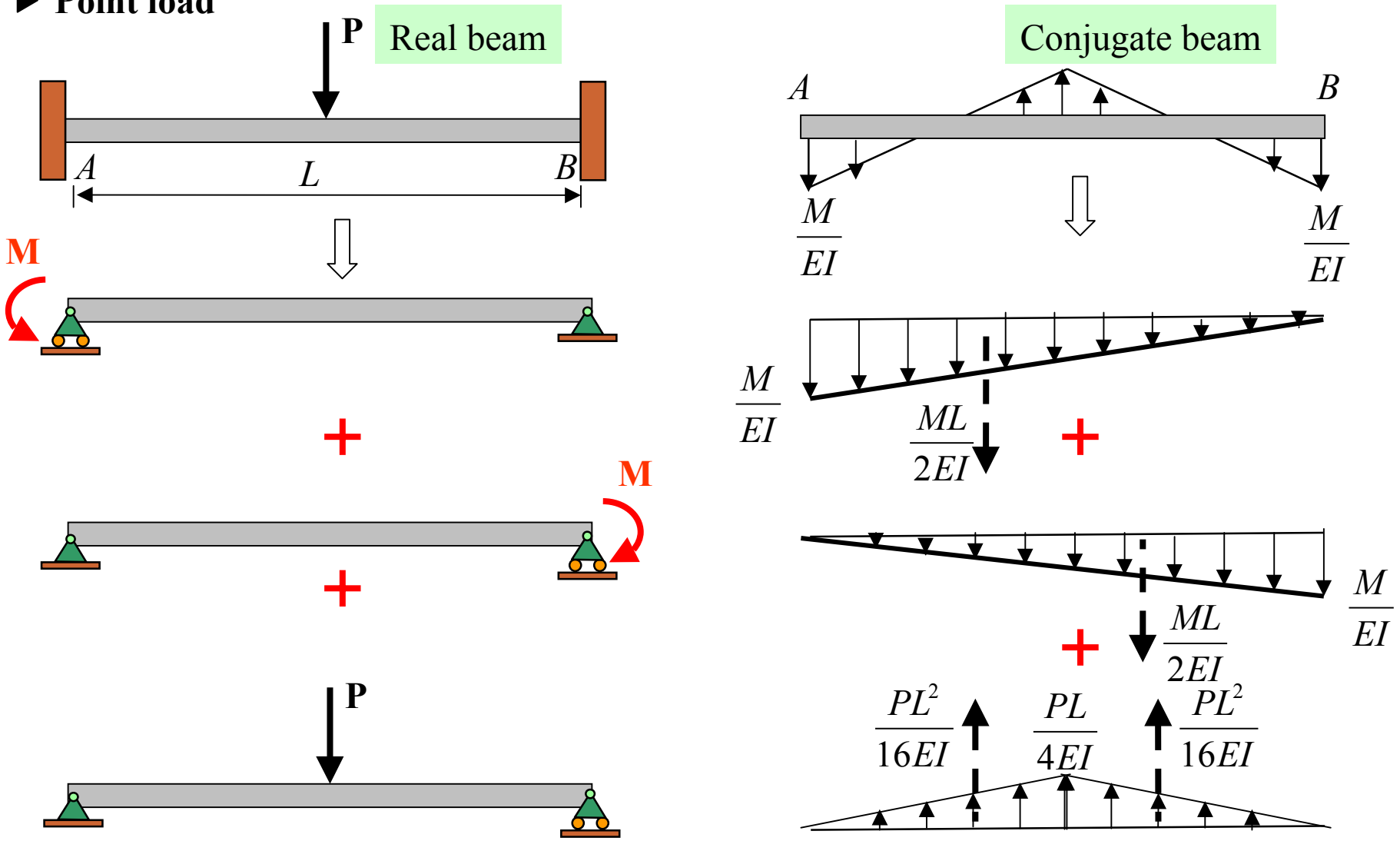
$$+\uparrow \quad \Sigma F_y = 0: \quad \theta_i - (\frac{M_i L}{2EI}) + (\frac{M_j L}{2EI}) = 0 \quad ---(2)$$

From (1) *and* (2);

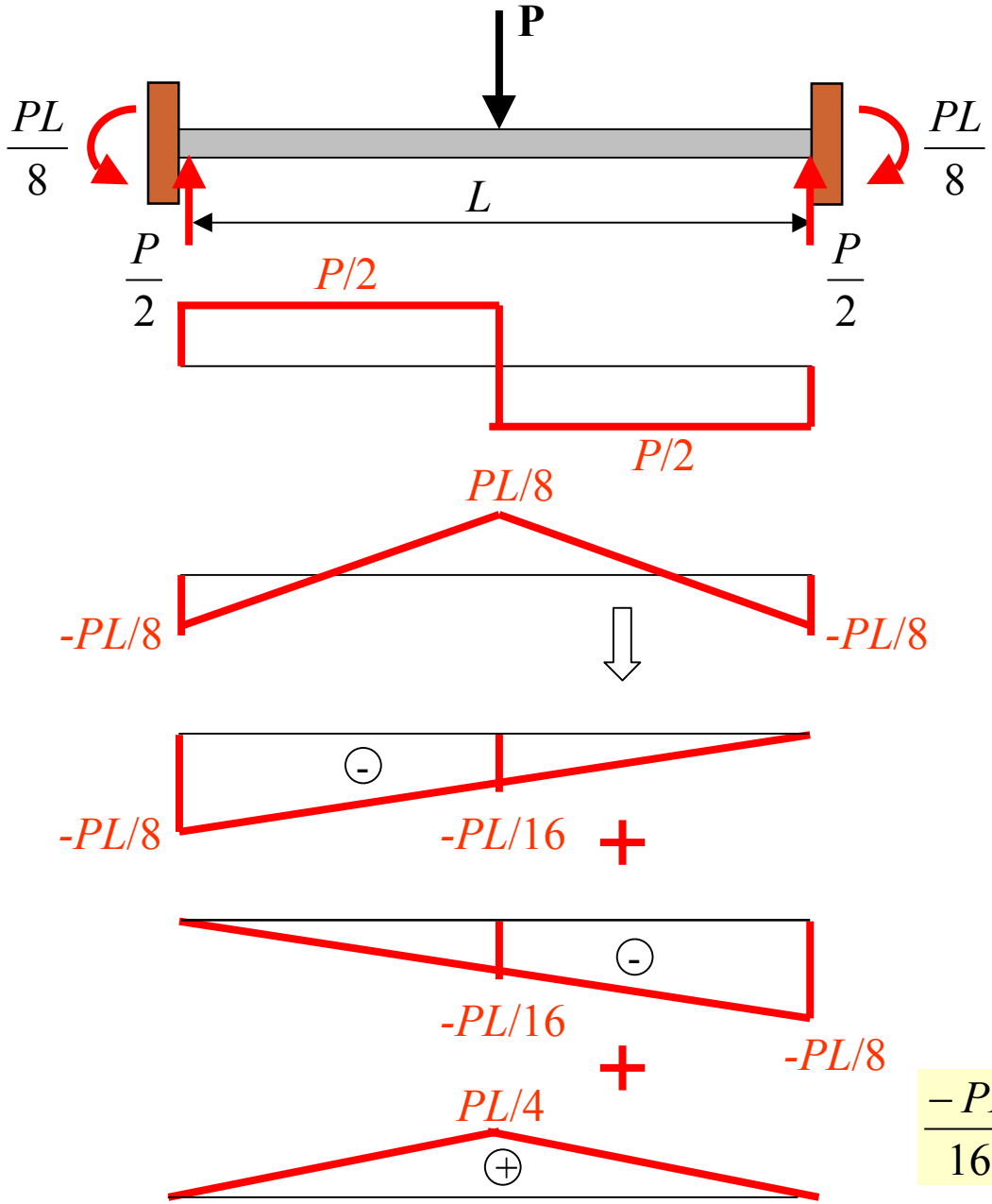
$$M_i = (\frac{4EI}{L})\theta_i$$

$$M_j = (\frac{2EI}{L})\theta_i$$

- **Derivation of Fixed-End Moment**
  - **Point load**

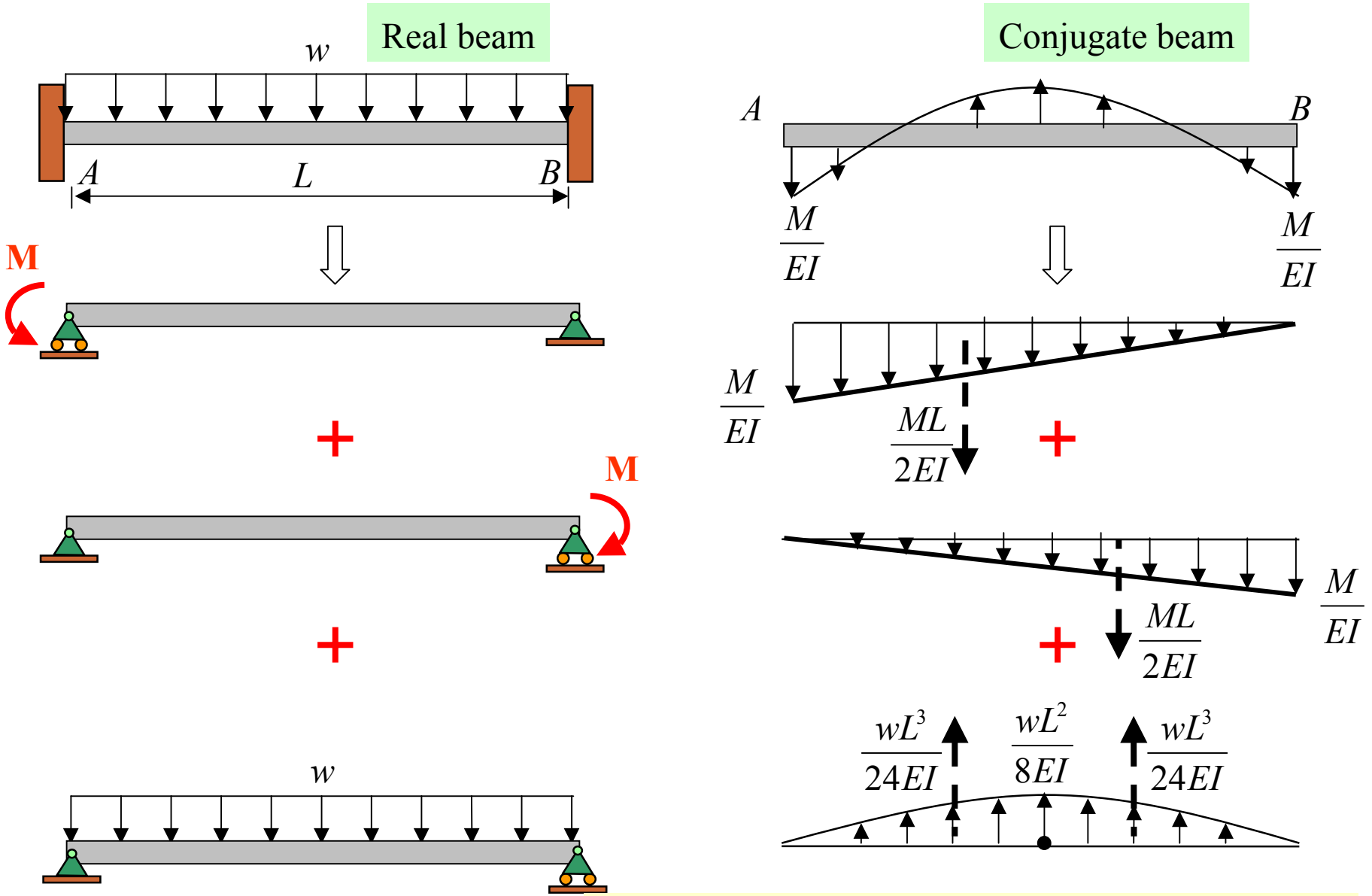


$$+\uparrow \quad \Sigma F_y = 0: \quad -\frac{ML}{2EI} - \frac{ML}{2EI} + \frac{2PL^2}{16EI} = 0, \quad M = \frac{PL}{8}$$



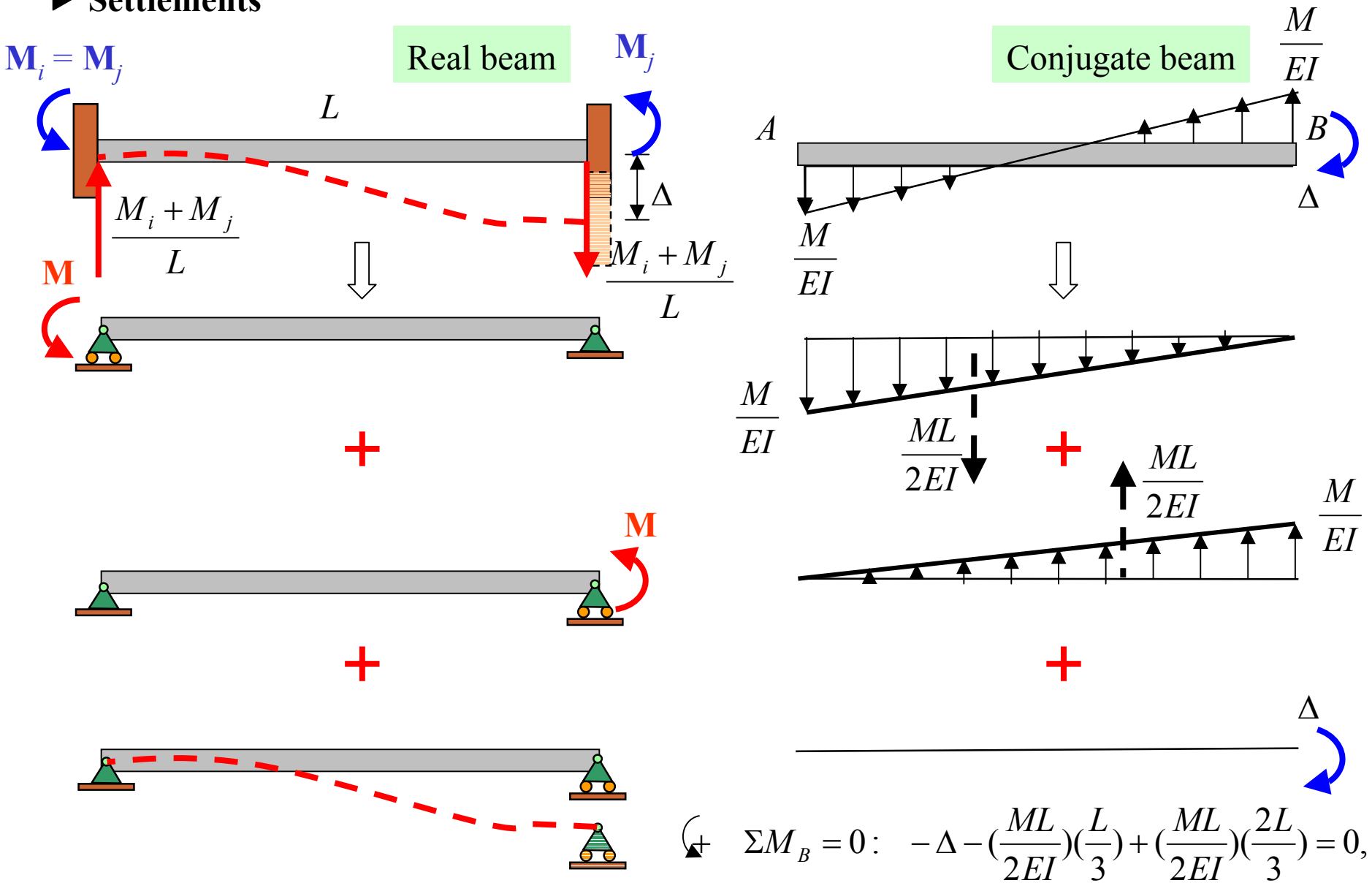
$$\frac{-PL}{16}+\frac{-PL}{16}+\frac{PL}{4}=\frac{PL}{8}$$

▶ Uniform load



$$+\uparrow \quad \Sigma F_y = 0: \quad -\frac{ML}{2EI} - \frac{ML}{2EI} + \frac{2wL^3}{24EI} = 0, \quad M = \frac{wL^2}{12}$$

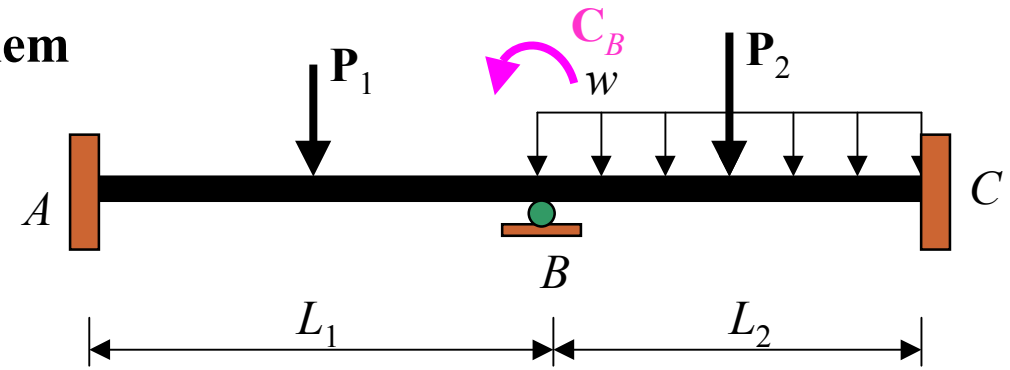
▶ **Settlements**

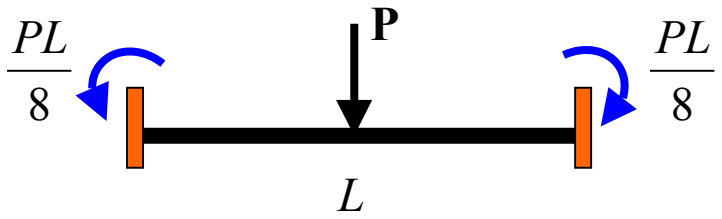


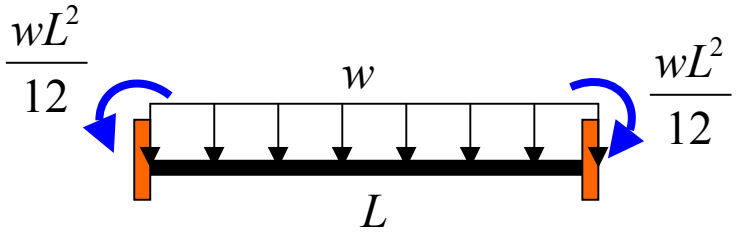
$$\circlearrowleft + \quad \Sigma M_B = 0: \quad -\Delta - (\frac{ML}{2EI})(\frac{L}{3}) + (\frac{ML}{2EI})(\frac{2L}{3}) = 0,$$

$$M = \frac{6EI\Delta}{L^2}$$

- **Typical Problem**





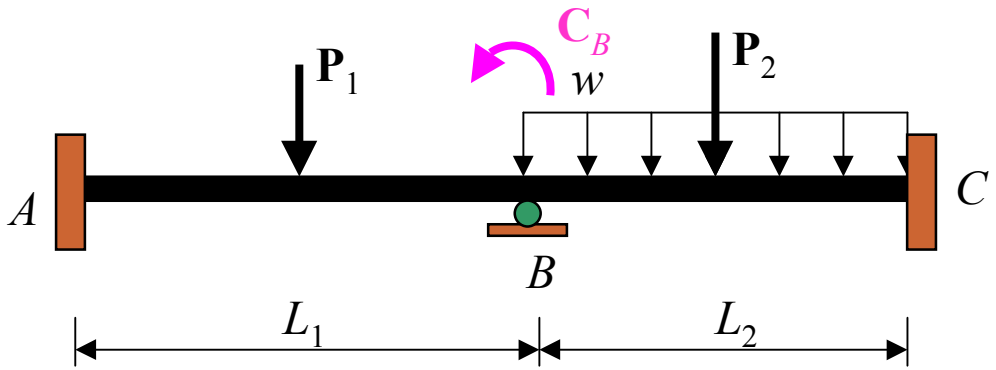


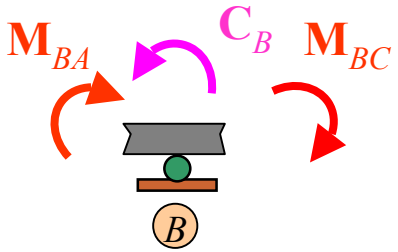
$$M_{AB} = \frac{4EI}{L_1}\overset{0}{\theta_A} + \frac{2EI}{L_1}\theta_B + 0 + \frac{P_1 L_1}{8}$$

$$M_{BA} = \frac{2EI}{L_1}\overset{0}{\theta_A} + \frac{4EI}{L_1}\theta_B + 0 - \frac{P_1 L_1}{8}$$

$$M_{BC} = \frac{4EI}{L_2}\theta_B + \frac{2EI}{L_2}\overset{0}{\theta_C} + 0 + \frac{P_2 L_2}{8} + \frac{wL_2^{\,2}}{12}$$

$$M_{CB} = \frac{2EI}{L_2}\theta_B + \frac{4EI}{L_2}\overset{0}{\theta_C} + 0 + \frac{-P_2 L_2}{8} - \frac{wL_2^{\,2}}{12}$$

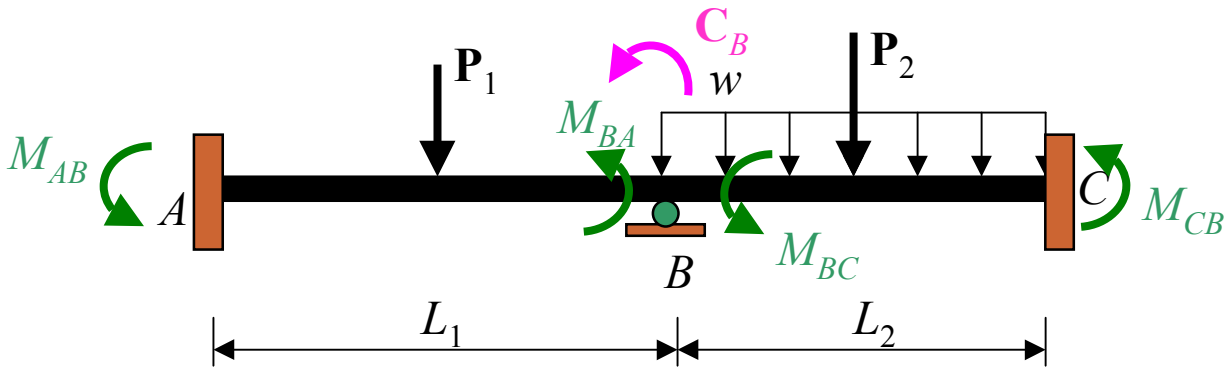




$$M_{BA} = \frac{2EI}{L_1}\theta_A + \frac{4EI}{L_1}\theta_B + 0 - \frac{P_1 L_1}{8}$$

$$M_{BC} = \frac{4EI}{L_2}\theta_B + \frac{2EI}{L_2}\theta_C + 0 + \frac{P_2 L_2}{8} + \frac{wL_2^{\,2}}{12}$$

$\circlearrowleft \quad \Sigma M_B = 0: \quad C_B - M_{BA} - M_{BC} = 0 \quad \rightarrow \quad Solve\ for\ \theta_B$



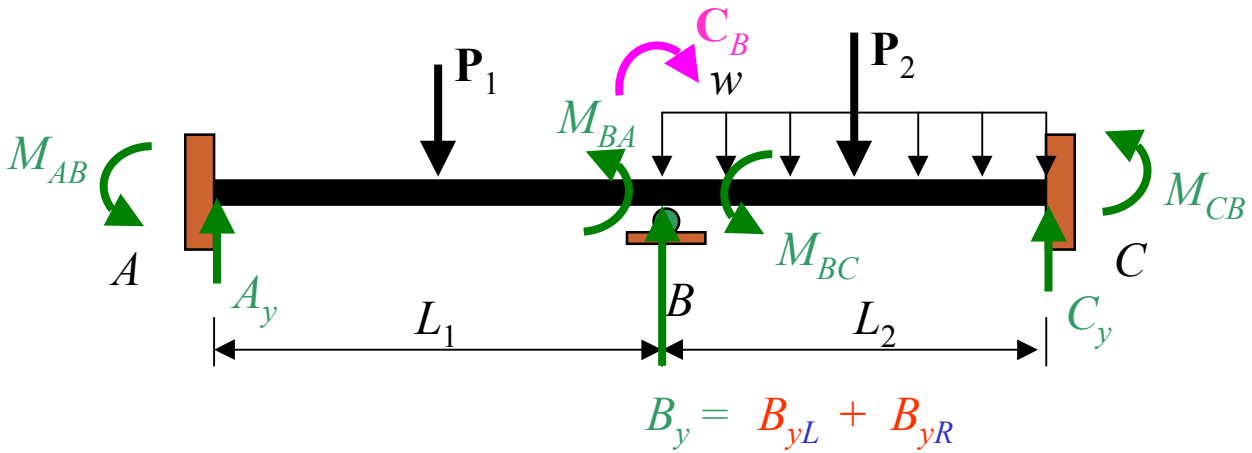
Substitute θ_B in M_{AB}, M_{BA}, M_{BC}, M_{CB}

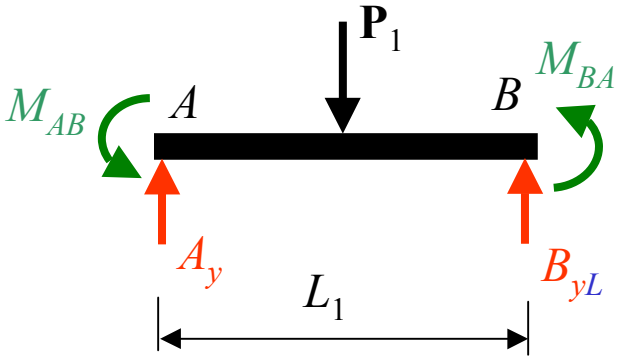
$$M_{AB} = \frac{4EI}{L_1}\overset{0}{\cancel{\theta_A}} + \frac{2EI}{L_1}\theta_B + 0 + \frac{P_1 L_1}{8}$$

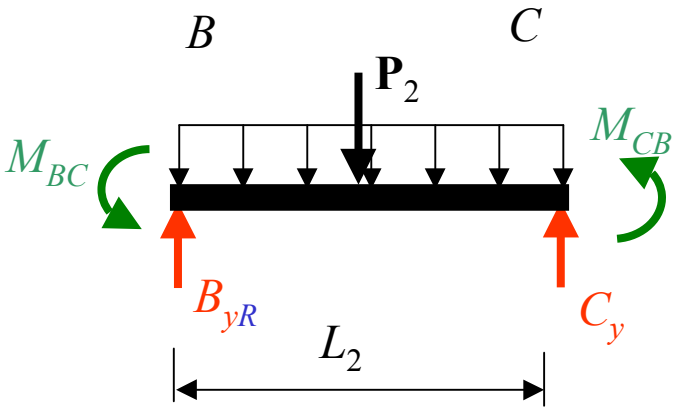
$$M_{BA} = \frac{2EI}{L_1}\overset{0}{\cancel{\theta_A}} + \frac{4EI}{L_1}\theta_B + 0 - \frac{P_1 L_1}{8}$$

$$M_{BC} = \frac{4EI}{L_2}\theta_B + \frac{2EI}{L_2}\overset{0}{\cancel{\theta_C}} + 0 + \frac{P_2 L_2}{8} + \frac{wL_2^{\,2}}{12}$$

$$M_{CB} = \frac{2EI}{L_2}\theta_B + \frac{4EI}{L_2}\overset{0}{\cancel{\theta_C}} + 0 + \frac{-P_2 L_2}{8} - \frac{wL_2^{\,2}}{12}$$

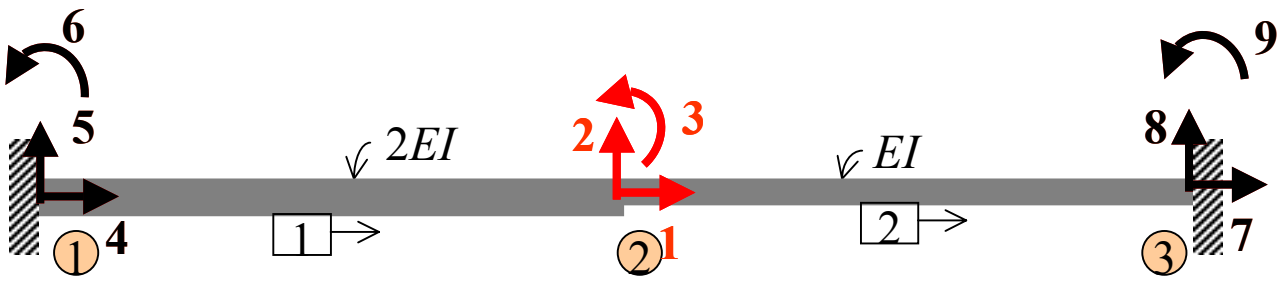

C_B
w
$\mathbf{P}_1$
$\mathbf{P}_2$
M_{BA}
M_{AB}
M_{CB}
M_{BC}
A
B
C
A_y
C_y
L_1
L_2
$B_y = B_{yL} + B_{yR}$


$\mathbf{P}_1$
M_{AB}
A
B
M_{BA}
A_y
B_{yL}
L_1

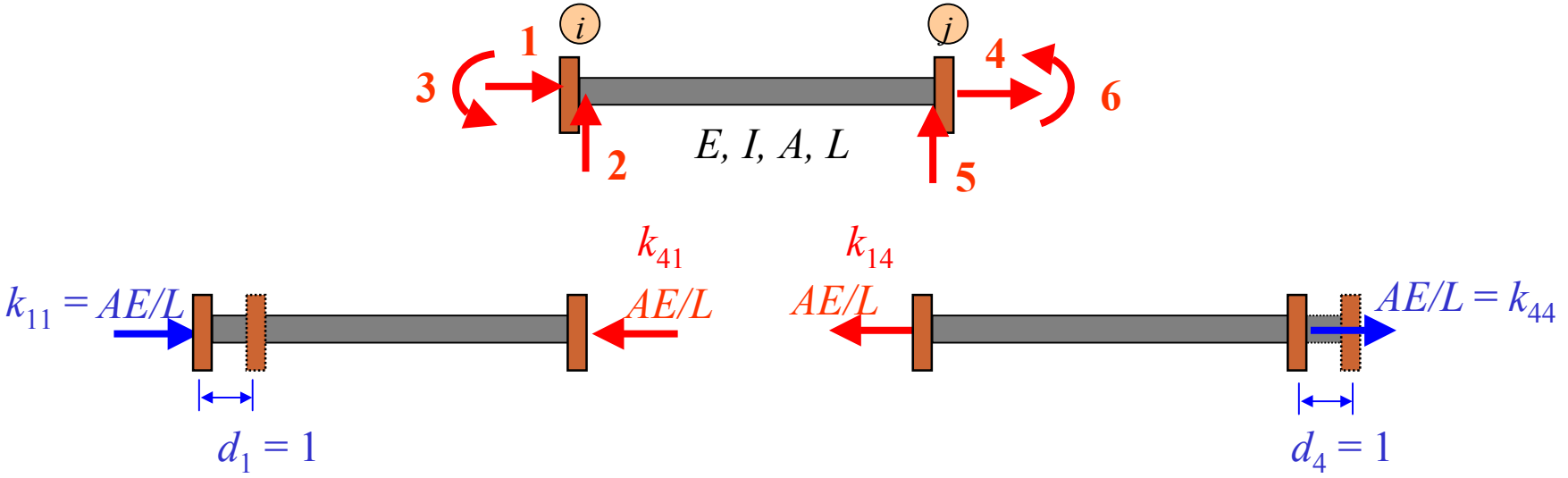

B
C
$\mathbf{P}_2$
M_{BC}
M_{CB}
B_{yR}
C_y
L_2

Stiffness Matrix

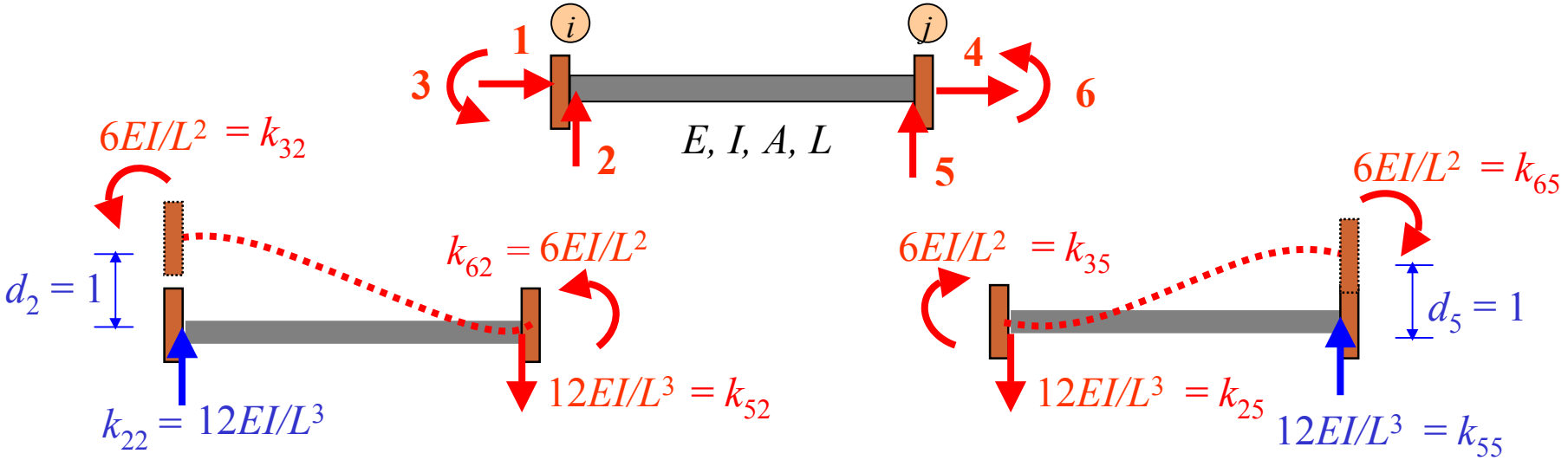
- **Node and Member Identification**
- **Global and Member Coordinates**
- **Degrees of Freedom**
  - **Known degrees of freedom D_4, D_5, D_6, D_7, D_8 and D_9**
  - **Unknown degrees of freedom D_1, D_2 and D_3**



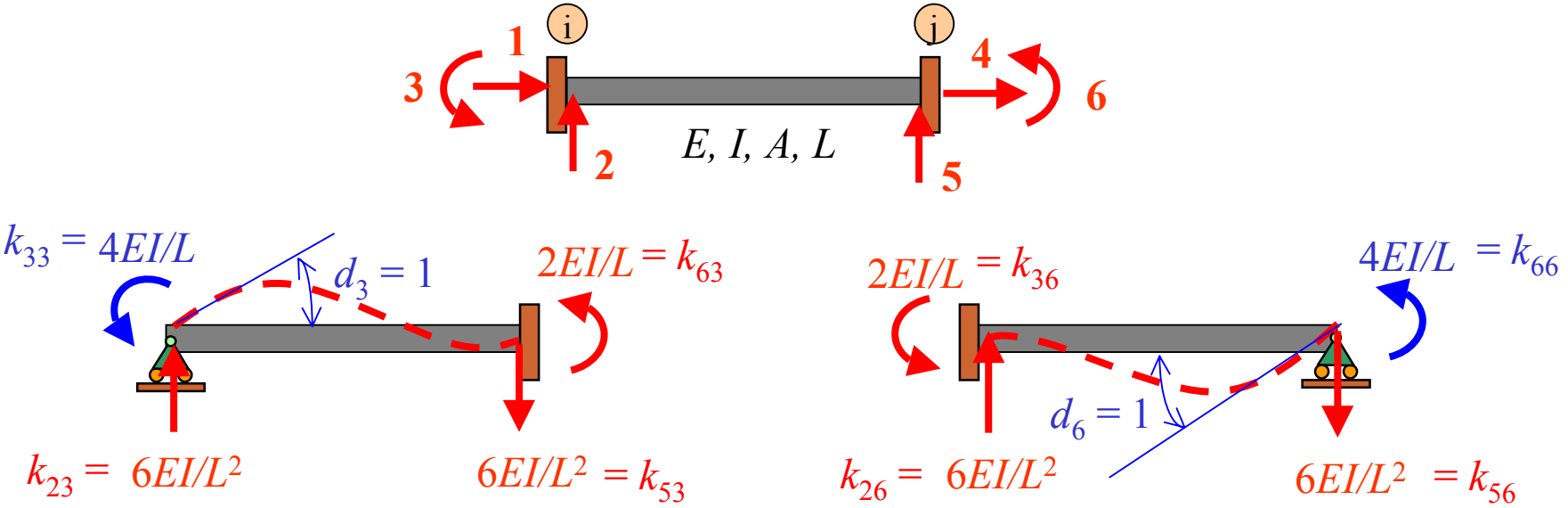
Beam-Member Stiffness Matrix



$$[k] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} AE/L & & & -AE/L & & \\ 0 & & & 0 & & \\ 0 & & & 0 & & \\ -AE/L & & & AE/L & & \\ 0 & & & 0 & & \\ 0 & & & 0 & & \end{pmatrix} \end{array}$$

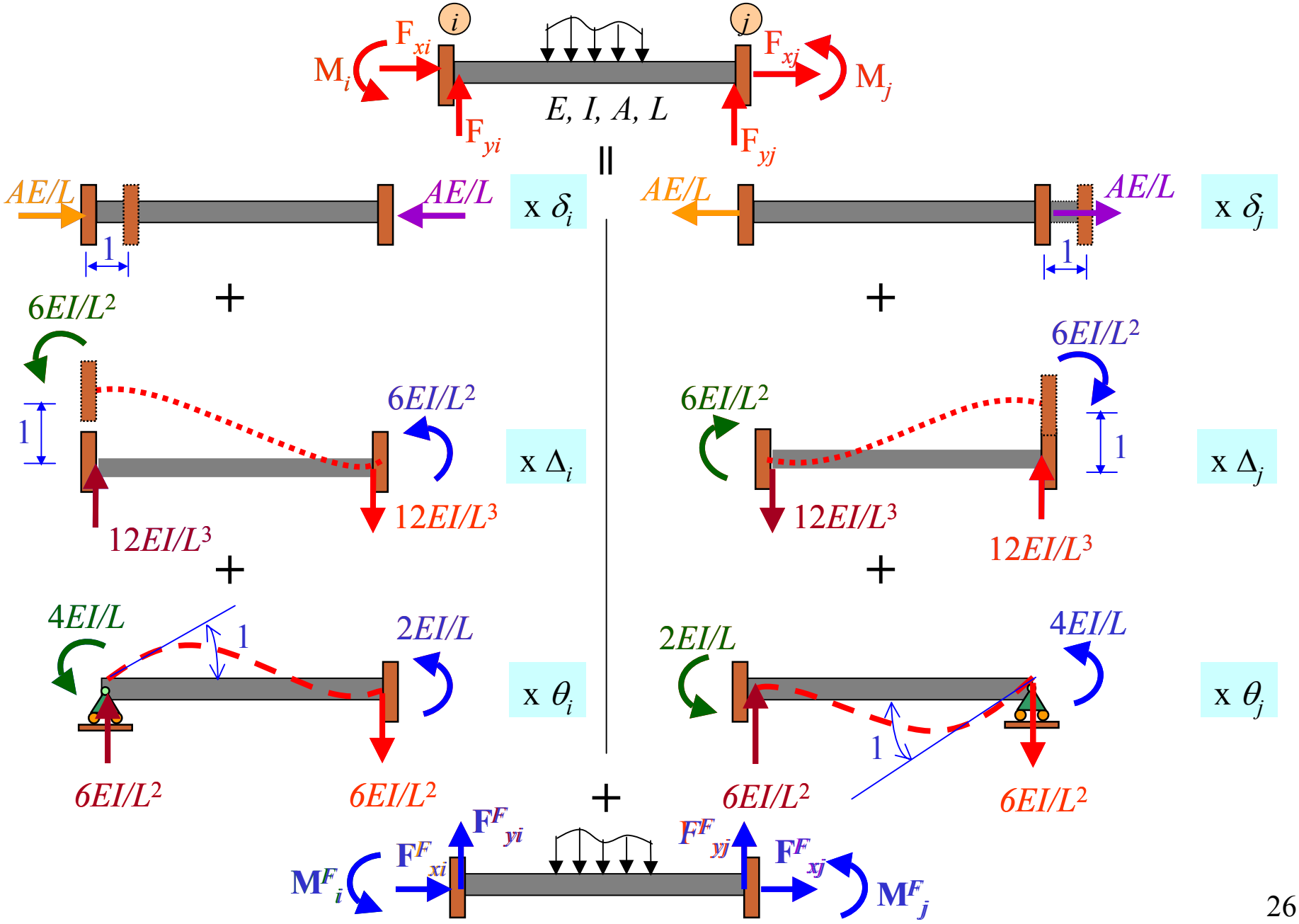


$$[k] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} AE/L & 0 & & -AE/L & 0 & \\ 0 & 12EI/L^3 & & 0 & -12EI/L^3 & \\ 0 & 6EI/L^2 & & 0 & -6EI/L^2 & \\ -AE/L & 0 & & AE/L & 0 & \\ 0 & -12EI/L^3 & & 0 & 12EI/L^3 & \\ 0 & 6EI/L^2 & & 0 & -6EI/L^2 & \end{pmatrix} \end{array}$$



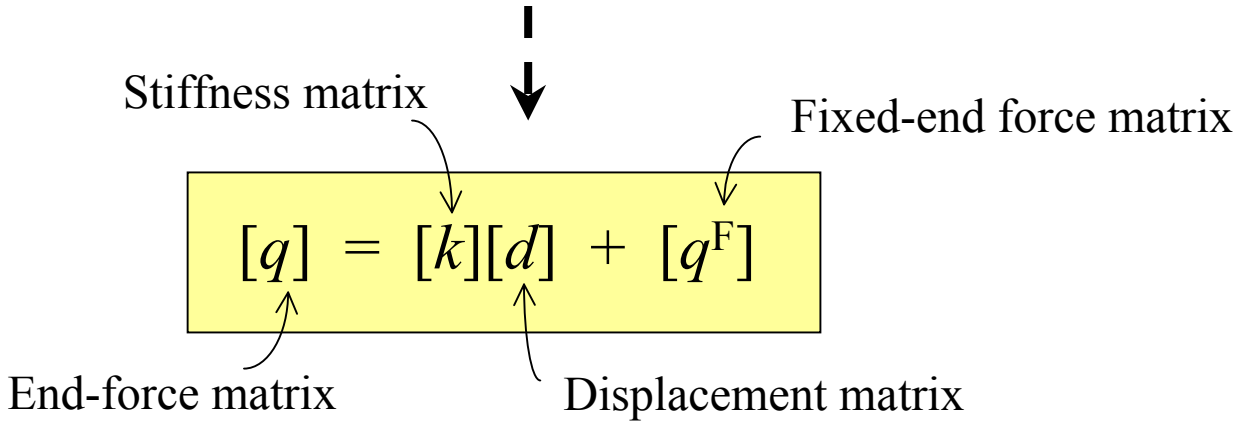
$$[k] = \begin{array}{c} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array} \\ \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

- **Member Equilibrium Equations**



$$
\begin{aligned}
F_{xi} &= (AE/L)\delta_i + (0)\Delta_i + (0)\theta_i + (-AE/L)\delta_j + (0)\Delta_j + (0)\theta_j + F_{xi}^F \\
F_{yi} &= (0)\delta_i + (12EI/L^3)\Delta_i + (6EI/L^2)\theta_i + (0)\delta_j + (-12EI/L^3)\Delta_j + (6EI/L^2)\theta_j + F_{yi}^F \\
M_{xi} &= (0)\delta_i + (6EI/L^2)\Delta_i + (4EI/L)\theta_i + (0)\delta_j + (-6EI/L^2)\Delta_j + (2EI/L)\theta_j + M_i^F \\
F_{xj} &= (-AE/L)\delta_i + (0)\Delta_i + (0)\theta_i + (AE/L)\delta_j + (0)\Delta_j + (0)\theta_j + F_{xi}^F \\
F_{yj} &= (0)\delta_i + (-12EI/L^3)\Delta_i + (-6EI/L^2)\theta_i + (0)\delta_j + (0)\Delta_j + (-6EI/L^2)\theta_j + F_{yj}^F \\
M_j &= (0)\delta_i + (6EI/L^2)\Delta_i + (2EI/L)\theta_i + (0)\delta_j + (-6EI/L^2)\Delta_j + (4EI/L)\theta_j + M_j^F
\end{aligned}
$$

$$
\begin{bmatrix} F_{xi} \\ F_{yj} \\ M_i \\ F_{xj} \\ F_{yj} \\ M_j \end{bmatrix} =
\begin{bmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-AE/L & 0 & 0 & AE/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\begin{bmatrix} \delta_i \\ \Delta_i \\ \theta_i \\ \delta_j \\ \Delta_j \\ \theta_j \end{bmatrix} +
\begin{bmatrix} F_{xi}^F \\ F_{yi}^F \\ M_i^F \\ F_{xj}^F \\ F_{yi}^F \\ M_j^F \end{bmatrix}
$$



▶ 6x6 Stiffness Matrix

$$
[k]_{6\times 6} = \begin{array}{c} \\ N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j \end{array}
\begin{array}{c}
\begin{array}{cccccc} \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \end{array} \\
\begin{bmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-AE/L & 0 & 0 & AE/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\end{array}
$$

▶ 4x4 Stiffness Matrix

$$
[k]_{4\times 4} = \begin{array}{c} \\ V_i \\ M_i \\ V_j \\ M_j \end{array}
\begin{array}{c}
\begin{array}{cccc} \Delta_i & \theta_i & \Delta_j & \theta_j \end{array} \\
\begin{bmatrix}
12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\
6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\
-12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\
6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L
\end{bmatrix}
\end{array}
$$

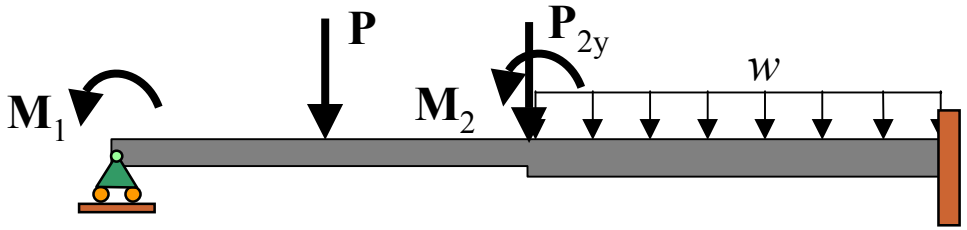
▶ **2x2 Stiffness Matrix**

$$[k]_{2\times 2} = \begin{matrix} \\ M_i \\ M_j \end{matrix} \begin{matrix} \begin{matrix} \theta_i & \theta_j \end{matrix} \\ \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix} \end{matrix}$$

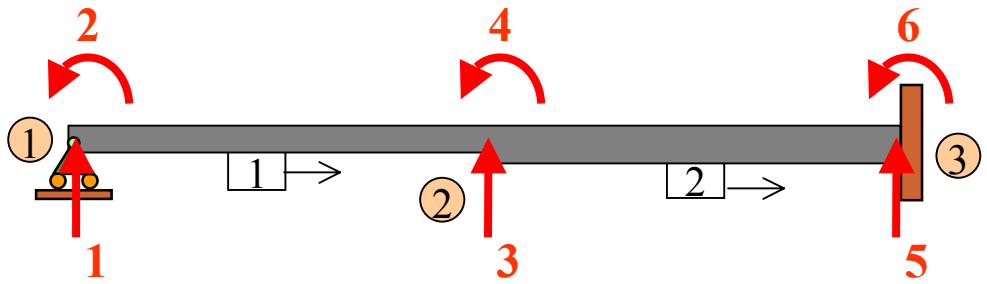
Comment:

- When use 4x4 stiffness matrix, specify settlement.
- When use 2x2 stiffness matrix, fixed-end forces must be included.

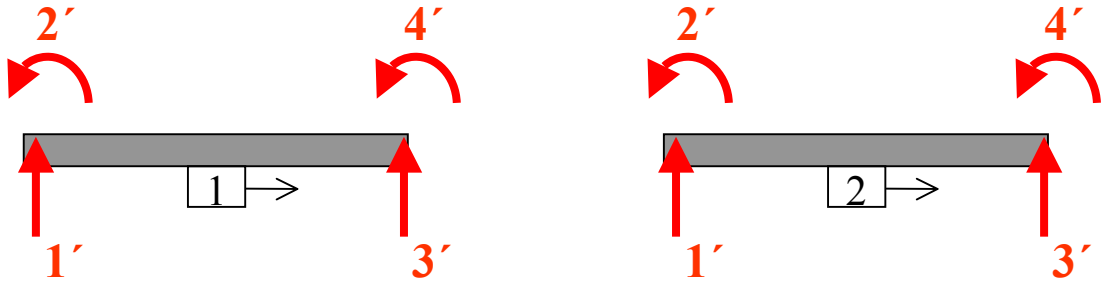
General Procedures: Application of the Stiffness Method for Beam Analysis

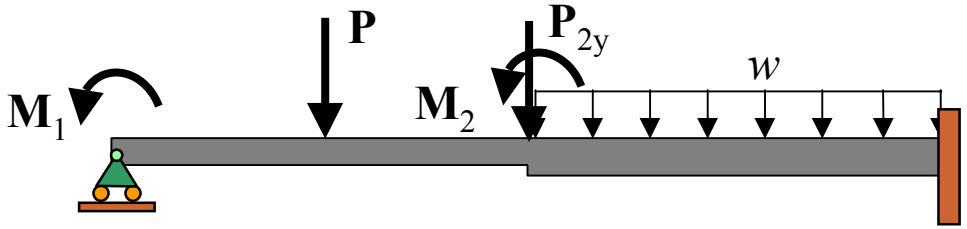


Global

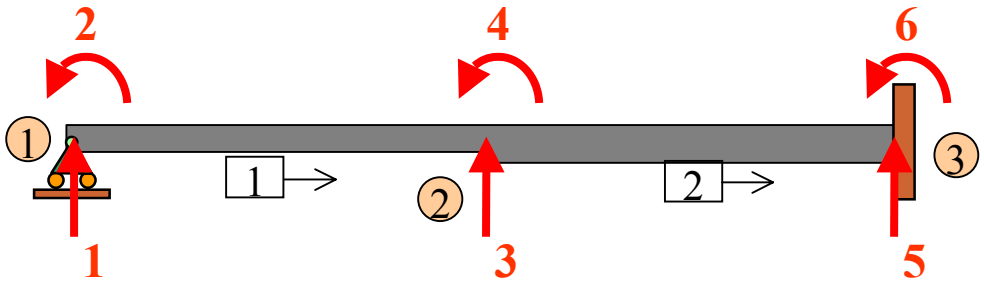


Local

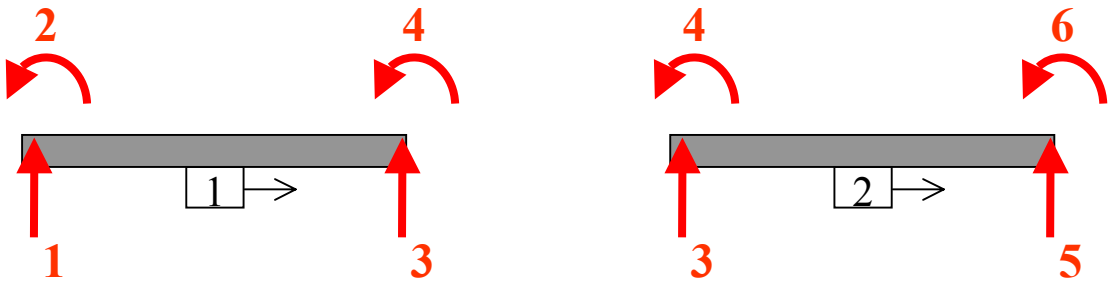




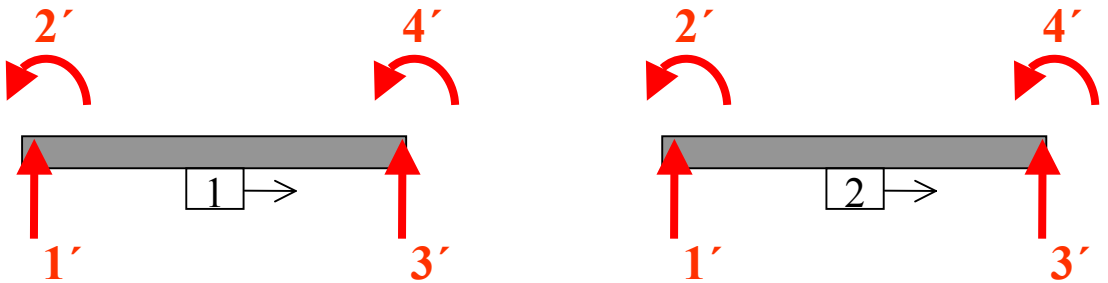
Global

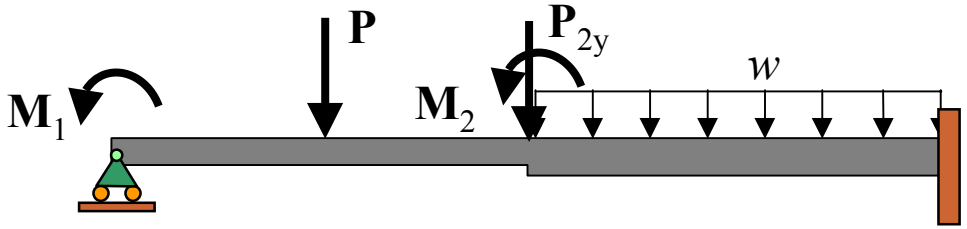


Member

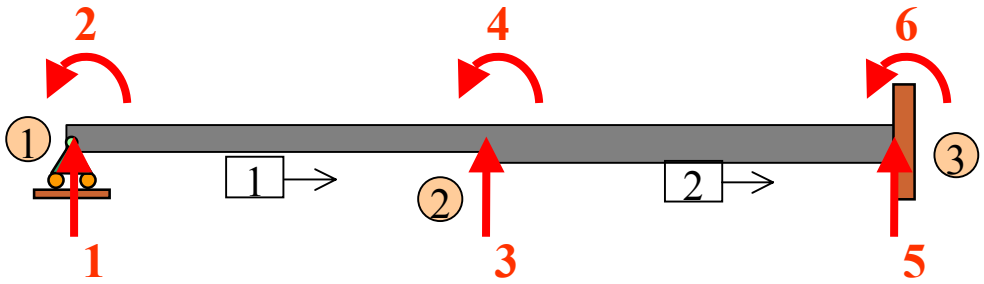


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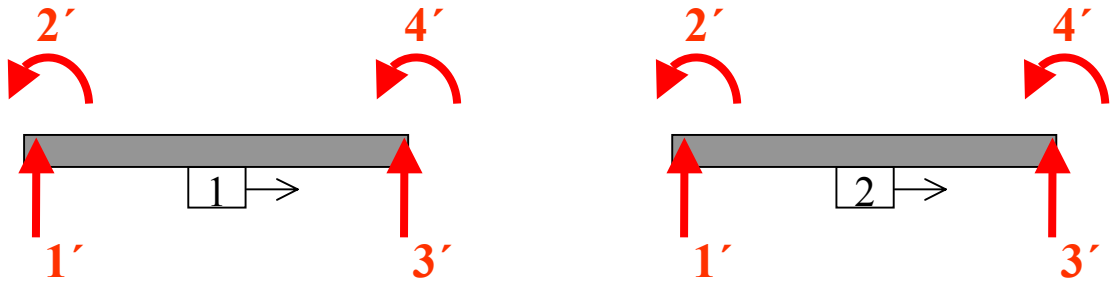




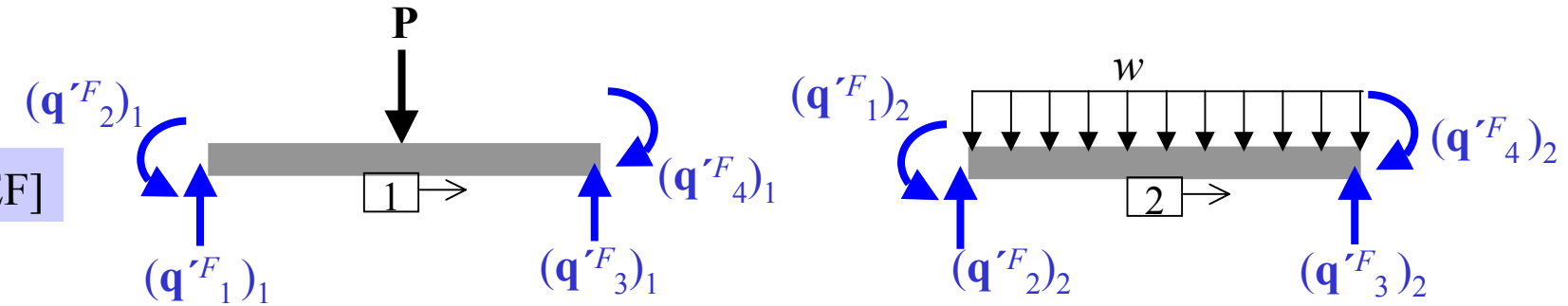
Global

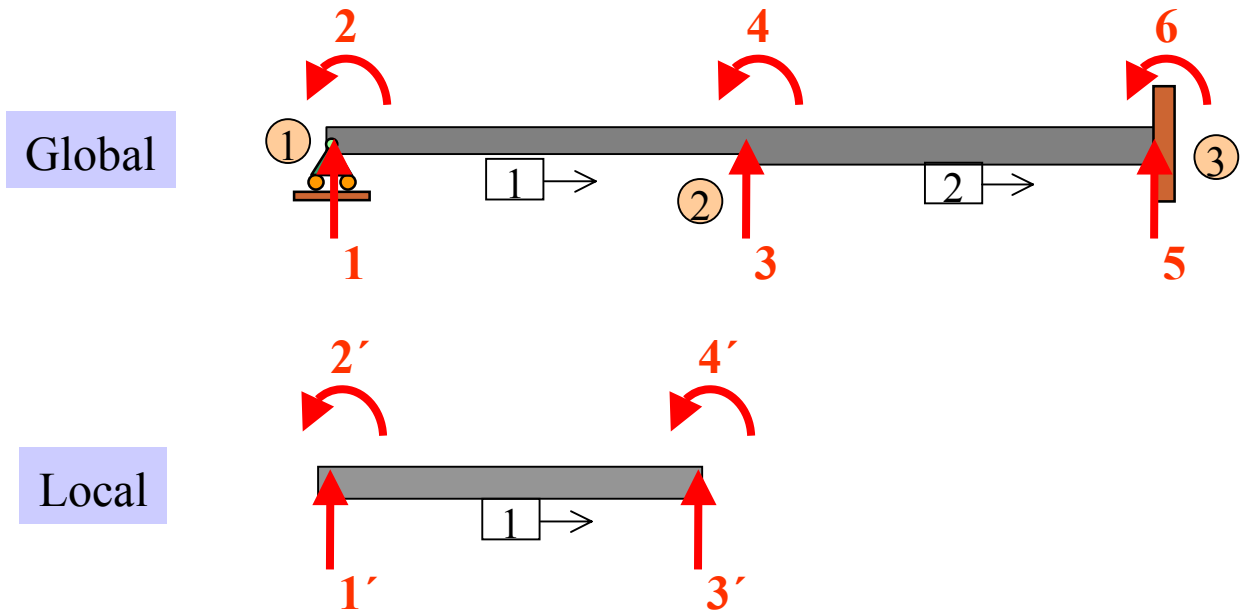


Local



[FEF]



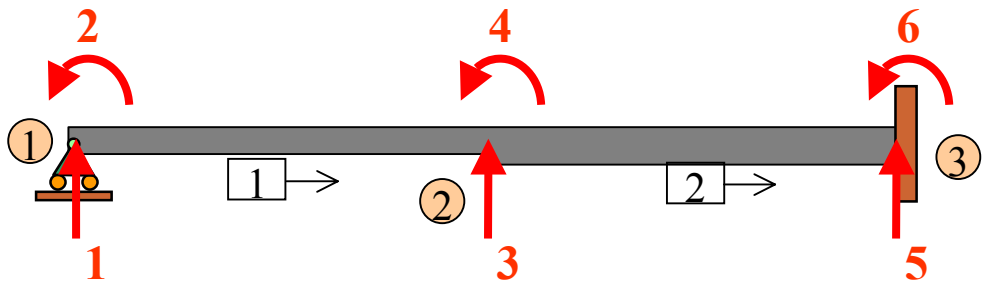


$$[q] = [T]^{T}[q']$$

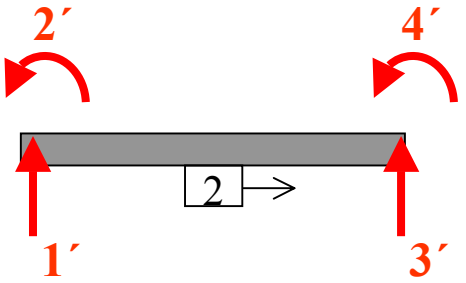
$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}_{1} = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} 1' & 2' & 3' & 4' \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \end{bmatrix}$$

$$[k] = [T]^{T}[q'][T]$$

Global



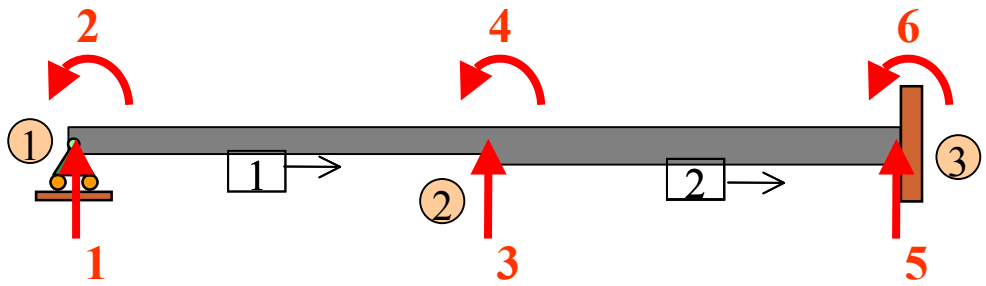
Local



$$[q] = [T]^{T}[q']$$

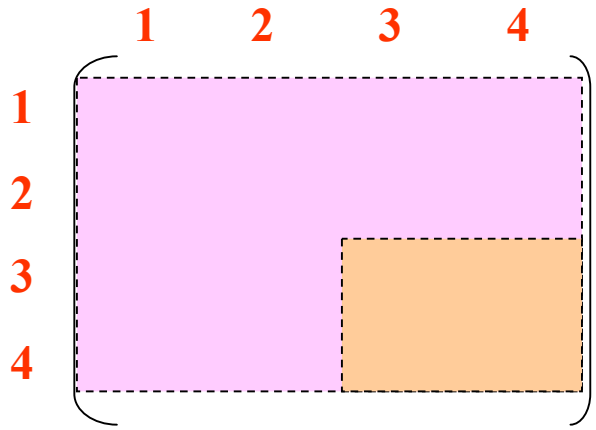
$$\begin{bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}_{\boxed{2}} = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 1' & 2' & 3' & 4' \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \end{bmatrix}$$

$$[k] = [T]^{T}[q'] \, [T]$$

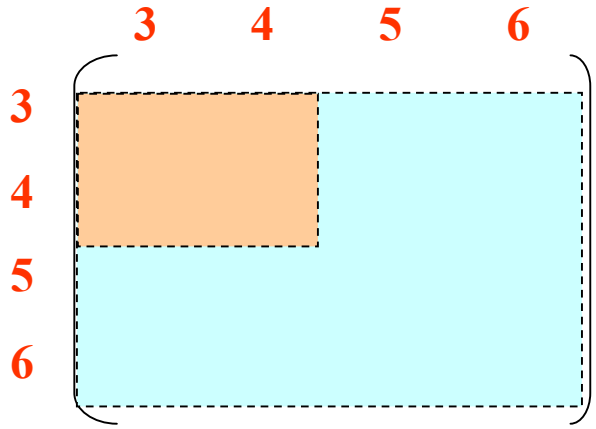


Stiffness Matrix:

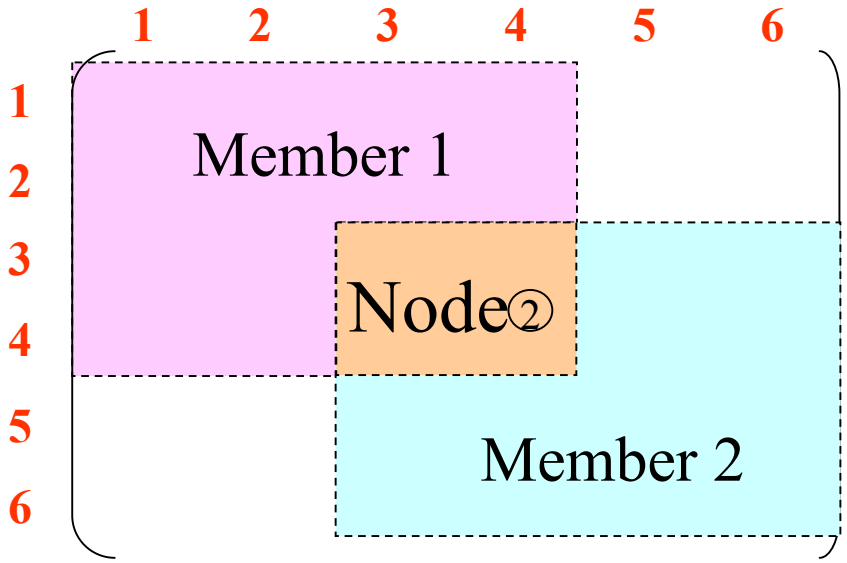
$[k]_1 =$

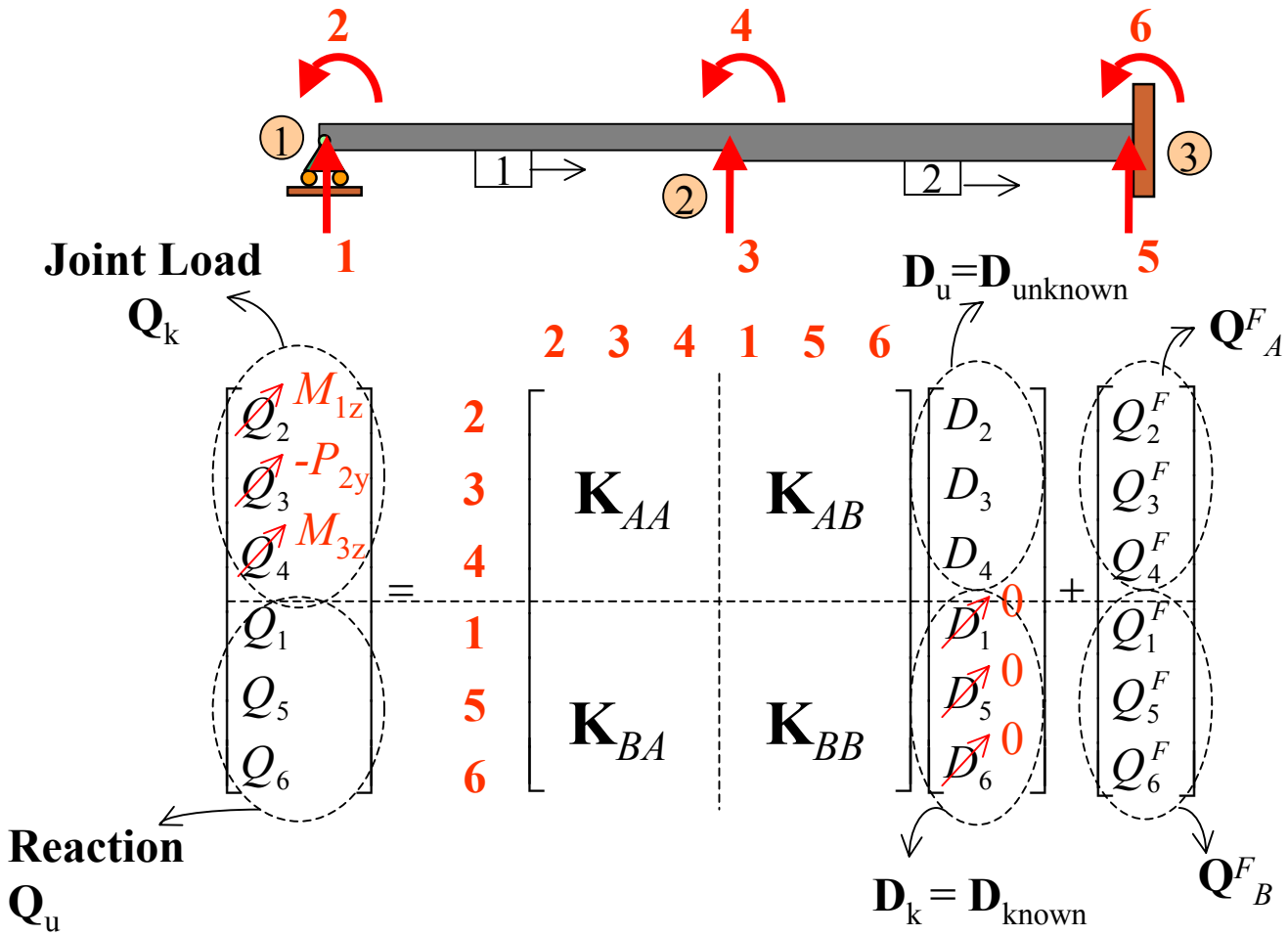


$[k]_2 =$



$[K] =$



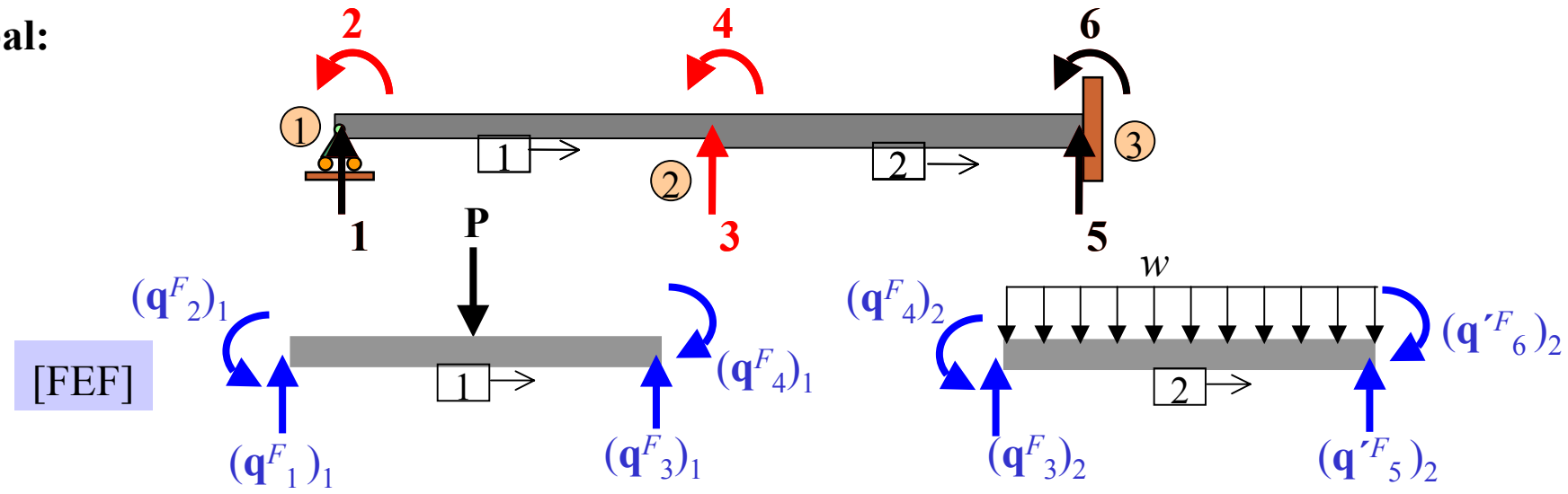


$$[Q_k] = [K_{AA}][D_u] + [Q_A^F]$$

$$[D_u] = [K_{AA}]^{-1} + ([Q_k] - [Q_A^F])$$

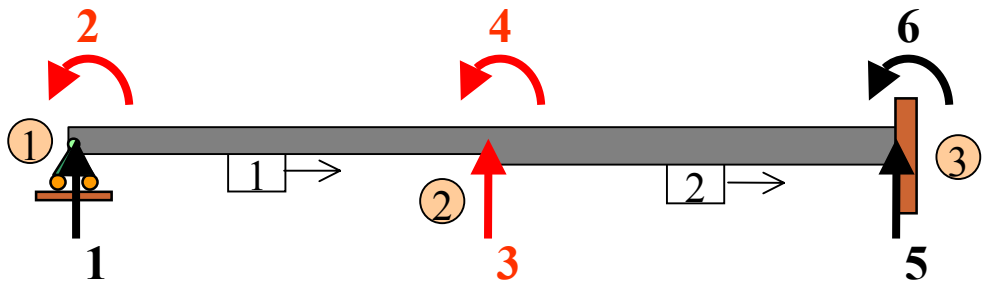
$$Member\ Force: [q] = [k][d] + [q^F]$$

Global:

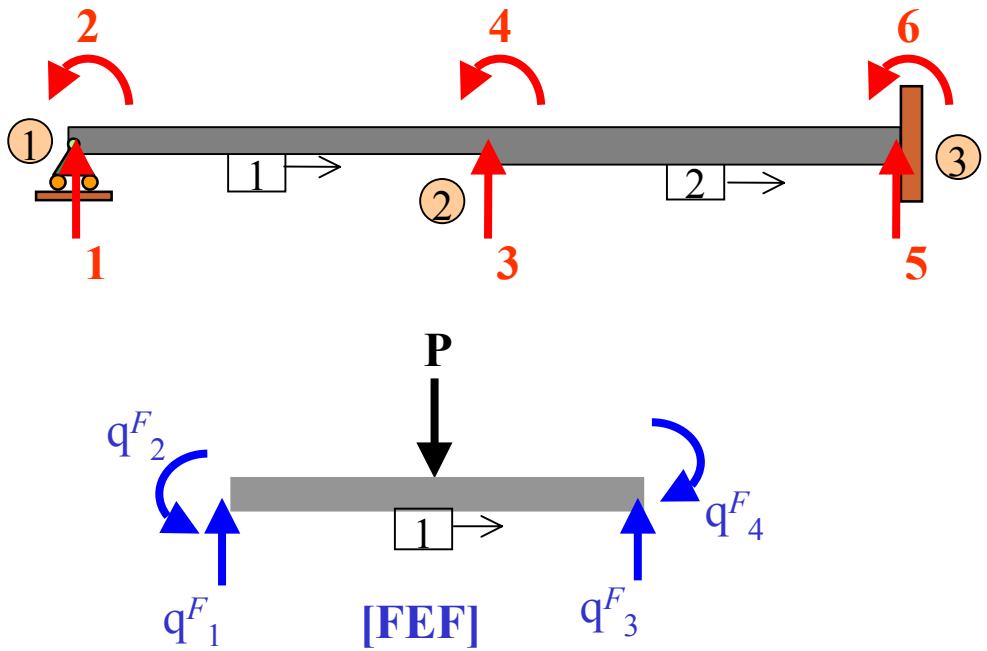


$$\begin{bmatrix} Q_1 \\ \cancel{Q_2}\ M_1 \\ \cancel{Q_3}\ -P_{2y} \\ \cancel{Q_4}\ M_2 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \overset{\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}}{\begin{bmatrix} & & & & & \\ & \text{Member 1} & & & & \\ & & \text{Node②} & & & \\ & & & \text{Member 2} & & \\ & & & & & \\ & & & & & \end{bmatrix}} \begin{bmatrix} \cancel{D_1}\ 0 \\ D_2 \\ D_3 \\ D_4 \\ \cancel{D_5}\ 0 \\ \cancel{D_6}\ 0 \end{bmatrix} + \begin{bmatrix} Q_1^F \\ Q_2^F \\ Q_3^F = (q_3^F)_1 + (q_3^F)_2 \\ Q_4^F = (q_4^F)_1 + (q_4^F)_2 \\ Q_5^F \\ Q_6^F \end{bmatrix}$$

$$\begin{bmatrix} Q_2 = M_1 \\ Q_3 = -P_{2y} \\ Q_4 = M_2 \end{bmatrix} = \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} \overset{\begin{matrix} 2 & 3 & 4 \end{matrix}}{\begin{bmatrix} & & \\ & & \\ & \text{Node②} & \end{bmatrix}} \begin{bmatrix} D_2 \\ D_3 \\ D_4 \end{bmatrix} + \begin{bmatrix} Q_2^F \\ Q_3^F \\ Q_4^F \end{bmatrix}$$

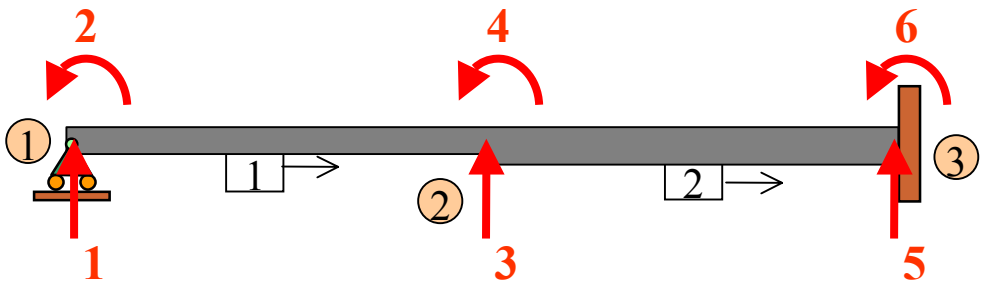


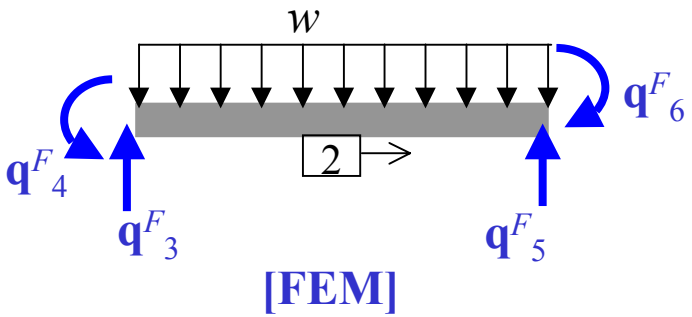
$$\begin{bmatrix} D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{matrix} 2 \\ 3 \\ 4 \end{matrix} \overset{\begin{matrix} 2 & 3 & 4 \end{matrix}}{\begin{bmatrix} & & \\ & \text{Node②} & \\ & & \end{bmatrix}} \left[\begin{bmatrix} Q_2 = M_1 \\ Q_3 = -P_{2y} \\ Q_4 = M_2 \end{bmatrix} - \begin{bmatrix} Q_2^F \\ Q_3^F \\ Q_4^F \end{bmatrix} \right]$$



Member 1:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \overset{\begin{matrix} 1 & 2 & 3 & 4 \end{matrix}}{\begin{bmatrix} & & & \\ & & k_1 & \\ & & & \\ & & & \end{bmatrix}} \begin{bmatrix} d_1 = 0 \\ d_2 = D_2 \\ d_3 = D_3 \\ d_4 = D_4 \end{bmatrix} + \begin{bmatrix} q_1^F \\ q_2^F \\ q_3^F \\ q_4^F \end{bmatrix}$$



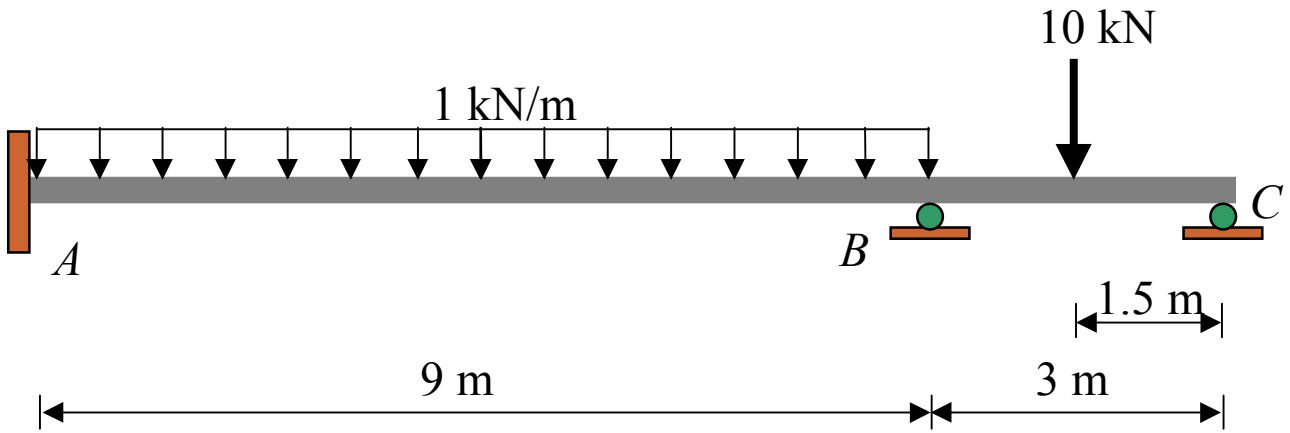


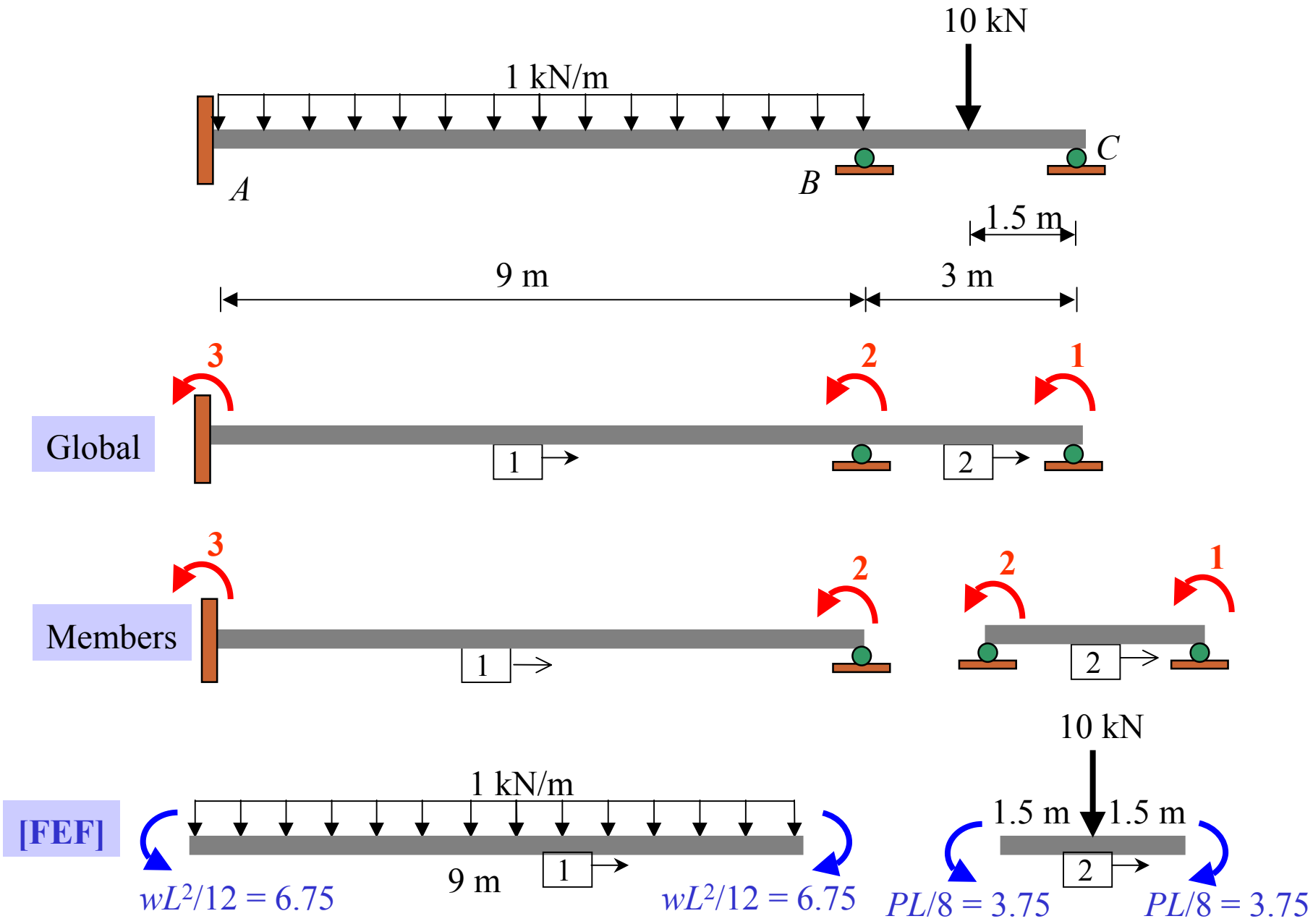
Member 2:

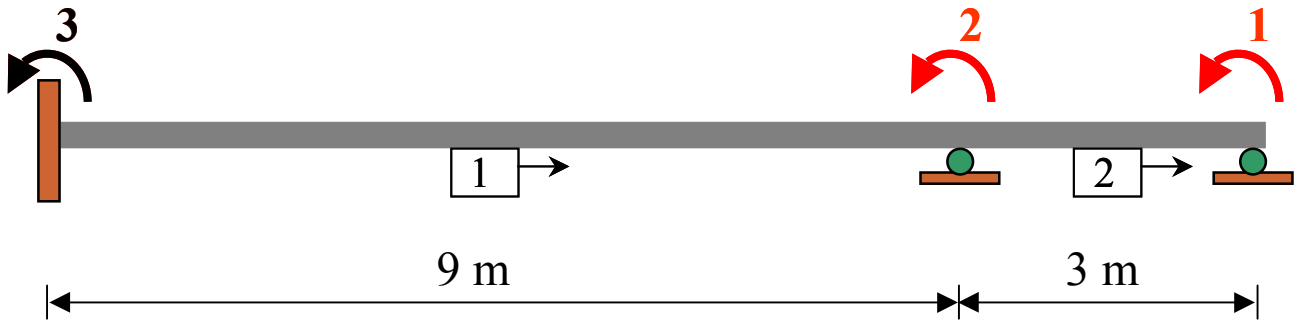
$$\begin{bmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} & & & \\ & k_1 & & \\ & & & \\ & & & \end{bmatrix} \end{matrix} \begin{bmatrix} d_3 = D_3 \\ d_4 = D_4 \\ d_5 = 0 \\ d_6 = 0 \end{bmatrix} + \begin{bmatrix} q_3^F \\ q_4^F \\ q_5^F \\ q_6^F \end{bmatrix}$$

Example 1

For the beam shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at *B*.
(b) Determine all the reactions at supports.
(c) Draw the **quantitative shear** and **bending moment diagrams.**




10 kN
1 kN/m
A
B
C
1.5 m
9 m
3 m
Global
3
2
1
1
2
Members
3
2
2
1
1
2
[FEF]
1 kN/m
9 m
1
wL²/12 = 6.75
wL²/12 = 6.75
10 kN
1.5 m
1.5 m
2
PL/8 = 3.75
PL/8 = 3.75



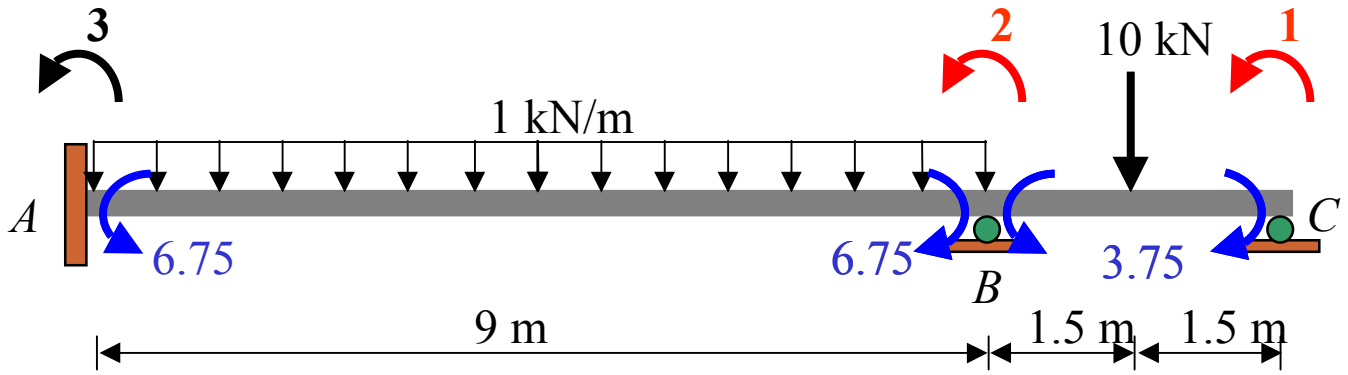
Stiffness Matrix:

$$[k]_{2\times 2} = \begin{matrix} \\ M_i \\ M_j \end{matrix} \begin{matrix} \theta_i & \theta_j \\ \end{matrix} \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix}$$

$$[k]_1 = EI \begin{matrix} \\ 3 \\ 2 \end{matrix} \begin{matrix} 3 & 2 \\ \end{matrix} \begin{bmatrix} 4/9 & 2/9 \\ 2/9 & 4/9 \end{bmatrix} \begin{matrix} \\ 3 \\ 2 \end{matrix}$$

$$[k]_2 = EI \begin{matrix} \\ 2 \\ 1 \end{matrix} \begin{matrix} 2 & 1 \\ \end{matrix} \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix} \begin{matrix} \\ 2 \\ 1 \end{matrix}$$

$$[K] = EI \begin{matrix} \\ 2 \\ 1 \end{matrix} \begin{matrix} 2 & 1 \\ \end{matrix} \begin{bmatrix} (4/9)+(4/3) & 2/3 \\ 2/3 & 4/3 \end{bmatrix} \begin{matrix} \\ 2 \\ 1 \end{matrix}$$



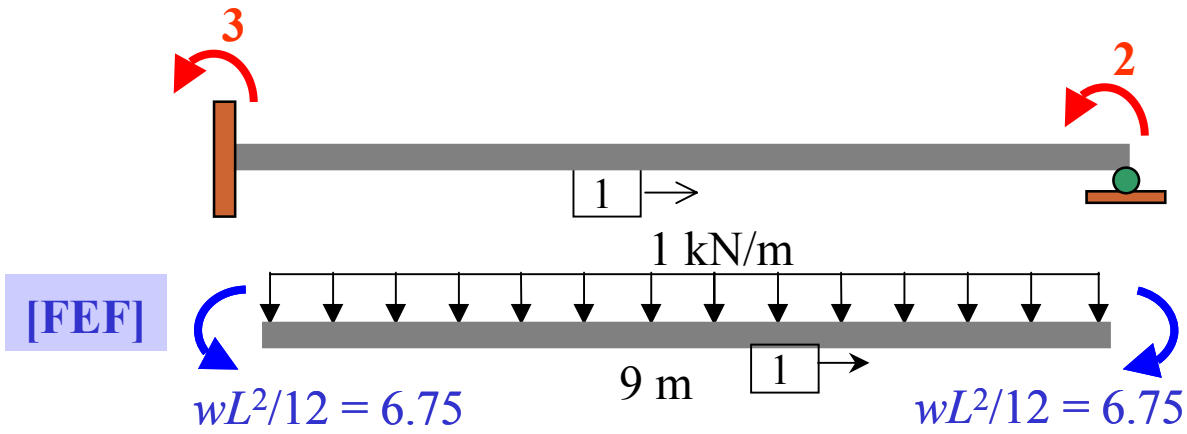
Equilibrium equations: $M_{CB} = 0$

$M_{BA} + M_{BC} = 0$

Global Equilibrium: $[\boldsymbol{Q}] = [\boldsymbol{K}][\boldsymbol{D}] + [\boldsymbol{Q}^{\mathrm{F}}]$

$$\begin{matrix} 2 \\ 1 \end{matrix}\begin{bmatrix} M_{BA} + M_{BC} = 0 \\ M_{CB} = 0 \end{bmatrix} = EI \begin{matrix} \\ 2 \\ 1 \end{matrix}\begin{matrix} \begin{matrix} 2 & 1 \end{matrix} \\ \begin{bmatrix} (4/9)+(4/3) & 2/3 \\ 2/3 & 4/3 \end{bmatrix} \end{matrix} \begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} + \begin{bmatrix} -6.75 + 3.75 = -3 \\ -3.75 \end{bmatrix}$$

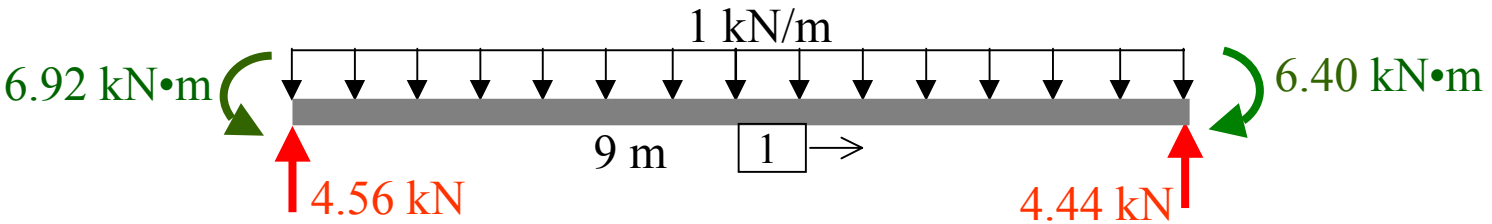
$$\begin{bmatrix} \theta_B \\ \theta_C \end{bmatrix} = \begin{bmatrix} 0.779/EI \\ 2.423/EI \end{bmatrix}$$

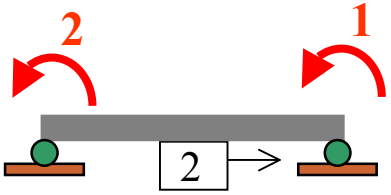


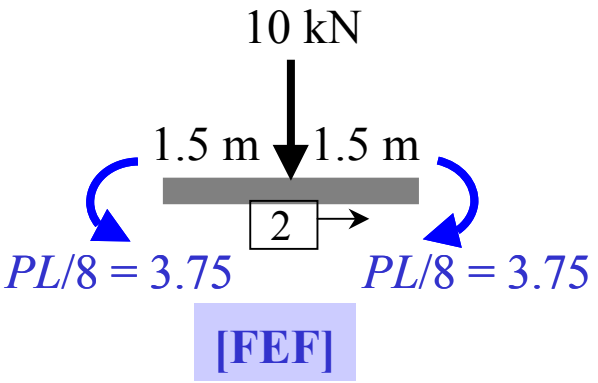
Substitute θ_B and θ_C in the member matrix,

Member 1 **:** $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{matrix} 3 \\ 2 \end{matrix}\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = EI \begin{matrix} \\ \end{matrix}\begin{matrix} 3 & 2 \end{matrix}\begin{bmatrix} 4/9 & 2/9 \\ 2/9 & 4/9 \end{bmatrix}\begin{matrix} 0 \\ \end{matrix}\begin{bmatrix} \theta_A \\ \theta_B = 0.779/EI \end{bmatrix} + \begin{bmatrix} 6.75 \\ -6.75 \end{bmatrix} = \begin{bmatrix} 6.92 \\ -6.40 \end{bmatrix}$$



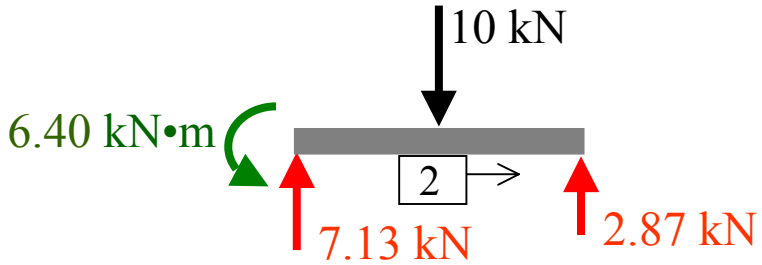


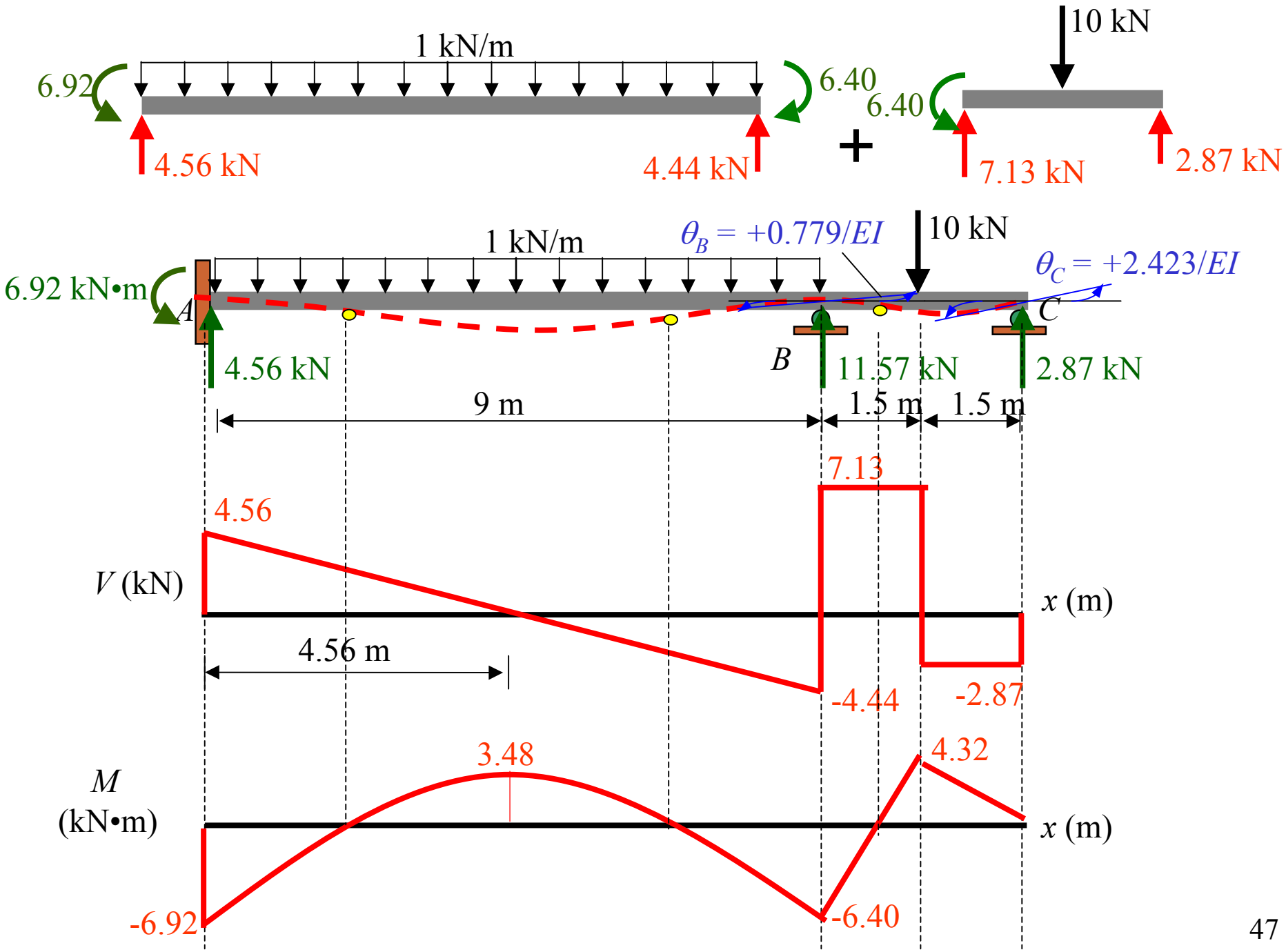


Substitute θ_B and θ_C in the member matrix,

Member 2 **:** $[q]_2 = [k]_2[d]_2 + [q^F]_2$

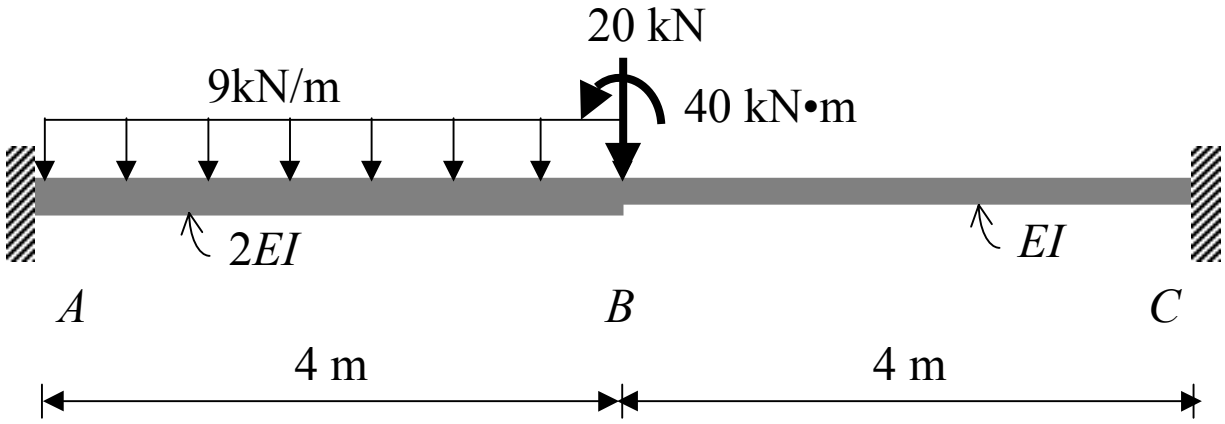
$$\begin{matrix} \\ 2 \\ 1 \end{matrix}\begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = EI \begin{matrix} \begin{matrix} 2 & 1 \end{matrix} \\ \begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix} \end{matrix} \begin{bmatrix} \theta_B = 0.779/EI \\ \theta_C = 2.423/EI \end{bmatrix} + \begin{bmatrix} 3.75 \\ -3.75 \end{bmatrix} = \begin{bmatrix} 6.40 \\ 0 \end{bmatrix}$$

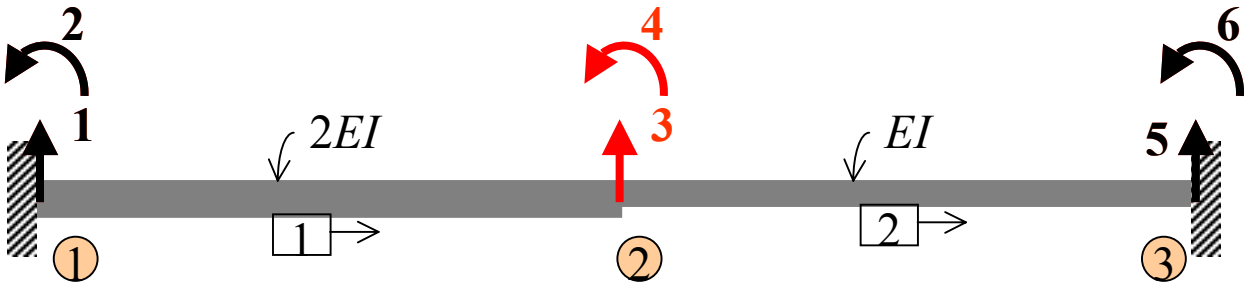



1 kN/m
6.92
6.40
4.56 kN
4.44 kN
+
10 kN
6.40
7.13 kN
2.87 kN
1 kN/m
$\theta_B = +0.779/EI$
10 kN
$\theta_C = +2.423/EI$
6.92 kN•m
A
B
C
4.56 kN
11.57 kN
2.87 kN
9 m
1.5 m
1.5 m
7.13
4.56
V (kN)
x (m)
4.56 m
-4.44
-2.87
3.48
4.32
M
(kN•m)
x (m)
-6.92
-6.40

Example 2

For the beam shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at ***B***.
(b) Determine all the reactions at supports.





Use 4x4 stiffness matrix,

$$[k]=\begin{array}{c} \\ V_i \\ M_i \\ V_j \\ M_j \end{array}\begin{array}{c} \begin{array}{cccc} \Delta_i & \theta_i & \Delta_j & \theta_j \end{array} \\ \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

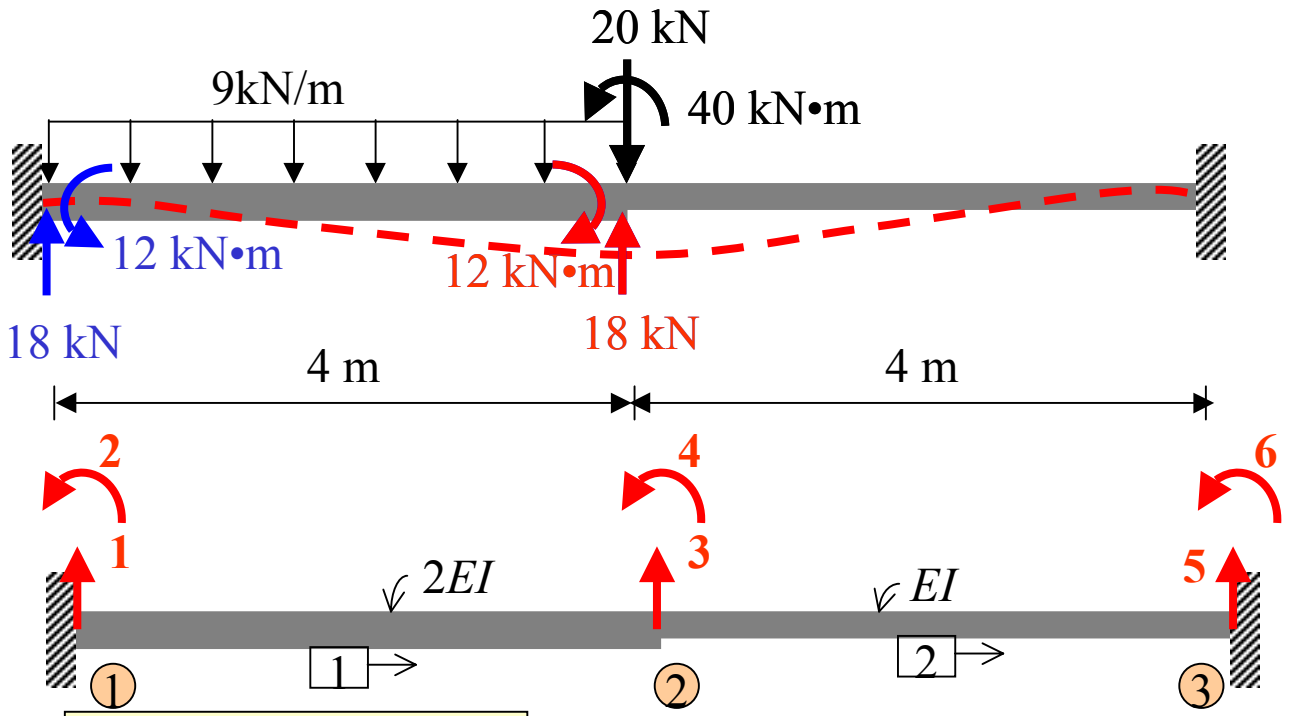
$$[K]=EI\begin{array}{c} \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 0.5625 & -0.375 \\ -0.375 & 3 \end{bmatrix} \end{array}$$

$$[k]_1$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI \end{pmatrix} \end{array}$$

$$[k]_2$$

$$\begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{pmatrix} \end{array}$$

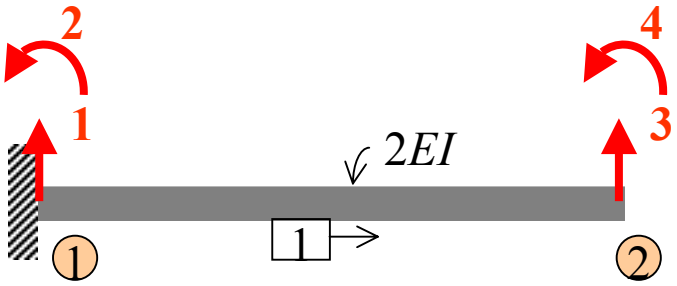


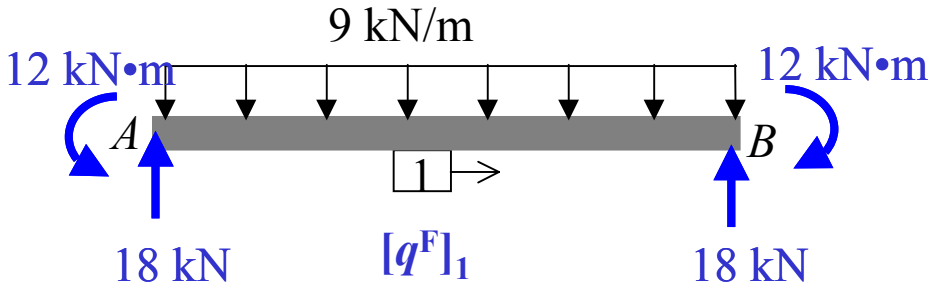
$$[Q] = [K][D] + [Q^F]$$

Global:

$$\begin{matrix} 3 \\ 4 \end{matrix}\begin{bmatrix} Q_3 = -20 \\ Q_4 = 40 \end{bmatrix} = EI \begin{matrix} \\ 3 \\ 4 \end{matrix}\overset{\begin{matrix} 3 & 4 \end{matrix}}{\begin{bmatrix} 0.5625 & -0.375 \\ -0.375 & 3 \end{bmatrix}} \begin{bmatrix} D_3 \\ D_4 \end{bmatrix} + \begin{bmatrix} 18 \\ -12 \end{bmatrix}\begin{matrix} 3 \\ 4 \end{matrix}$$

$$\begin{bmatrix} D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} -61.09/EI \\ 9.697/EI \end{bmatrix}$$

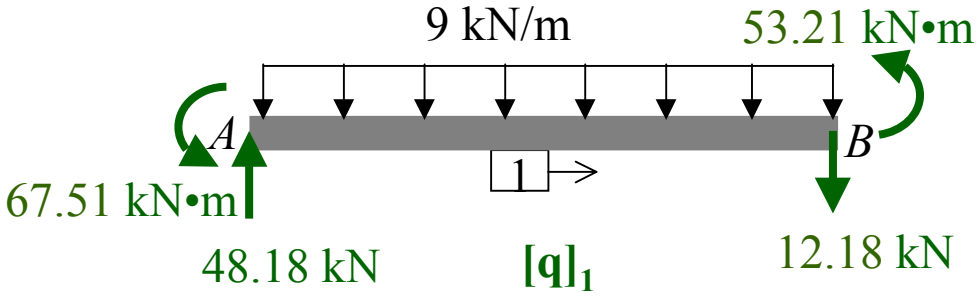


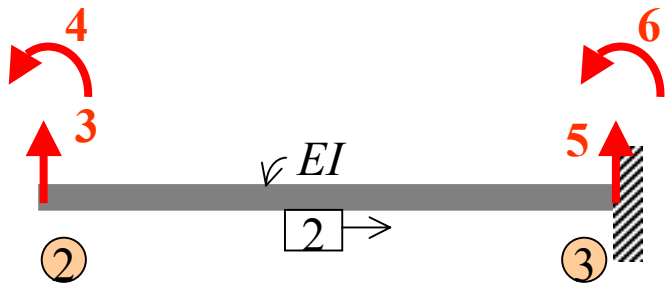


Member 1:

$$[q]_1 = [k]_1[d]_1 + [q^F]_1$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_{3L} \\ q_{4L} \end{pmatrix} = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{pmatrix} 12(2EI)/4^3 & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI \end{pmatrix} \end{matrix} \begin{pmatrix} d_1 = 0 \\ d_2 = 0 \\ d_3 = -61.09/EI \\ d_4 = 9.697/EI \end{pmatrix} + \begin{pmatrix} 18 \\ 12 \\ 18 \\ -12 \end{pmatrix} = \begin{pmatrix} 48.18 \\ 67.51 \\ -12.18 \\ 53.21 \end{pmatrix}$$

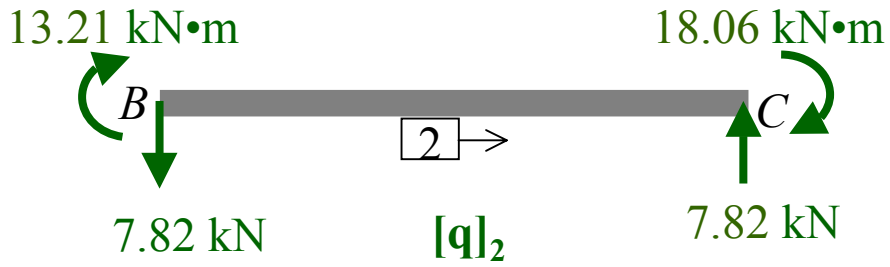


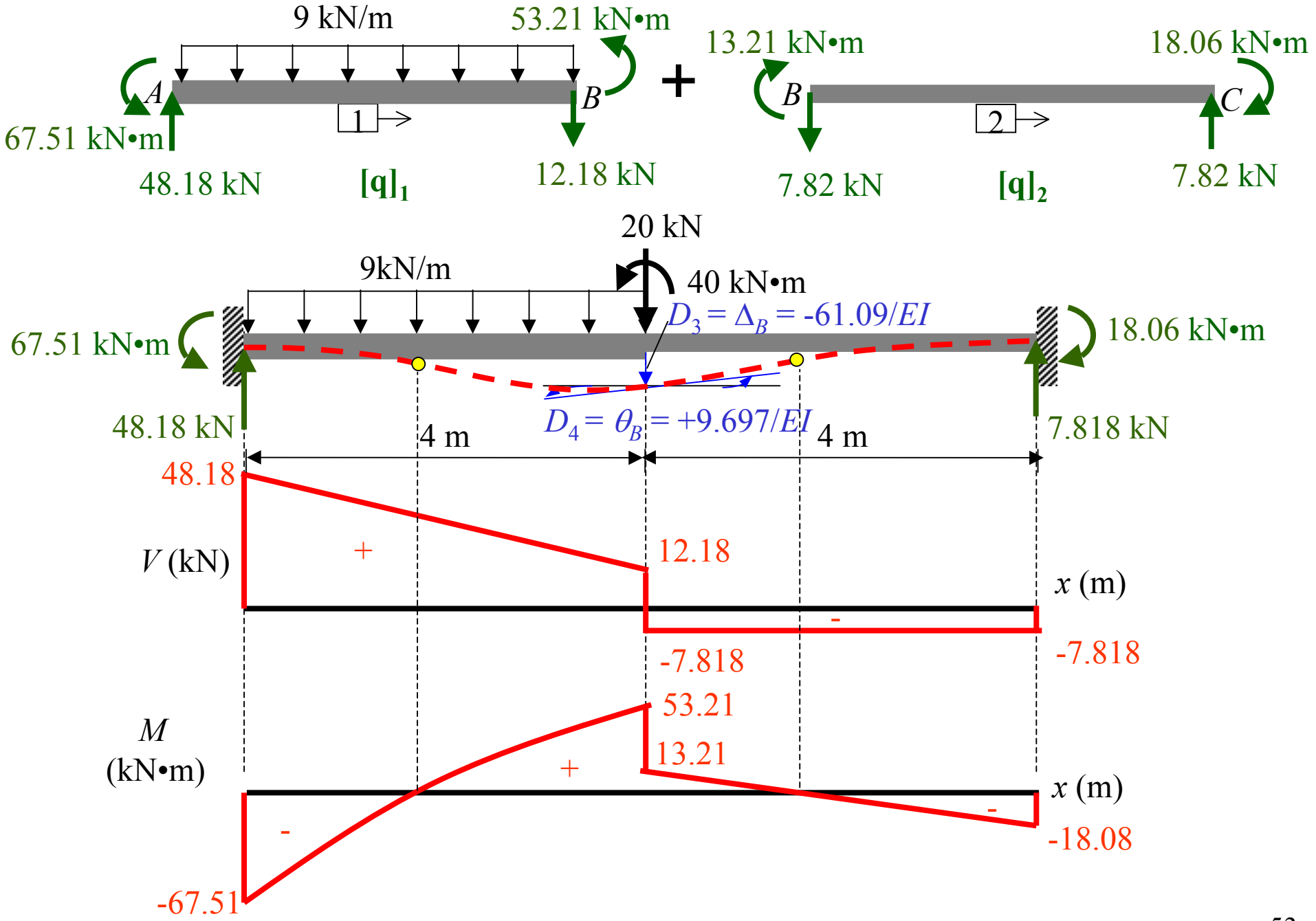


Member 2:

$$[q]_2 = [k]_2[d]_2 + [q^F]_2$$

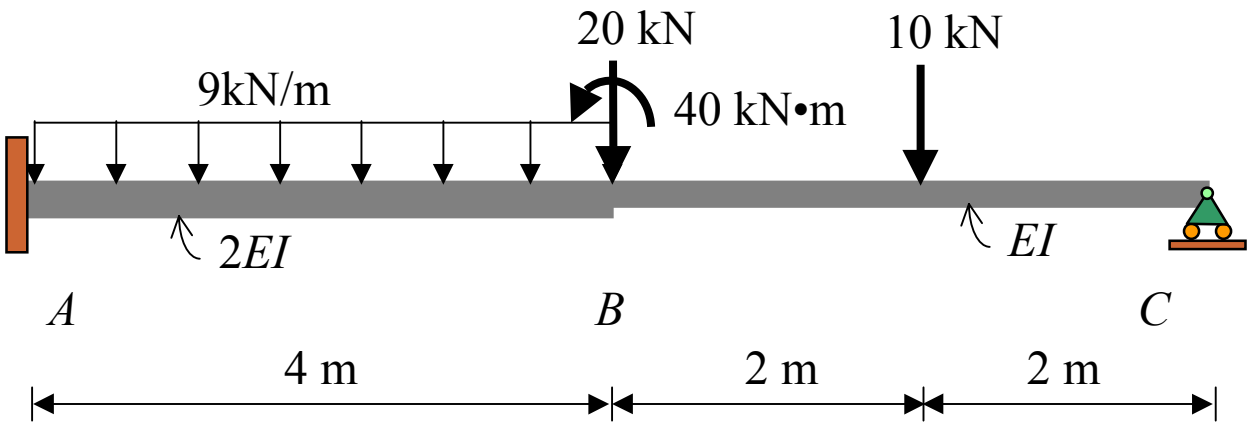
$$\begin{pmatrix} q_{3R} \\ q_{4R} \\ q_5 \\ q_6 \end{pmatrix} = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{pmatrix} \end{array} \begin{pmatrix} d_3 = -61.09/EI \\ d_4 = 9.697/EI \\ d_5 = 0 \\ d_6 = 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -7.818 \\ -13.21 \\ 7.818 \\ -18.06 \end{pmatrix}$$

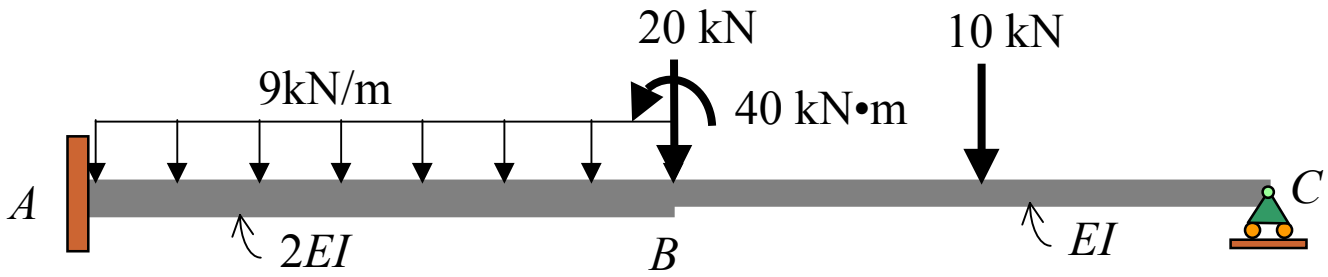



9 kN/m
53.21 kN•m
A
B
1
67.51 kN•m
48.18 kN
[q]1
12.18 kN
13.21 kN•m
18.06 kN•m
B
C
2
7.82 kN
[q]2
7.82 kN
20 kN
9kN/m
40 kN•m
D3 = ΔB = -61.09/EI
67.51 kN•m
18.06 kN•m
D4 = θB = +9.697/EI
48.18 kN
4 m
4 m
7.818 kN
48.18
V (kN)
+
12.18
x (m)
-
-7.818
-7.818
53.21
M
(kN•m)
13.21
+
x (m)
-
-
-18.08
-67.51

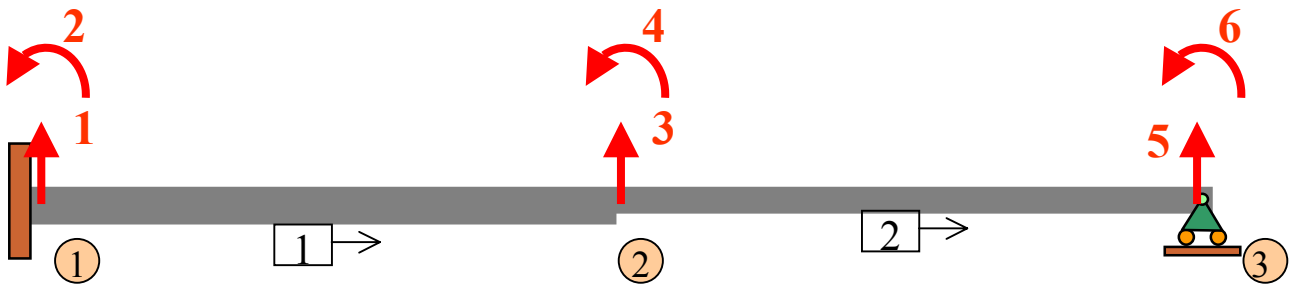
Example 3

For the beam shown, use the stiffness method to:
(a) Determine the **deflection** and **rotation** at ***B***.
(b) Determine all the reactions at supports.

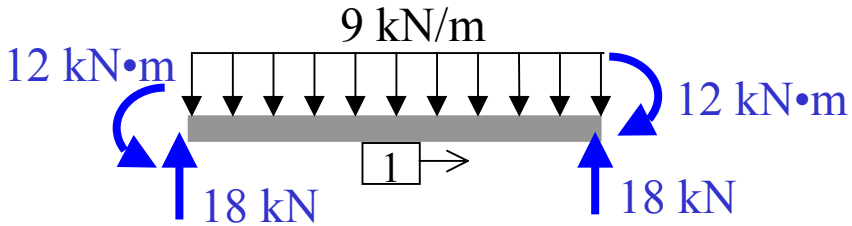


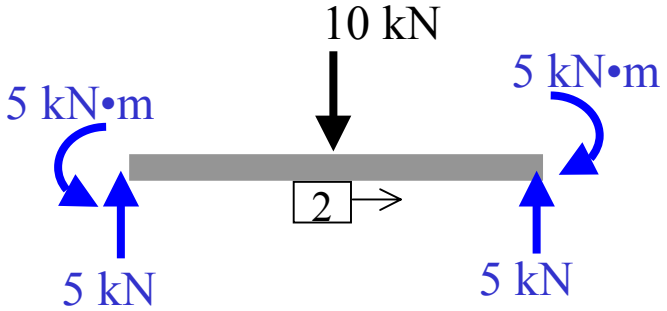


Global

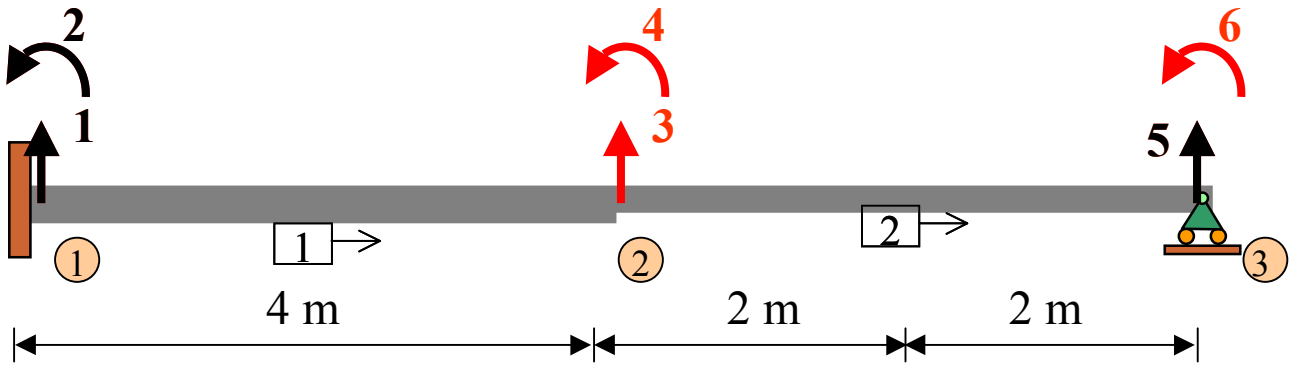


[FEF]





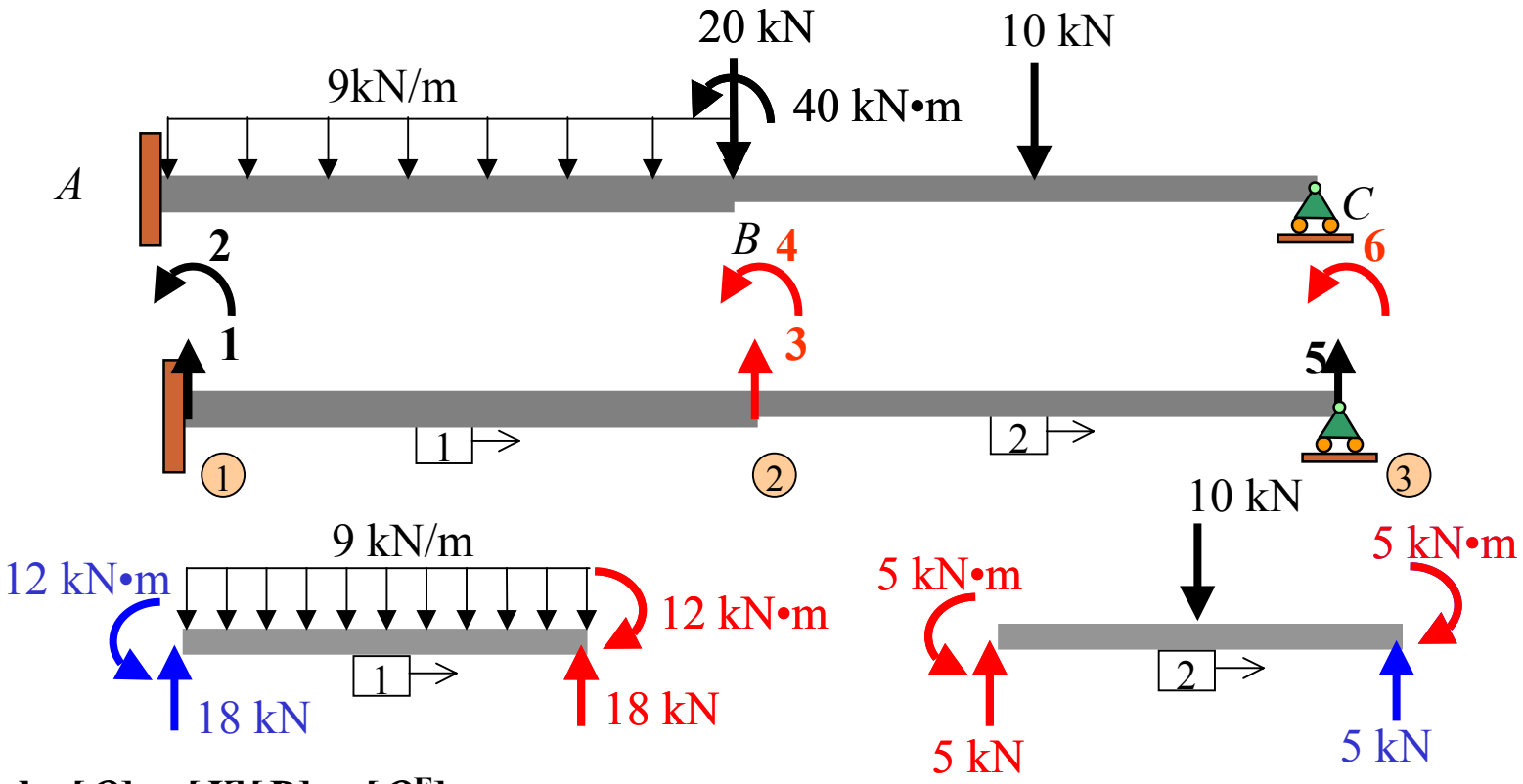
$$[k]_{4\times4} = \begin{array}{c} \\ V_i \\ M_i \\ V_j \\ M_j \end{array} \begin{array}{c} \begin{array}{cccc} \Delta_i & \theta_i & \Delta_j & \theta_j \end{array} \\ \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$



$$[k]_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 12(2EI)/4^3 & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI/L \end{pmatrix} \end{array}$$

$$[K] = EI \begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 4 & 6 \end{array} \\ \begin{pmatrix} 0.5625 & -0.375 & 0.375 \\ -0.375 & 3 & 0.5 \\ 0.375 & 0.5 & 1 \end{pmatrix} \end{array}$$

$$[k]_2 = \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{pmatrix} \end{array}$$

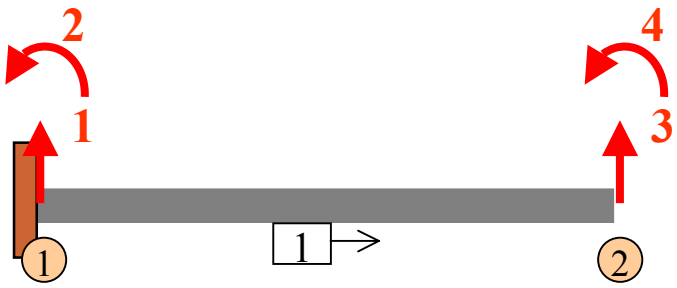


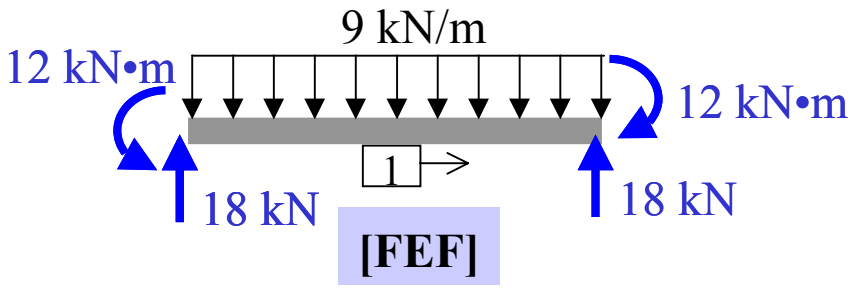
[FEF]

Global: $[Q] = [K][D] + [Q^F]$

$$\begin{pmatrix} V_{BL}+V_{BR} = -20 \\ M_{BA}+M_{BC} = 40 \\ M_{CB} = 0 \end{pmatrix} = EI \begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \quad 6 \\ \begin{pmatrix} 0.5625 & -0.375 & 0.375 \\ -0.375 & 3 & 0.5 \\ 0.375 & 0.5 & 1 \end{pmatrix} \end{array} \begin{pmatrix} \Delta_B \\ \theta_B \\ \theta_C \end{pmatrix} + \begin{pmatrix} 18+5 = 23 \\ -12+5 = -7 \\ -5 \end{pmatrix}$$

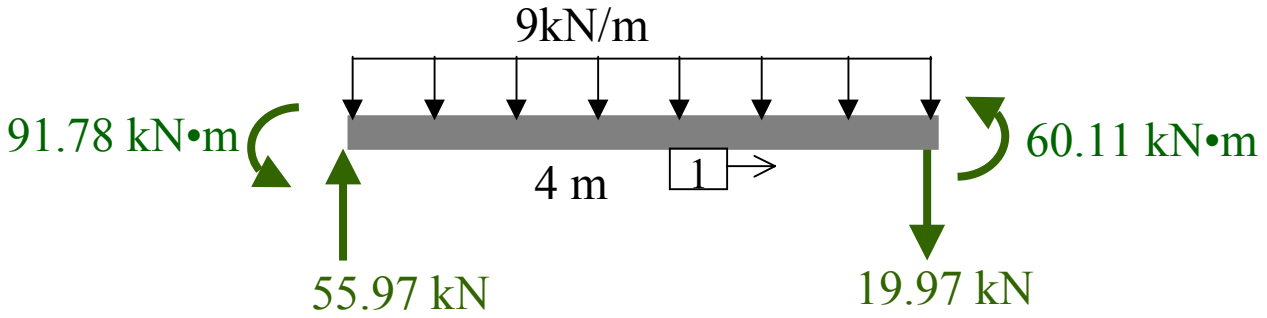
$$\begin{pmatrix} \Delta_B \\ \theta_B \\ \theta_C \end{pmatrix} = \begin{pmatrix} -116.593/EI \\ -7.667/EI \\ 52.556/EI \end{pmatrix}$$

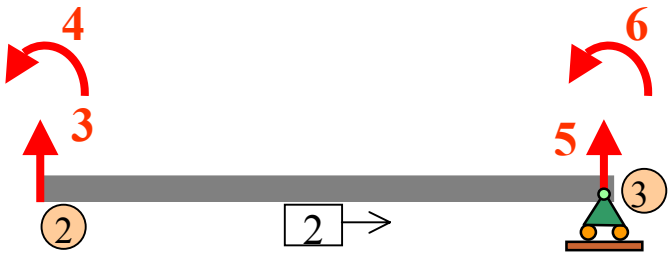


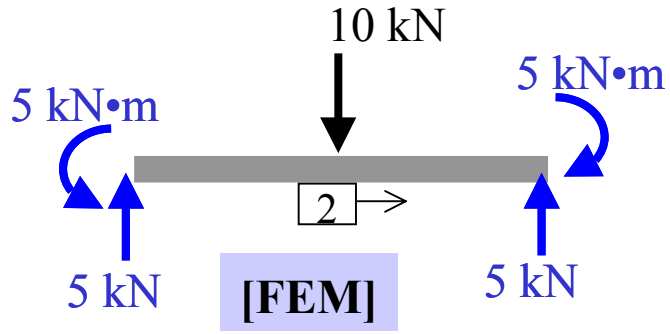


Member 1: $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{pmatrix} V_A \\ M_{AB} \\ V_{BL} \\ M_{BA} \end{pmatrix} = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \overset{\begin{matrix} 1 & 2 & 3 & 4 \end{matrix}}{\begin{pmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI/L \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ \Delta_B = -116.593/EI \\ \theta_B = -7.667/EI \end{pmatrix} + \begin{pmatrix} 18 \\ 12 \\ 18 \\ -12 \end{pmatrix} = \begin{pmatrix} 55.97 \\ 91.78 \\ -19.97 \\ 60.11 \end{pmatrix}$$

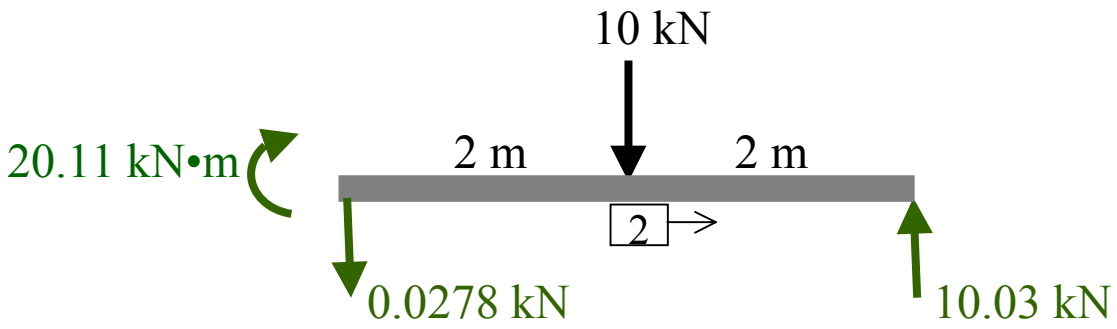


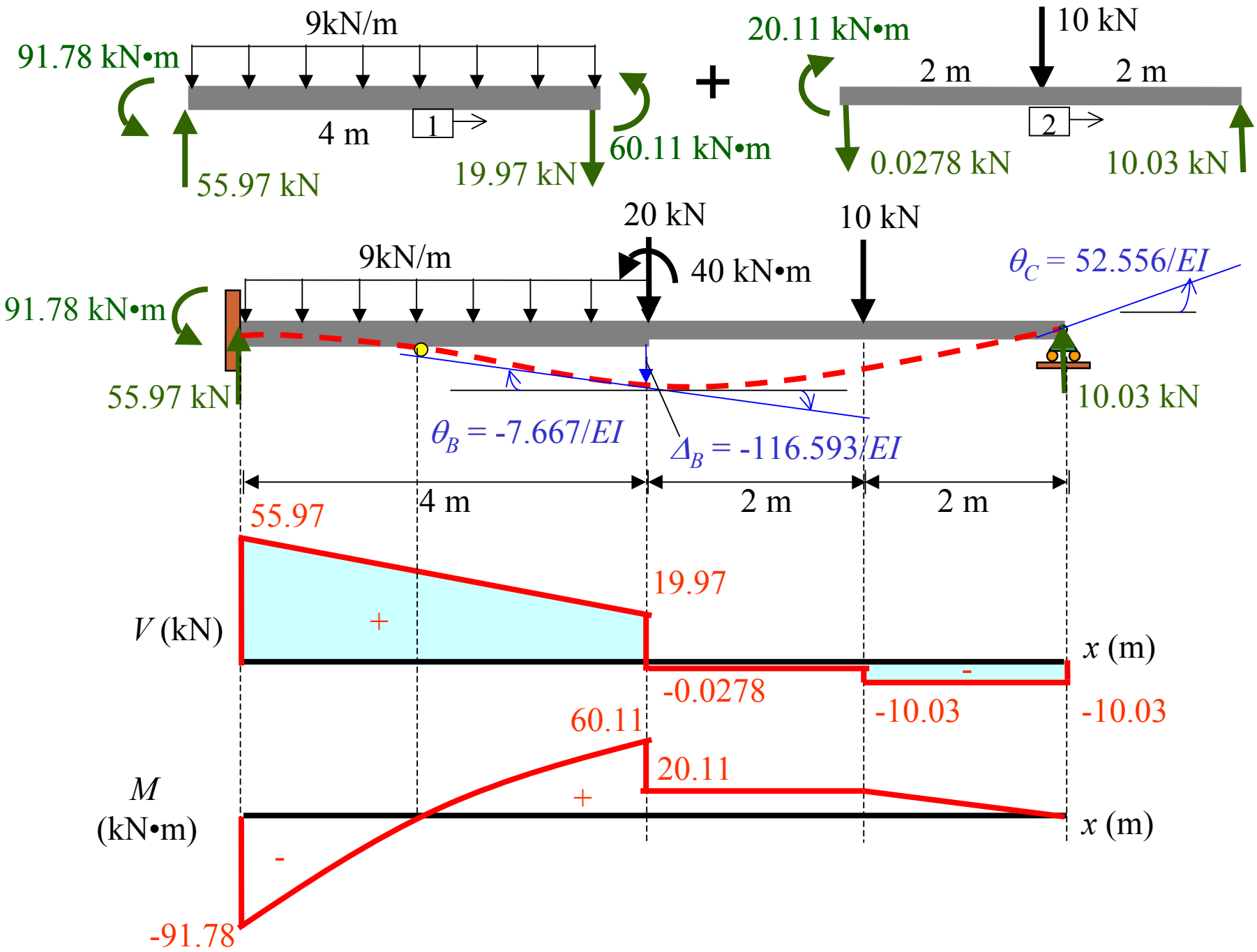




Member 2: $[q]_1 = [k]_1[d]_1 + [q^F]_1$

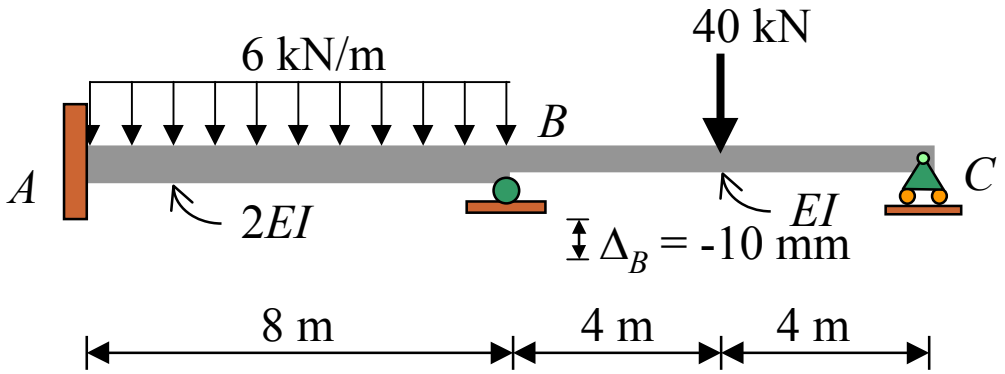
$$\begin{pmatrix} V_{BR} \\ M_{BC} \\ V_C \\ M_{CB} \end{pmatrix} = \begin{matrix} \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{matrix} \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{pmatrix} \end{matrix} \begin{pmatrix} \Delta_B = -116.593/EI \\ \theta_B = -7.667/EI \\ 0 \\ \theta_C = 52.556/EI \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \\ 5 \\ -5 \end{pmatrix} = \begin{pmatrix} -0.0278 \\ -20.11 \\ 10.03 \\ 0 \end{pmatrix}$$

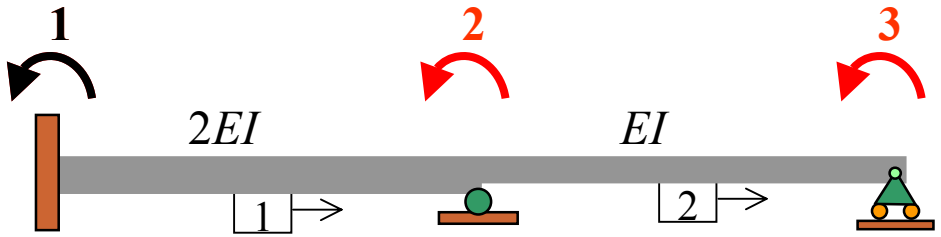



9kN/m
91.78 kN•m
4 m
1
55.97 kN
19.97 kN
60.11 kN•m
+
20.11 kN•m
10 kN
2 m
2 m
2
0.0278 kN
10.03 kN
20 kN
10 kN
9kN/m
40 kN•m
$\theta_C = 52.556/EI$
91.78 kN•m
55.97 kN
10.03 kN
$\theta_B = -7.667/EI$
$\Delta_B = -116.593/EI$
4 m
2 m
2 m
55.97
19.97
V (kN)
+
x (m)
-
-0.0278
-10.03
-10.03
60.11
20.11
M
(kN•m)
+
x (m)
-
-91.78

Example 4

For the beam shown:
(a) Use the stiffness method to determine all the **reactions** at supports.
(b) Draw the **quantitative free-body diagram** of member.
(c) Draw the **quantitative shear diagram, bending moment diagram** and **qualitative deflected shape.**
Take $I = 200(10^6)$ mm^4 and $E = 200$ GPa and support B settlement 10 mm.





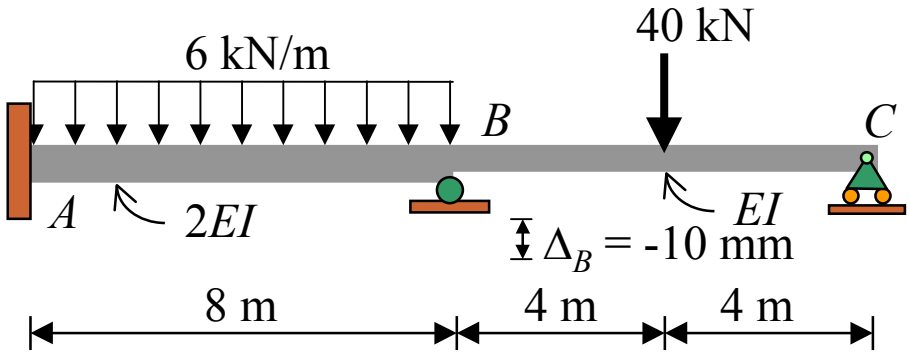
Use 2x2 stiffness matrix:

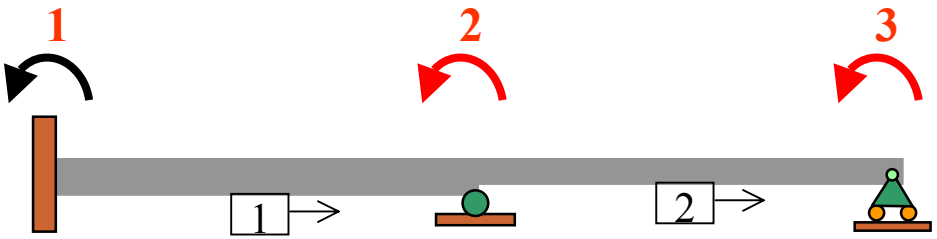
$$[k]_{2\times 2} = \begin{matrix} \\ M_i \\ M_j \end{matrix} \begin{matrix} \theta_i & \theta_j \\ \end{matrix} \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix}$$

$$[k]_1 = \frac{EI}{8} \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} 1 & 2 \\ \end{matrix} \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix}$$

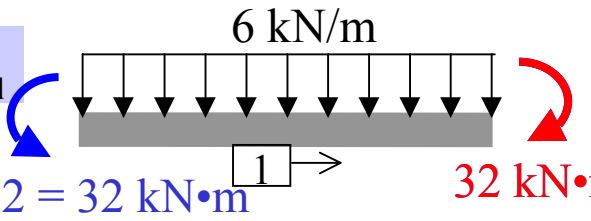
$$[k]_2 = \frac{EI}{8} \begin{matrix} \\ 2 \\ 3 \end{matrix} \begin{matrix} 2 & 3 \\ \end{matrix} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

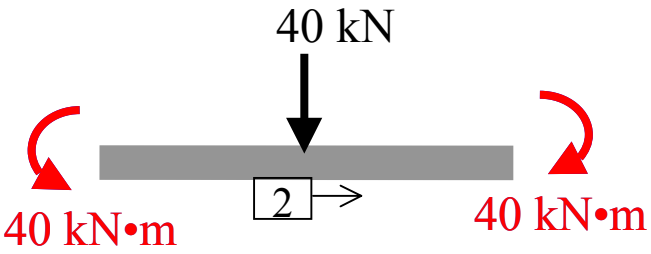
$$[K] = \frac{EI}{8} \begin{matrix} \\ 2 \\ 3 \end{matrix} \begin{matrix} 2 & 3 \\ \end{matrix} \begin{pmatrix} 12 & 2 \\ 2 & 4 \end{pmatrix}$$



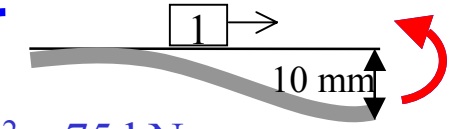


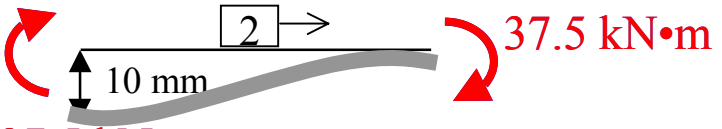
[FEM]$_{load}$





[FEM]$_{\Delta}$

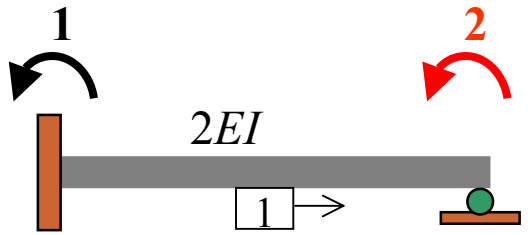


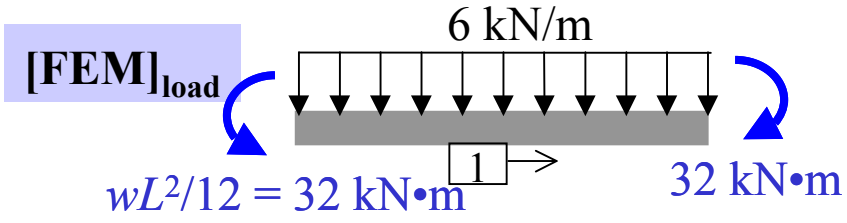


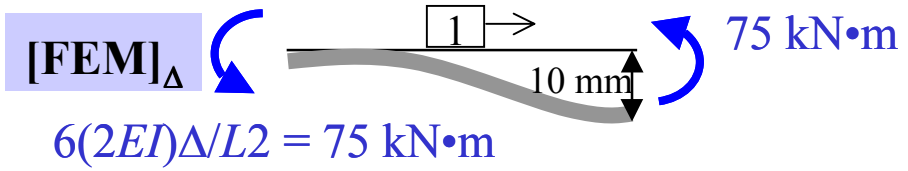
Global: $[Q] = [K][D] + [Q^F]$ 3

$$\begin{pmatrix} Q_2 = 0 \\ Q_3 = 0 \end{pmatrix} = \frac{EI}{8} \begin{matrix} 2 \\ 3 \end{matrix} \begin{pmatrix} 12 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} D_2 \\ D_3 \end{pmatrix} + \begin{pmatrix} -32 + 40 + 75 - 37.5 = 45.5 \\ -40 - 37.5 = -77.5 \end{pmatrix}$$

$$\begin{pmatrix} D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} -61.27/EI & \text{rad} \\ 185.64/EI & \text{rad} \end{pmatrix}$$

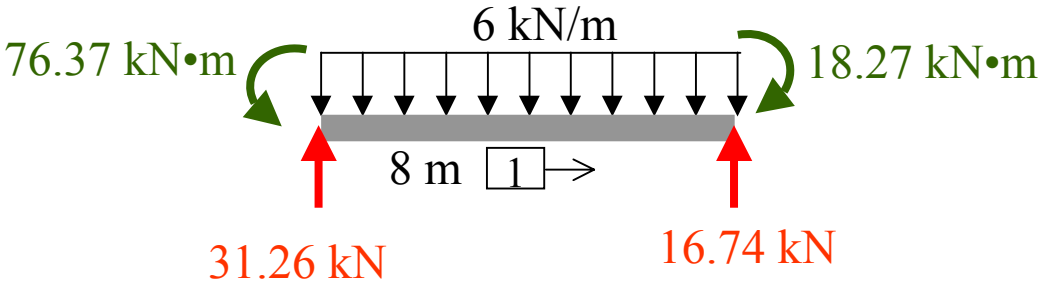


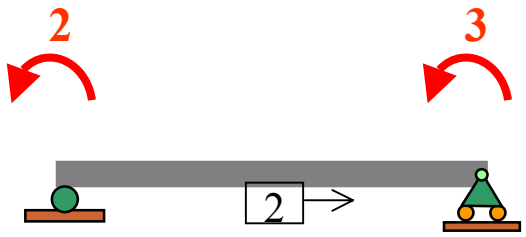


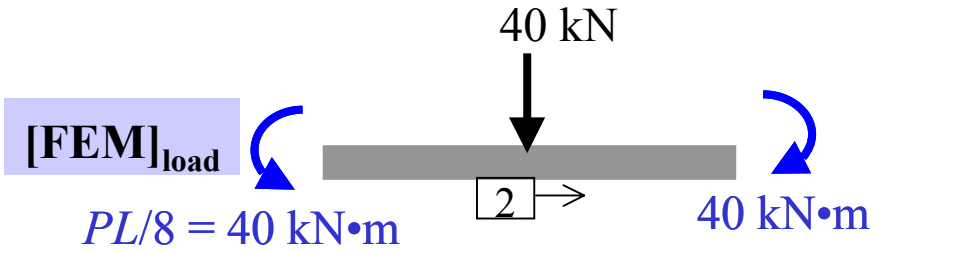


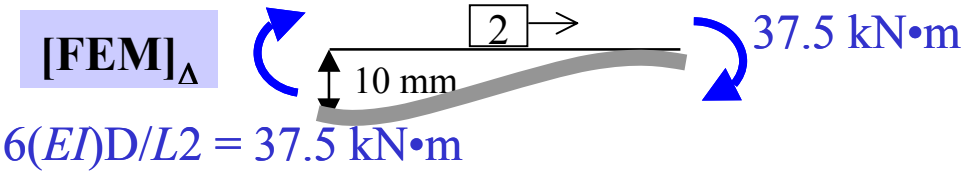
Member 1: $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{EI}{8} \begin{matrix} 1 \\ 2 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{pmatrix} 8 & 4 \\ 4 & 8 \end{pmatrix} \end{matrix} \begin{pmatrix} d_1 = 0 \\ d_2 = -61.27/EI \end{pmatrix} + \begin{pmatrix} 32 + 75 = 107 \\ -32 + 75 = 43 \end{pmatrix} = \begin{pmatrix} 76.37 \text{ kN•m} \\ -18.27 \text{ kN•m} \end{pmatrix}$$



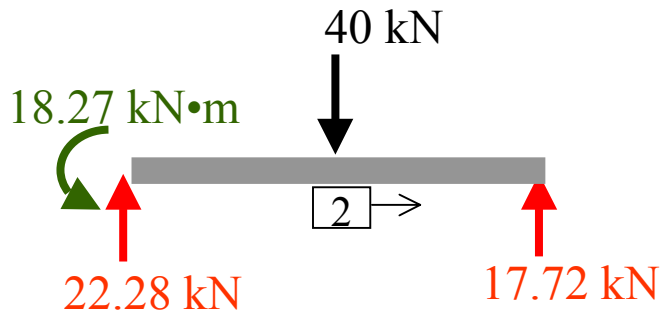


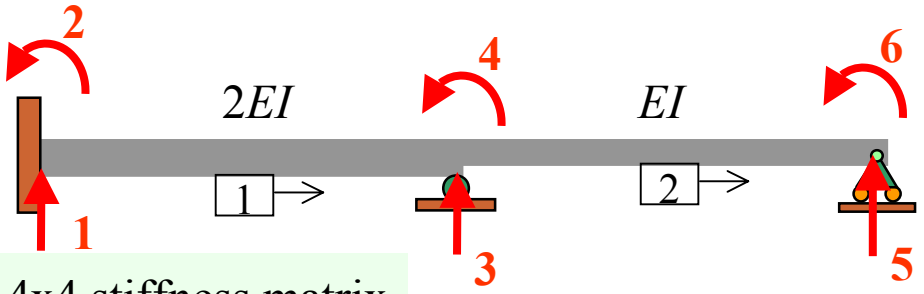




Member 2: $[q]_2 = [k]_2[d]_2 + [q^F]_2$

$$\begin{pmatrix} q_2 \\ q_3 \end{pmatrix} = \frac{EI}{8} \begin{matrix} 2 \\ 3 \end{matrix} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{matrix} \begin{pmatrix} 2 & 3 \end{matrix} \begin{pmatrix} d_2 = -61.27/EI \\ d_3 = 185.64/EI \end{pmatrix} + \begin{pmatrix} 40 - 37.5 = 2.5 \\ -40 - 37.5 = -77.5 \end{pmatrix} = \begin{pmatrix} 18.27 \text{ kN•m} \\ 0 \quad \text{kN•m} \end{pmatrix}$$





Alternate method: Use 4x4 stiffness matrix

$$[k]_1 = \frac{EI}{8}$$

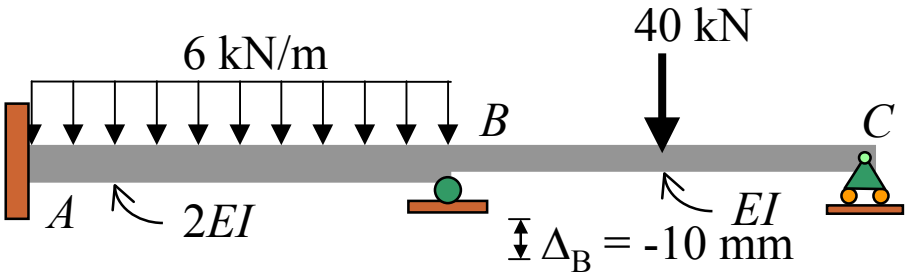
	1	2	3	4
1	12(2)/8²	1.5	-0.375	1.5
2	1.5	8	-1.5	4
3	-0.375	-1.5	0.375	-1.5
4	1.5	4	-1.5	8

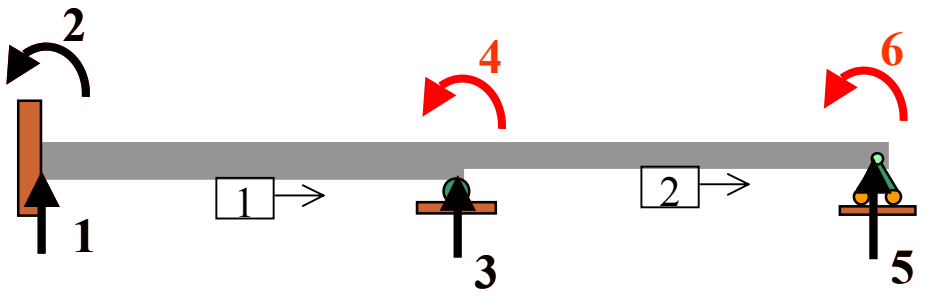
$$[k]_2 = \frac{EI}{8}$$

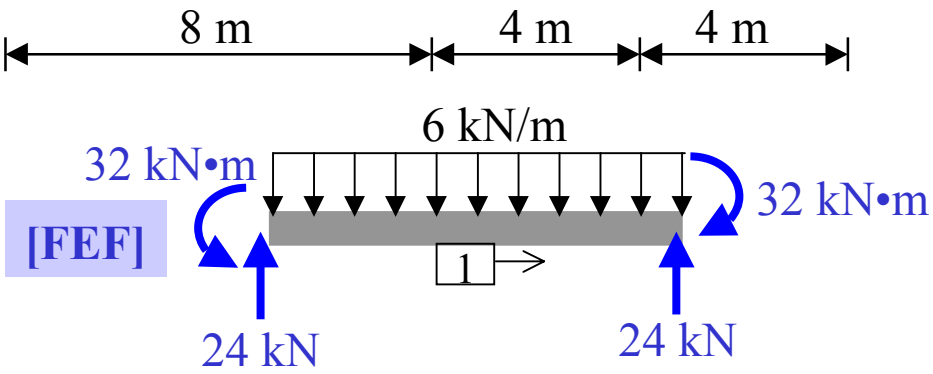
	3	4	5	6
3	12/8²	0.75	-0.1875	0.75
4	0.75	4	-0.75	2
5	-0.1875	-0.75	0.1875	-0.75
6	0.75	2	-0.75	4

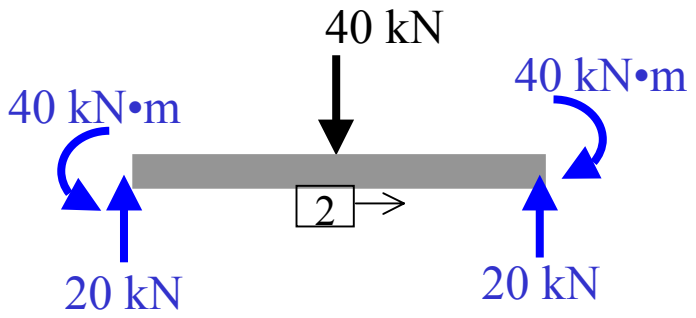
$$[K] = \frac{EI}{8}$$

	1	2	3	4	5	6
1	0.375	1.5	-0.375	1.5	0	0
2	1.5	8	-1.5	4	0	0
3	-0.375	-1.5	0.5625	-0.75	-0.1875	0.75
4	1.5	4	-0.75	12	-0.75	2
5	0	0	-0.1875	-0.75	0.1875	-0.75
6	0	0	0.75	2	-0.75	4







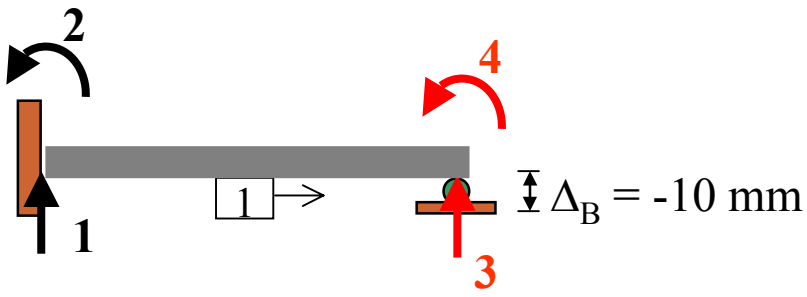


Global: $[Q] = [K][D] + [Q^F]$

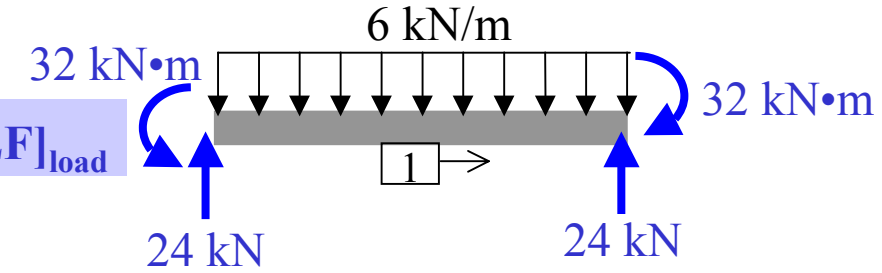
$$\begin{bmatrix} Q_4 = 0 \\ Q_6 = 0 \end{bmatrix} = \frac{EI}{8} \begin{matrix} 4 \\ 6 \end{matrix} \begin{bmatrix} 12 & 2 \\ 2 & 4 \end{bmatrix} \begin{matrix} 4 \\ 6 \end{matrix} \begin{bmatrix} D_4 \\ D_6 \end{bmatrix} + \left(\frac{200 \times 200}{8}\right) \begin{matrix} 4 \\ 6 \end{matrix} \begin{bmatrix} -0.75 \\ -0.75 \end{bmatrix} \begin{matrix} 5 \\ [D_5 = -0.01] \end{matrix} + \begin{bmatrix} 8 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} Q_4 = 0 \\ Q_6 = 0 \end{bmatrix} = \frac{EI}{8} \begin{matrix} 4 \\ 6 \end{matrix} \begin{bmatrix} 12 & 2 \\ 2 & 4 \end{bmatrix} \begin{matrix} 4 \\ 6 \end{matrix} \begin{bmatrix} D_4 \\ D_6 \end{bmatrix} + \begin{bmatrix} (200x200/8)(-0.75)(-0.01) = 37.5 \\ (200x200/8)(0.75)(-0.01) = -37.5 \end{bmatrix} + \begin{bmatrix} 8 \\ -40 \end{bmatrix}$$

$$\begin{bmatrix} D_4 \\ D_6 \end{bmatrix} = \begin{bmatrix} -61.27/EI = -1.532x10^{-3} & \text{rad} \\ 185.64/EI = 4.641x10^{-3} & \text{rad} \end{bmatrix}$$



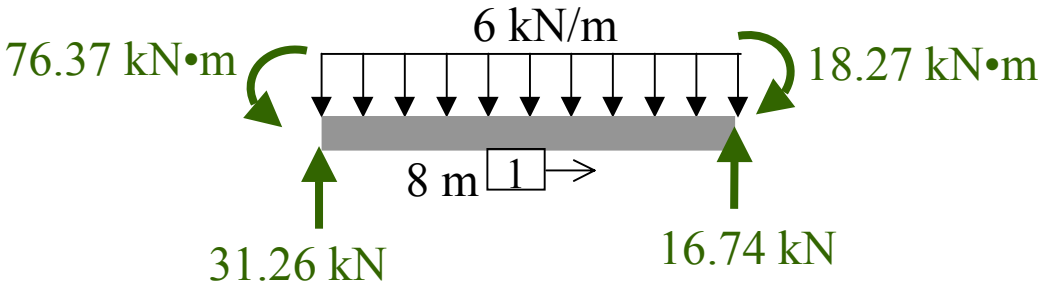
$[FEF]_{load}$

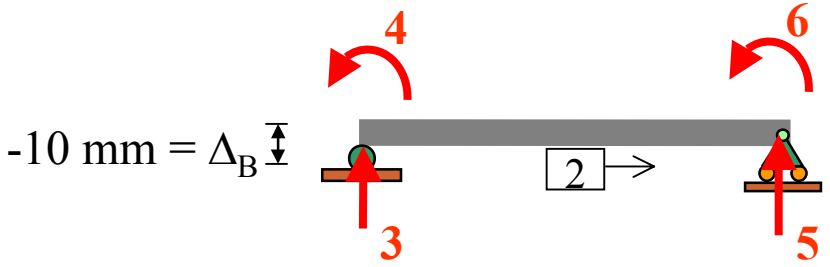


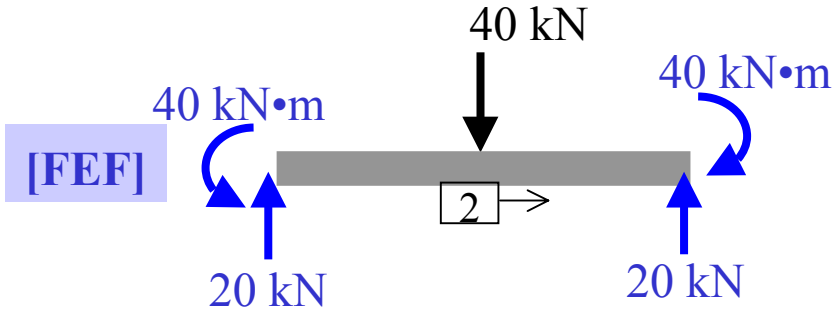
Member 1: $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \frac{(200\text{x}200)}{8} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 12(2)/8^2 & 1.5 & -0.375 & 1.5 \\ 1.5 & 8 & -1.5 & 4 \\ -0.375 & -1.5 & 0.375 & -1.5 \\ 1.5 & 4 & -1.5 & 8 \end{pmatrix} \end{array} \begin{pmatrix} d_1 = 0 \\ d_2 = 0 \\ d_3 = -0.01 \\ d_4 = -1.532\text{x}10^{-3} \end{pmatrix} + \begin{pmatrix} 24 \\ 32 \\ 24 \\ -32 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} 31.26 & \text{kN} \\ 76.37 & \text{kN•m} \\ 16.74 & \text{kN} \\ -18.27 & \text{kN•m} \end{pmatrix}$$



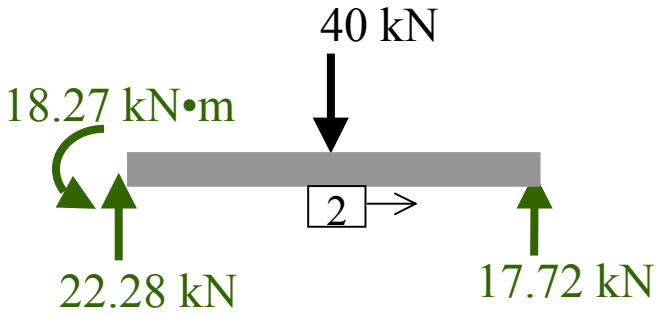


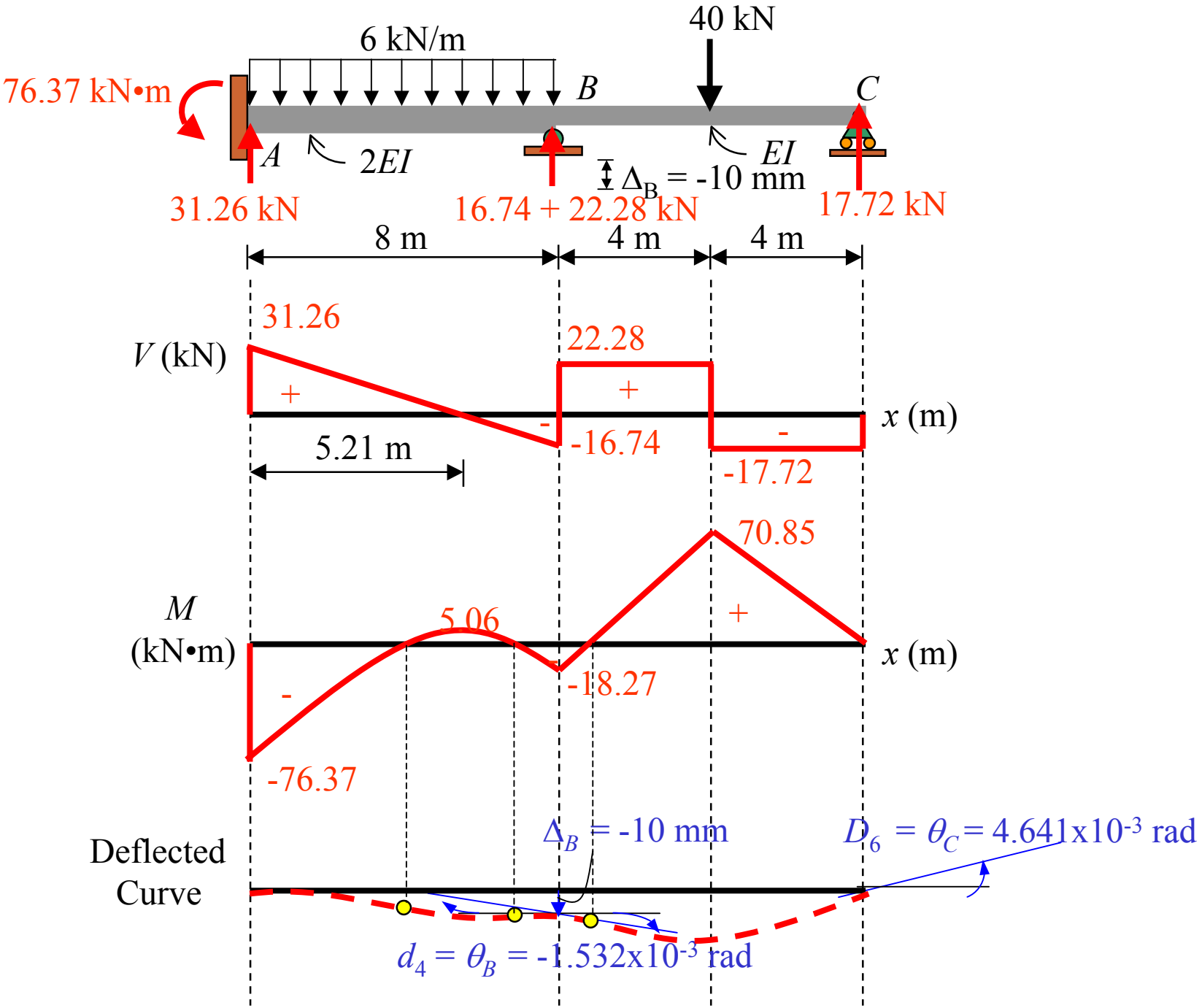


Member 2:

$$\begin{pmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = \frac{(200\text{x}200)}{8} \begin{matrix} \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{matrix} \begin{matrix} 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 12/8^2 & 0.75 & -0.1875 & 0.75 \\ 0.75 & 4 & -0.75 & 2 \\ -0.1875 & -0.75 & 0.1875 & -0.75 \\ 0.75 & 2 & -0.75 & 4 \end{pmatrix} \end{matrix} \begin{pmatrix} d_3 = -0.01 \\ d_4 = -1.532\text{x}10^{-3} \\ d_5 = 0 \\ d_6 = 4.641\text{x}10^{-3} \end{pmatrix} + \begin{pmatrix} 20 \\ 40 \\ 20 \\ -40 \end{pmatrix}$$

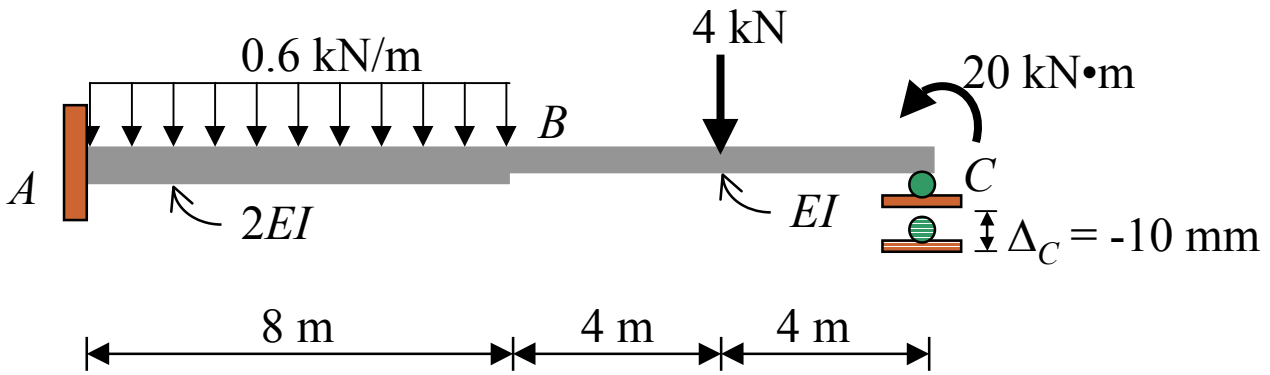
$$\begin{pmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = \begin{pmatrix} 22.28 & \text{kN} \\ 18.27 & \text{kN•m} \\ 17.72 & \text{kN} \\ 0 & \text{kN•m} \end{pmatrix}$$

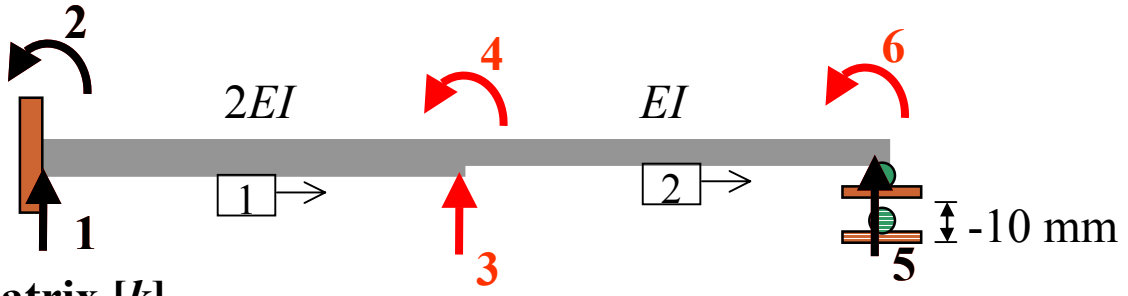



6 kN/m
40 kN
76.37 kN•m
B
C
A
2EI
EI
Δ_B = -10 mm
31.26 kN
16.74 + 22.28 kN
17.72 kN
8 m
4 m
4 m
V (kN)
31.26
22.28
+
+
-
-16.74
-
-17.72
x (m)
5.21 m
M
(kN•m)
70.85
5.06
+
-18.27
-
-76.37
x (m)
Deflected
Curve
Δ_B = -10 mm
D_6 = θ_C = 4.641x10^-3 rad
d_4 = θ_B = -1.532x10^-3 rad

Example 5

For the beam shown:
(a) Use the stiffness method to determine all the **reactions** at supports.
(b) Draw the **quantitative free-body diagram** of member.
(c) Draw the **quantitative shear diagram, bending moment diagram** and **qualitative deflected shape.**
Take $I = 200(10^6)$ mm^4 and $E = 200$ GPa and support C settlement 10 mm.





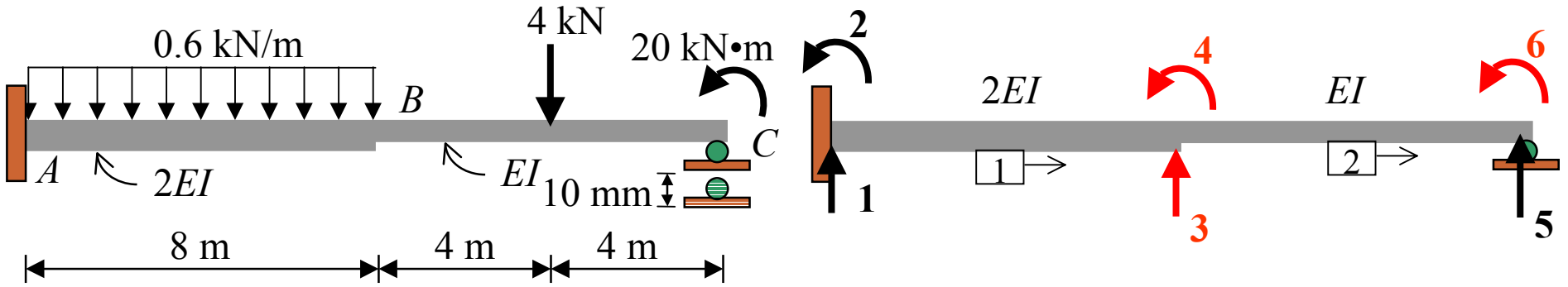
•**Member stiffness matrix $[k]_{4x4}$**

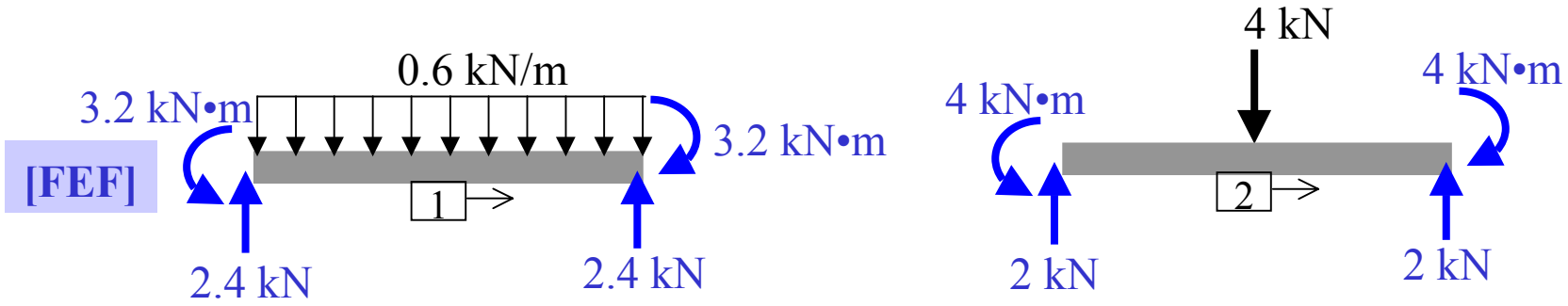
$$[k]_1 = \frac{EI}{8}\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 12(2)/8^2 & 1.5 & -0.375 & 1.5 \\ 1.5 & 8 & -1.5 & 4 \\ -0.375 & -1.5 & 0.375 & -1.5 \\ 1.5 & 4 & -1.5 & 8 \end{pmatrix} \end{array}$$

$$[k]_2 = \frac{EI}{8}\begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 12/8^2 & 0.75 & -0.1875 & 0.75 \\ 0.75 & 4 & -0.75 & 2 \\ -0.1875 & -0.75 & 0.1875 & -0.75 \\ 0.75 & 2 & -0.75 & 4 \end{pmatrix} \end{array}$$

• **Global: $[Q] = [K][D] + [Q^F]$**

$$\begin{pmatrix} Q_3 \\ Q_4 \\ Q_6 \end{pmatrix} = \frac{EI}{8}\begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccc} 3 & 4 & 6 \end{array} \\ \begin{pmatrix} 0.5625 & -0.75 & 0.75 \\ -0.75 & 12 & 2 \\ 0.75 & 2 & 4 \end{pmatrix} \end{array}\begin{pmatrix} D_3 \\ D_4 \\ D_6 \end{pmatrix} + \left(\frac{200\times 200}{8}\right)\begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array}\begin{array}{c} 5 \\ \begin{pmatrix} -0.1875 \\ -0.75 \\ -0.75 \end{pmatrix} \end{array}\left[D_5 = -0.01\right] + \begin{pmatrix} Q^F_3 \\ Q^F_4 \\ Q^F_6 \end{pmatrix}$$

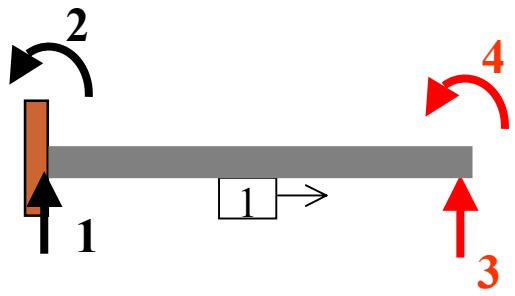


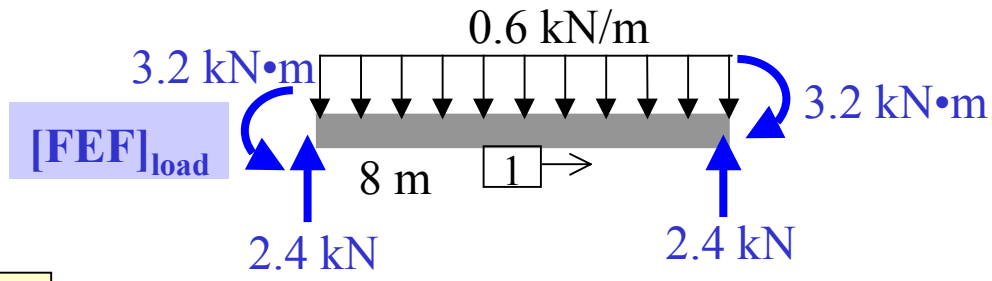


Global: $[Q] = [K][D] + [Q^F]$

$$\begin{pmatrix} Q_3 = 0 \\ Q_4 = 0 \\ Q_6 = 20 \end{pmatrix} = \frac{EI}{8} \begin{array}{c} \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 4 & 6 \end{array} \\ \begin{pmatrix} 0.5625 & -0.75 & 0.75 \\ -0.75 & 12 & 2 \\ 0.75 & 2 & 4 \end{pmatrix} \end{array} \begin{pmatrix} D_3 \\ D_4 \\ D_6 \end{pmatrix} + \begin{pmatrix} 9.375 \\ 37.5 \\ 37.5 \end{pmatrix} + \begin{pmatrix} 2.4+2 = 4.4 \\ -3.2+4 = 0.8 \\ -4.0 \end{pmatrix}$$

$$\begin{pmatrix} D_3 \\ D_4 \\ D_6 \end{pmatrix} = \begin{pmatrix} -377.30/EI = -9.433\text{x}10^{-3}\ \text{m} \\ -61.53/EI = -1.538\text{x}10^{-3}\ \text{rad} \\ +74.50/EI = +1.863\text{x}10^{-3}\ \text{rad} \end{pmatrix}$$

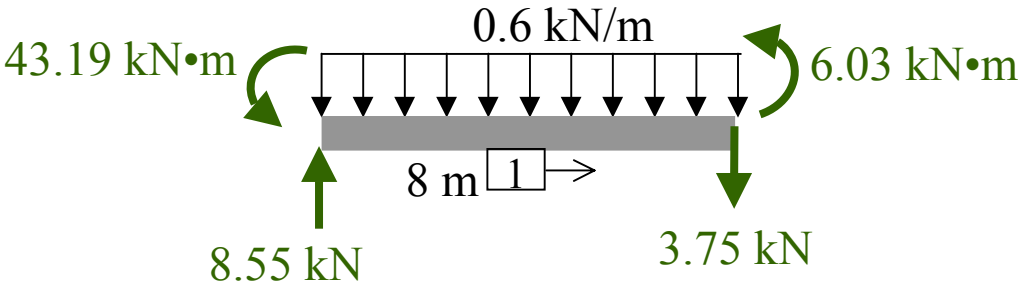


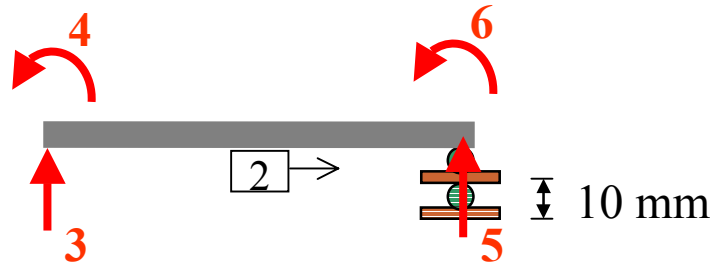


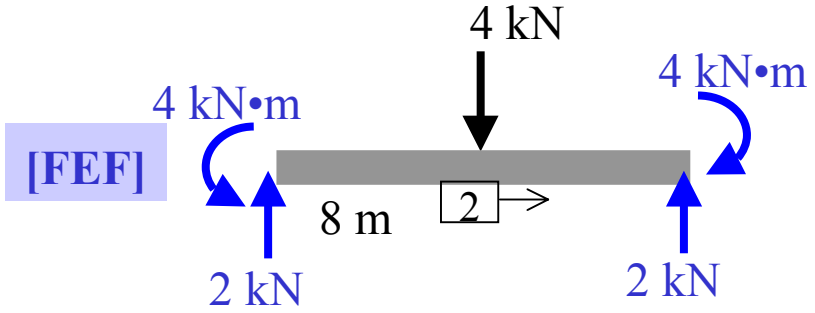
Member 1: $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \frac{200\times200}{8} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 12(2)/8^2 & 1.5 & -0.375 & 1.5 \\ 1.5 & 8 & -1.5 & 4 \\ -0.375 & -1.5 & 0.375 & -1.5 \\ 1.5 & 4 & -1.5 & 8 \end{pmatrix} \end{array} \begin{pmatrix} d_1 = 0 \\ d_2 = 0 \\ d_3 = -9.433\text{x}10^{-3} \\ d_4 = -1.538\text{x}10^{-3} \end{pmatrix} + \begin{pmatrix} 2.4 \\ 3.2 \\ 2.4 \\ -3.2 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} 8.55 & \text{kN} \\ 43.19 & \text{kN•m} \\ -3.75 & \text{kN} \\ 6.03 & \text{kN•m} \end{pmatrix}$$



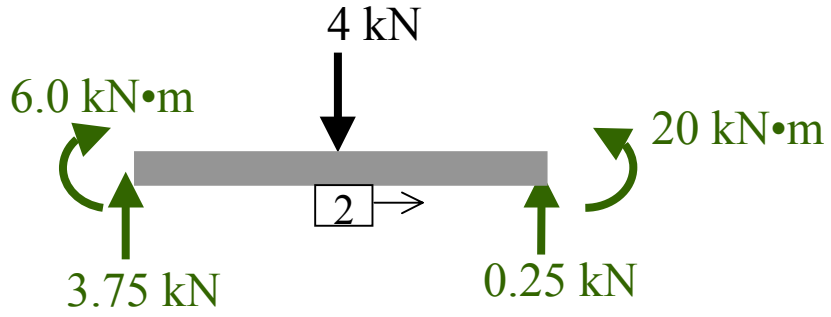


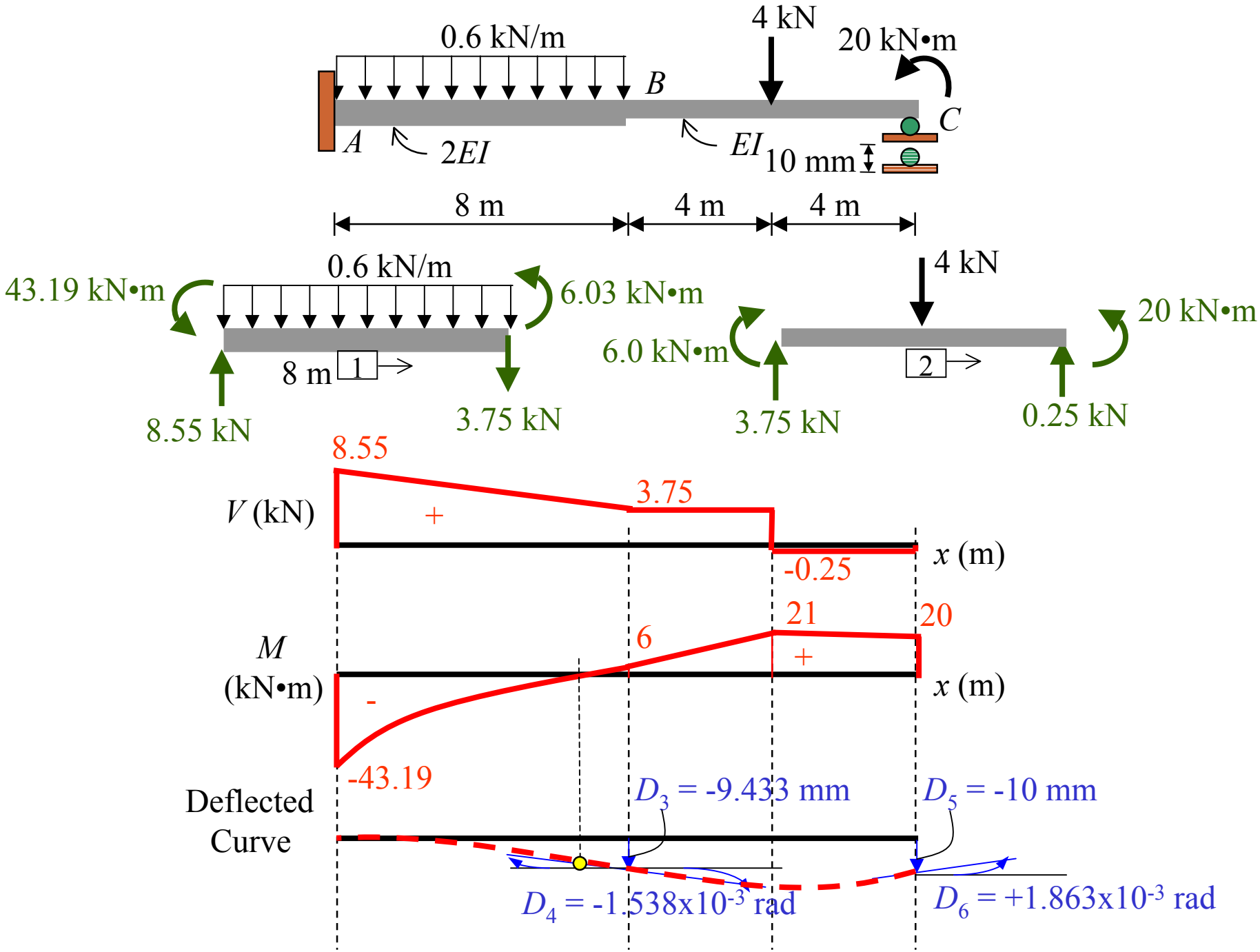


Member 2:

$$\begin{pmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = \frac{200\times 200}{8} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 12/8^2 & 0.75 & -0.1875 & 0.75 \\ 0.75 & 4 & -0.75 & 2 \\ -0.1875 & -0.75 & 0.1875 & -0.75 \\ 0.75 & 2 & -0.75 & 4 \end{pmatrix} \end{array} \begin{pmatrix} d_3 = -9.433\text{x}10^{-3} \\ d_4 = -1.538\text{x}10^{-3} \\ d_5 = -0.01 \\ d_6 = 1.863\text{x}10^{-3} \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 2 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = \begin{pmatrix} 3.75 & \text{kN} \\ -6.0 & \text{kN•m} \\ 0.25 & \text{kN} \\ 20.0 & \text{kN•m} \end{pmatrix}$$

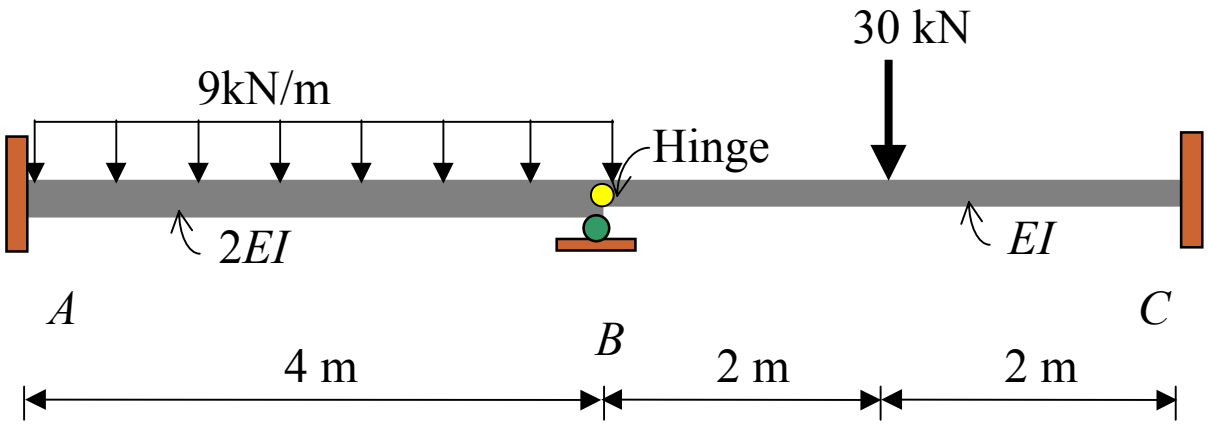


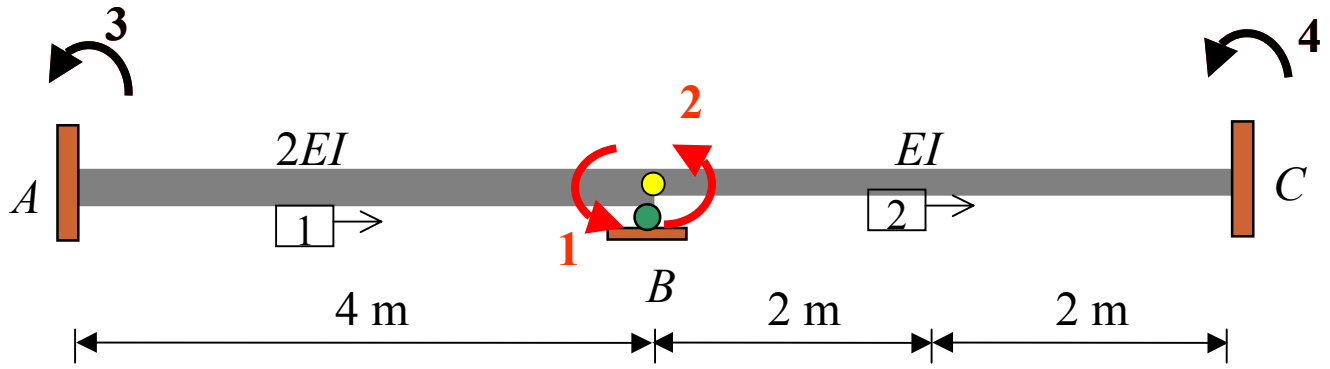

0.6 kN/m
4 kN
20 kN•m
B
C
A
2EI
EI
10 mm
8 m
4 m
4 m
0.6 kN/m
43.19 kN•m
6.03 kN•m
8 m
1
8.55 kN
3.75 kN
4 kN
20 kN•m
6.0 kN•m
2
3.75 kN
0.25 kN
8.55
3.75
V (kN)
+
x (m)
-0.25
21
20
6
M
(kN•m)
+
x (m)
-
-43.19
Deflected
Curve
D3 = -9.433 mm
D5 = -10 mm
D4 = -1.538x10-3 rad
D6 = +1.863x10-3 rad

Internal Hinges

Example 6

For the beam shown, use the stiffness method to:
(a) Determine all the reactions at supports.
(b) Draw the **quantitative shear** and **bending moment diagrams** and **qualitative deflected shape.**
E = 200 GPa, $I = 50\text{x}10^{-6}$ m^4.



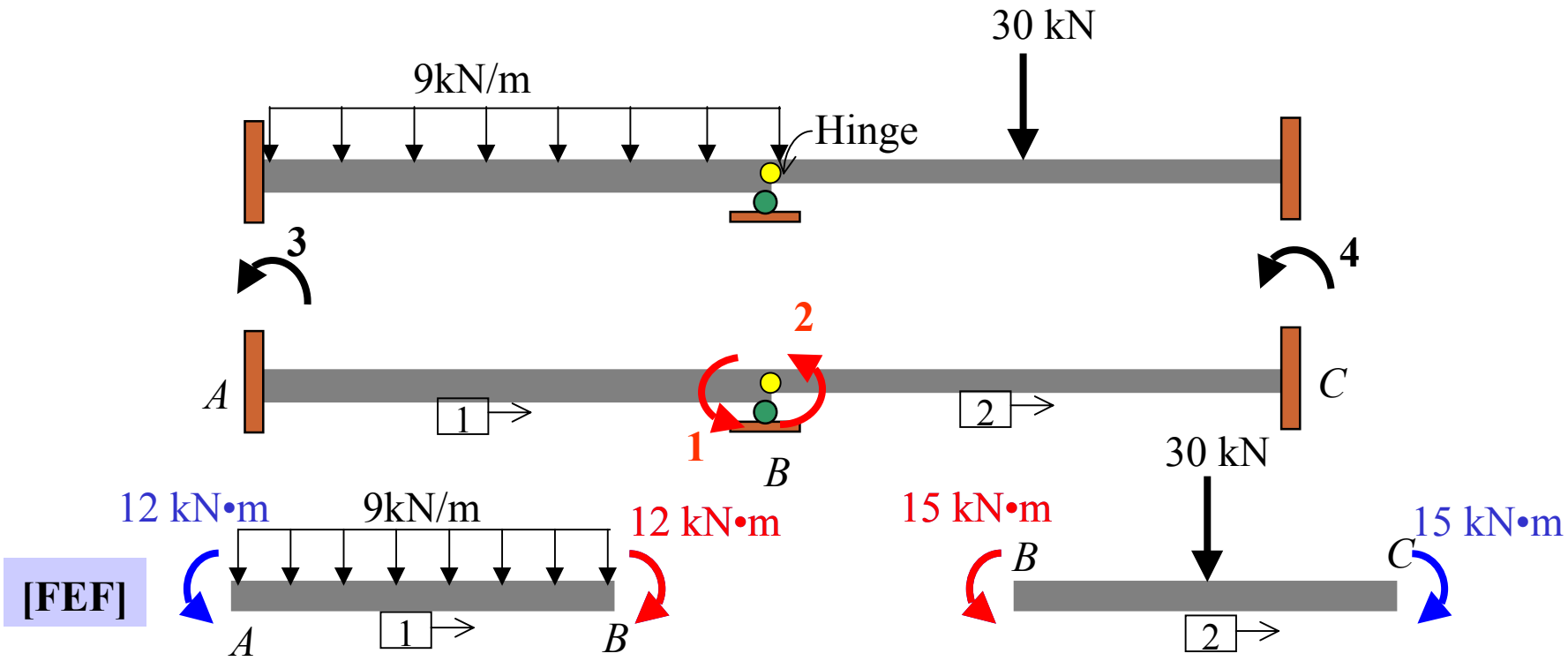


Use 2x2 stiffness matrix,

$$[k]_{2\times 2} = \begin{matrix} \\ M_i \\ M_j \end{matrix} \begin{matrix} \begin{matrix} \theta_i & \theta_j \end{matrix} \\ \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix} \end{matrix}$$

$$[k]_1 = \begin{matrix} \\ 3 \\ 1 \end{matrix} \begin{matrix} \begin{matrix} 3 & 1 \end{matrix} \\ \begin{bmatrix} 2EI & EI \\ EI & 2EI \end{bmatrix} \end{matrix} \qquad [k]_2 = \begin{matrix} \\ 2 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 2 & 4 \end{matrix} \\ \begin{bmatrix} 1EI & 0.5EI \\ 0.5EI & 1EI \end{bmatrix} \end{matrix}$$

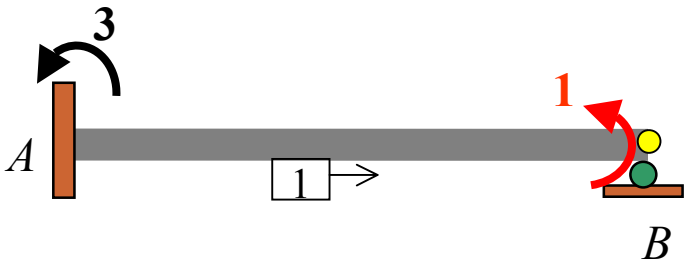
$$[\boldsymbol{K}] = EI \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \end{matrix}$$



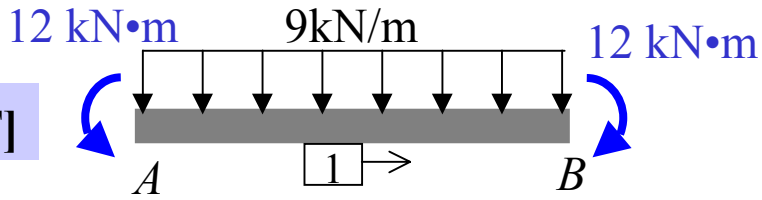
Global matrix:

$$\begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix} = EI \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 2.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix} \end{array} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} + \begin{pmatrix} -12 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} 0.0006 & \text{rad} \\ -0.0015 & \text{rad} \end{pmatrix}$$

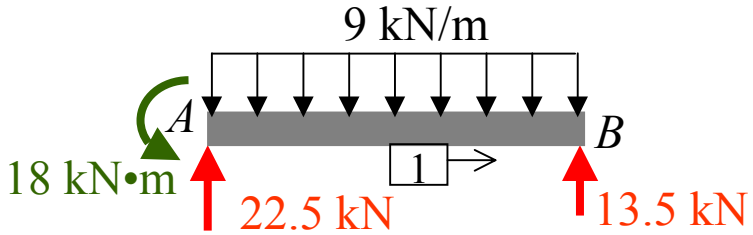


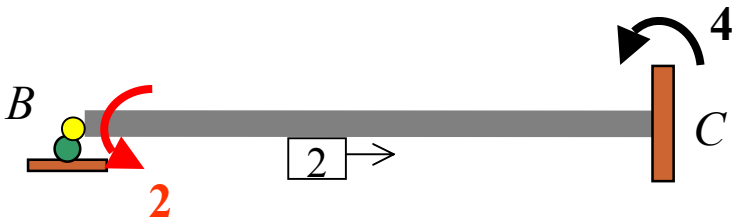
[FEF]

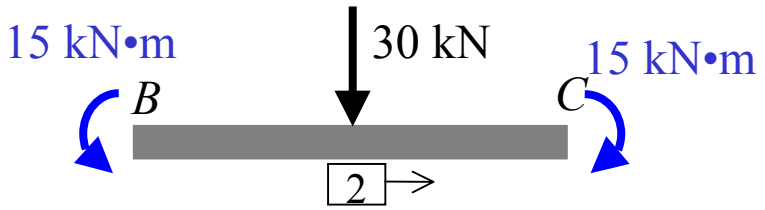


Member 1:

$$\begin{bmatrix} q_3 \\ q_1 \end{bmatrix} = \begin{matrix} 3 \\ 1 \end{matrix} \begin{bmatrix} 2EI & EI \\ EI & 2EI \end{bmatrix} \begin{bmatrix} d_3 = 0.0 \\ d_1 = 0.0006 \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \end{bmatrix} = \begin{bmatrix} 18 \\ 0.0 \end{bmatrix}$$

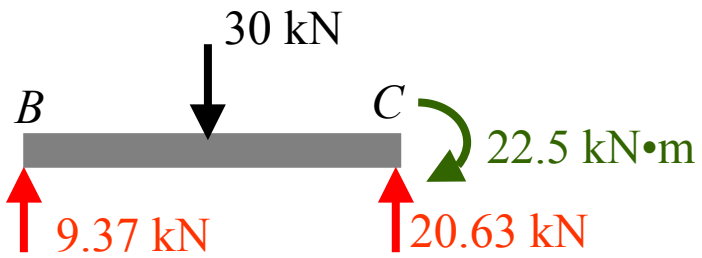


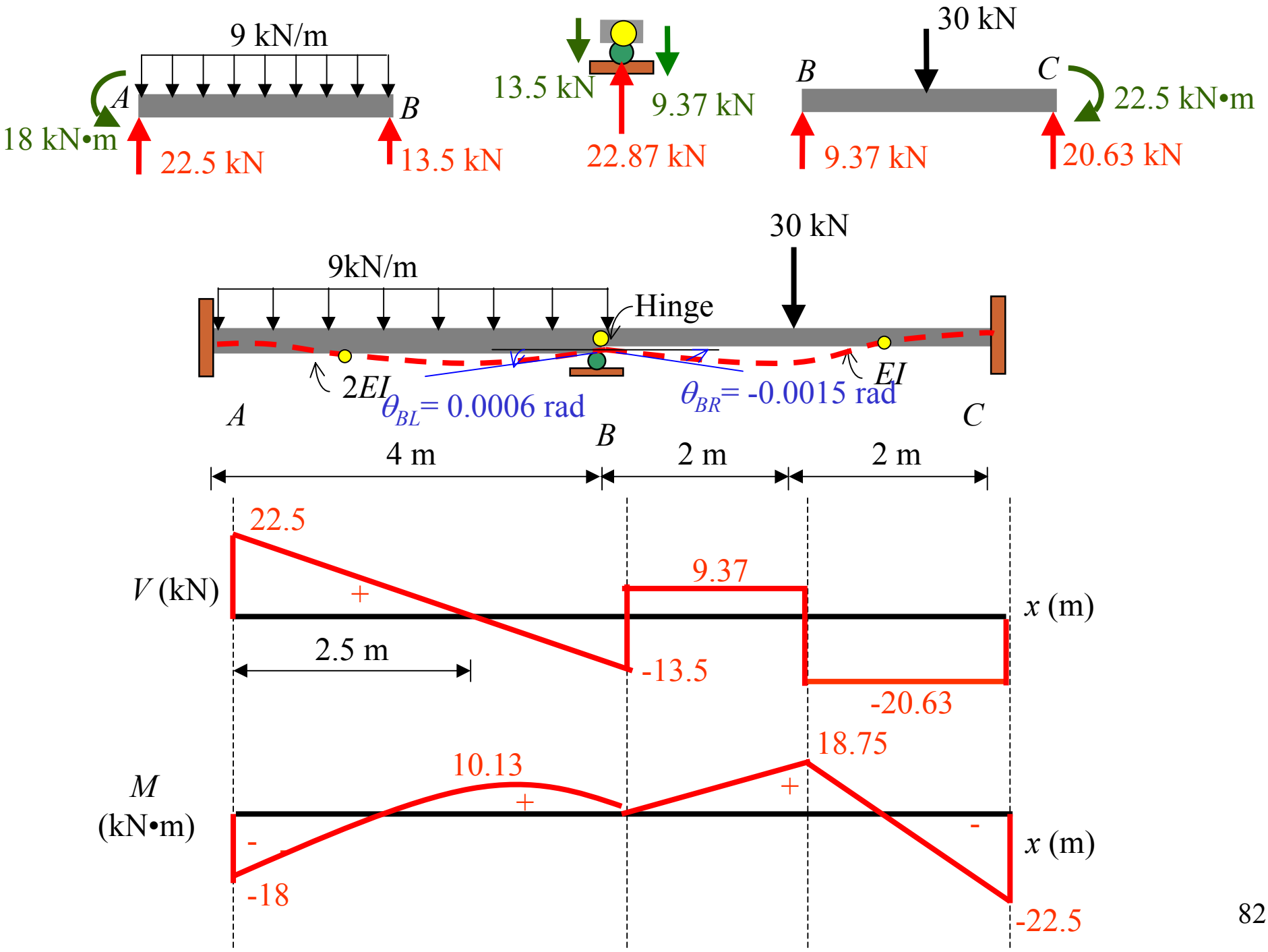




Member 2:

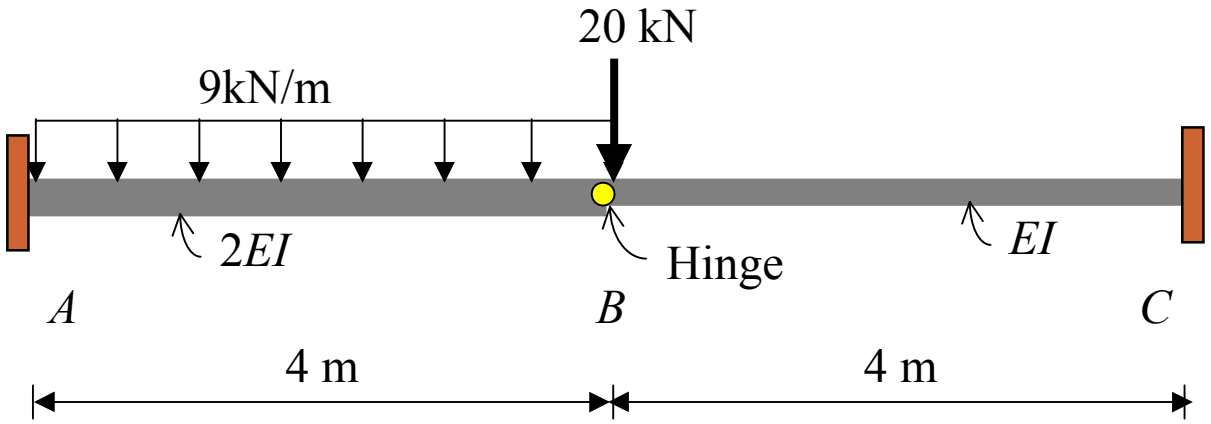
$$\begin{matrix} & & 2 & 4 \\ \begin{bmatrix} q_2 \\ q_4 \end{bmatrix} = & \begin{matrix} 2 \\ 4 \end{matrix} & \begin{bmatrix} 1EI & 0.5EI \\ 0.5EI & 1EI \end{bmatrix} \end{matrix} \begin{bmatrix} d_2 = -0.0015 \\ d_4 = 0.0 \end{bmatrix} + \begin{bmatrix} 15 \\ -15 \end{bmatrix} = \begin{bmatrix} 0.0 \\ -22.5 \end{bmatrix}$$

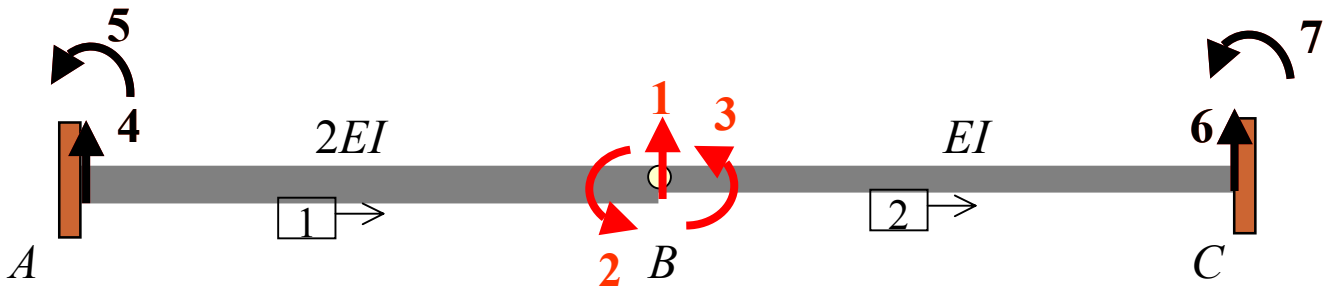



9 kN/m
A
B
18 kN•m
22.5 kN
13.5 kN
13.5 kN
9.37 kN
22.87 kN
30 kN
B
C
22.5 kN•m
9.37 kN
20.63 kN
30 kN
9kN/m
Hinge
2EI
EI
θBL= 0.0006 rad
θBR= -0.0015 rad
A
B
C
4 m
2 m
2 m
22.5
V (kN)
+
9.37
x (m)
2.5 m
-13.5
-20.63
18.75
10.13
M
(kN•m)
+
+
-
-
x (m)
-18
-22.5

Example 7

For the beam shown, use the stiffness method to:
(a) Determine all the reactions at supports.
(b) Draw the **quantitative shear** and **bending moment diagrams** and **qualitative deflected shape.**
$E = 200$ GPa, $I = 50 \times 10^{-6}$ m^4.

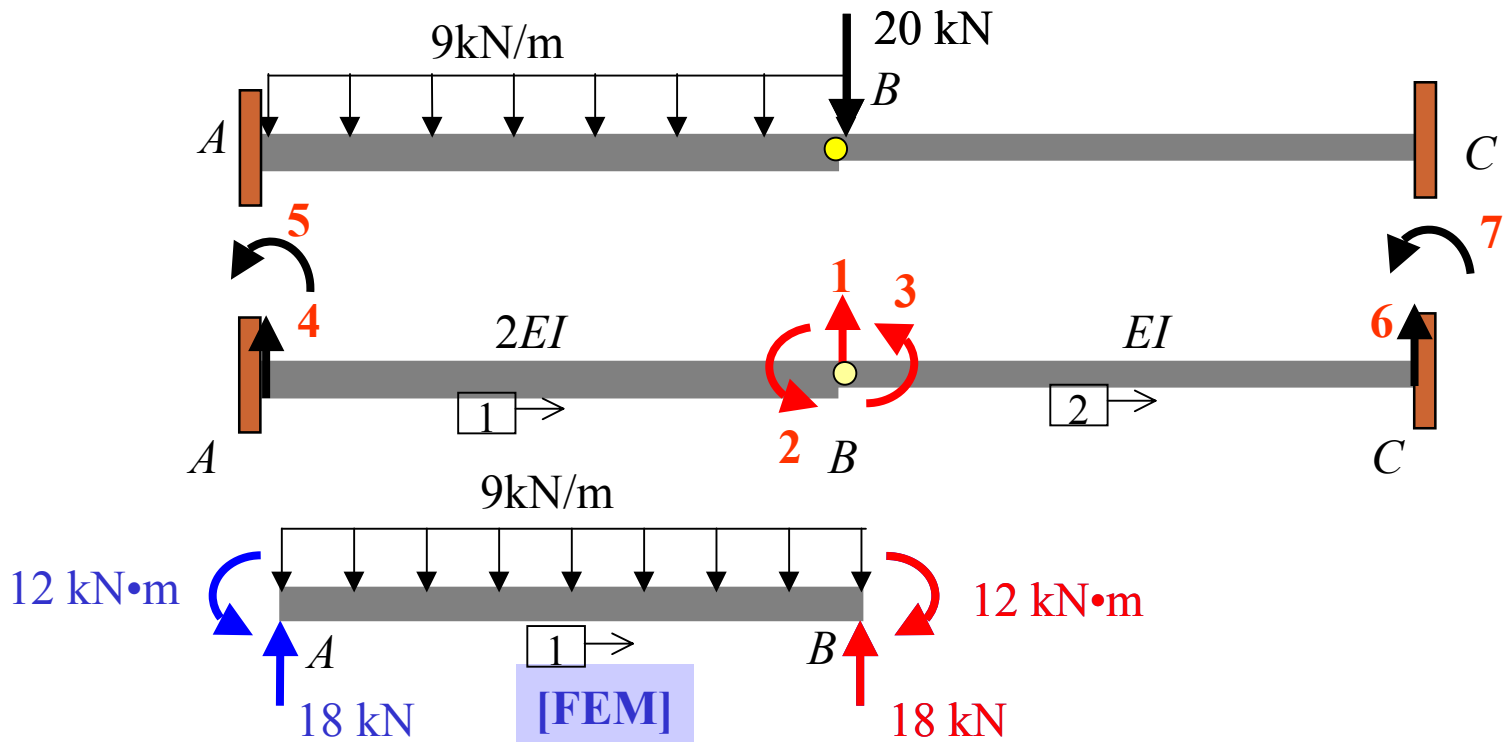




$$[k]_1 = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI/L \end{bmatrix} \end{array}$$

$$[K] = EI \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5625 & -0.75 & 0.375 \\ -0.75 & 2.0 & 0 \\ 0.375 & 0.0 & 1.0 \end{bmatrix} \end{array}$$

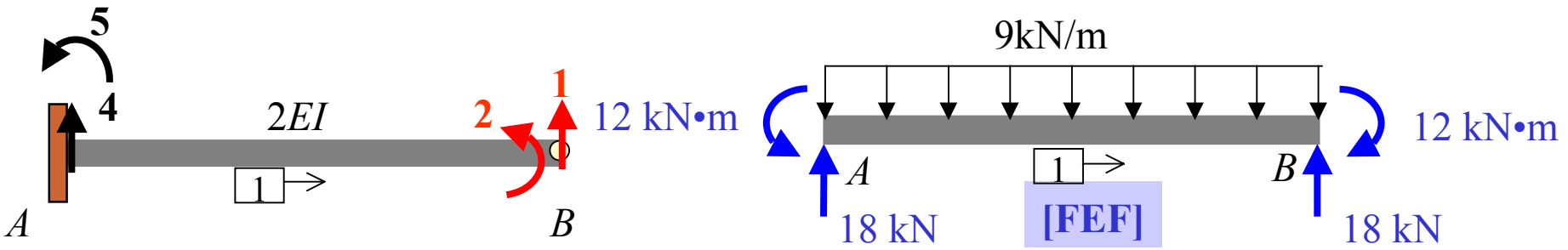
$$[k]_2 = \begin{array}{c} \\ 1 \\ 3 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 6 & 7 \end{array} \\ \begin{bmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{bmatrix} \end{array}$$



Global:

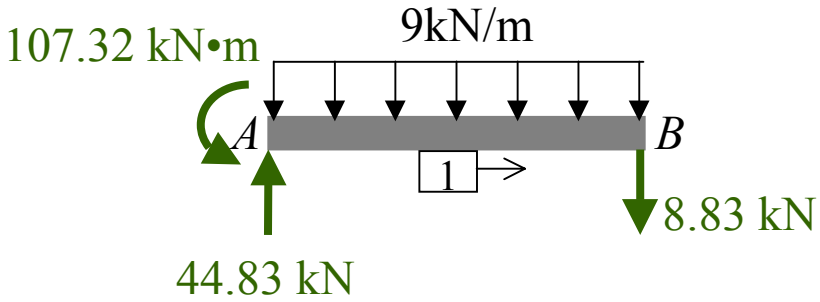
$$\begin{pmatrix} Q_1 = -20 \\ Q_2 = 0.0 \\ Q_3 = 0.0 \end{pmatrix} = EI \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5625 & -0.75 & 0.375 \\ -0.75 & 2.0 & 0 \\ 0.375 & 0.0 & 1.0 \end{bmatrix} \end{array} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} + \begin{pmatrix} 18 \\ -12 \\ 0.0 \end{pmatrix}$$

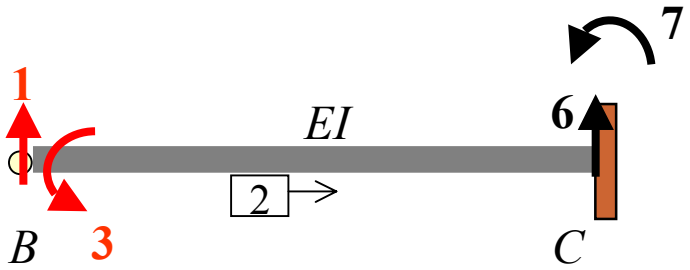
$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} -0.02382 \text{ m} \\ -0.008333 \text{ rad} \\ 0.008933 \text{ rad} \end{pmatrix}$$



Member 1:

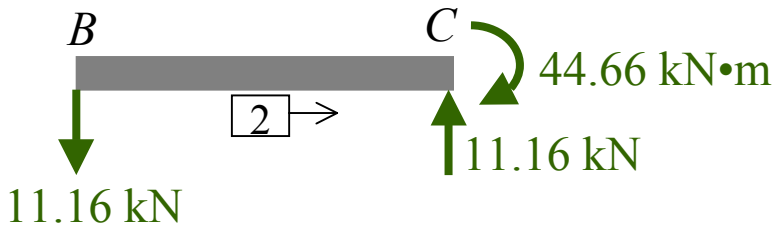
$$\begin{pmatrix} q_4 \\ q_5 \\ q_1 \\ q_2 \end{pmatrix} = \begin{matrix} 4 \\ 5 \\ 1 \\ 2 \end{matrix} \begin{matrix} \begin{matrix} 4 & 5 & 1 & 2 \end{matrix} \\ \begin{pmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI/L \end{pmatrix} \end{matrix} \begin{pmatrix} d_4 = 0.0 \\ d_5 = 0.0 \\ d_1 = -0.02382 \\ d_2 = -0.00833 \end{pmatrix} + \begin{pmatrix} 18 \\ 12 \\ 18 \\ -12 \end{pmatrix} = \begin{pmatrix} 44.83 \\ 107.32 \\ -8.83 \\ 0.0 \end{pmatrix}$$

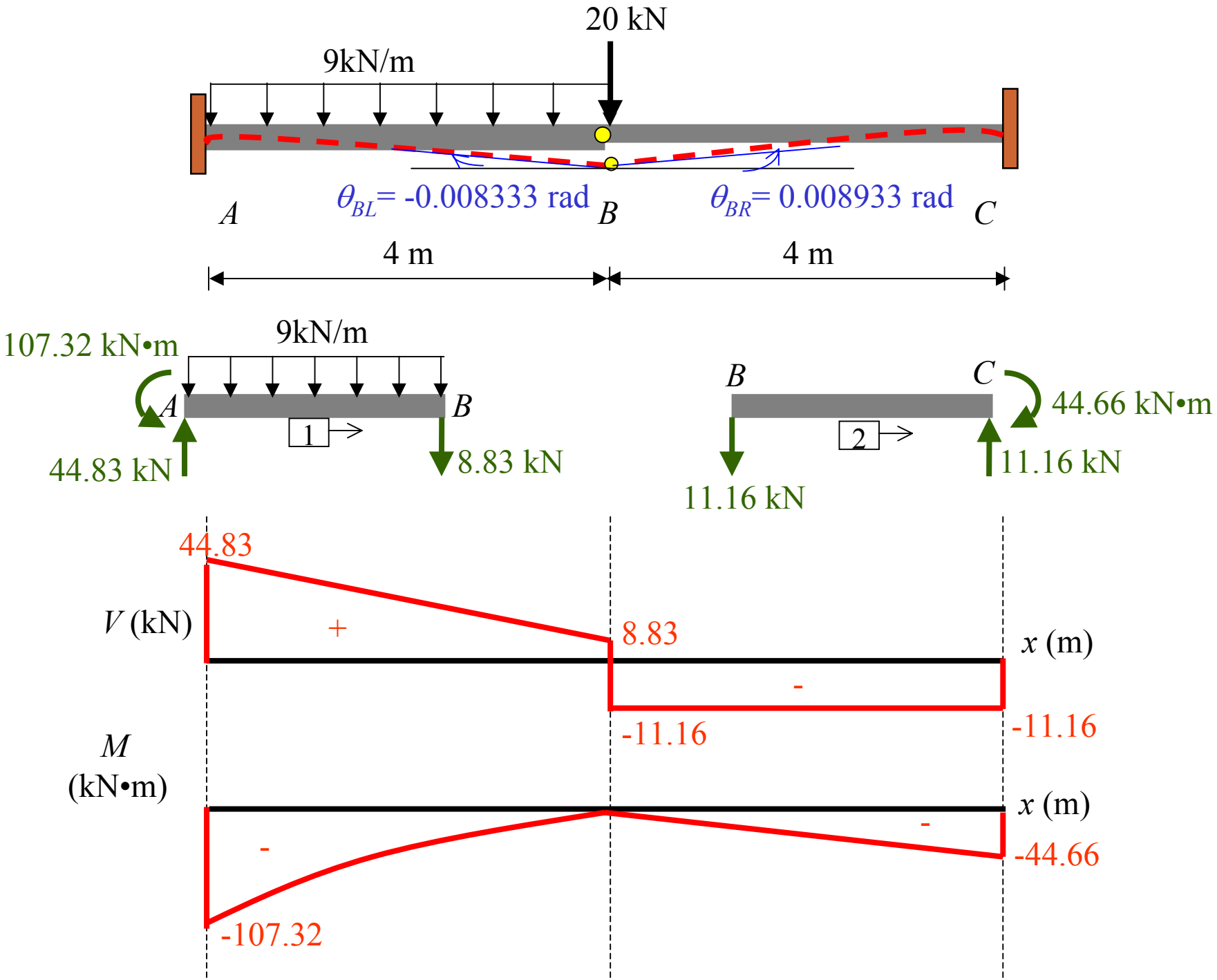




Member 2:

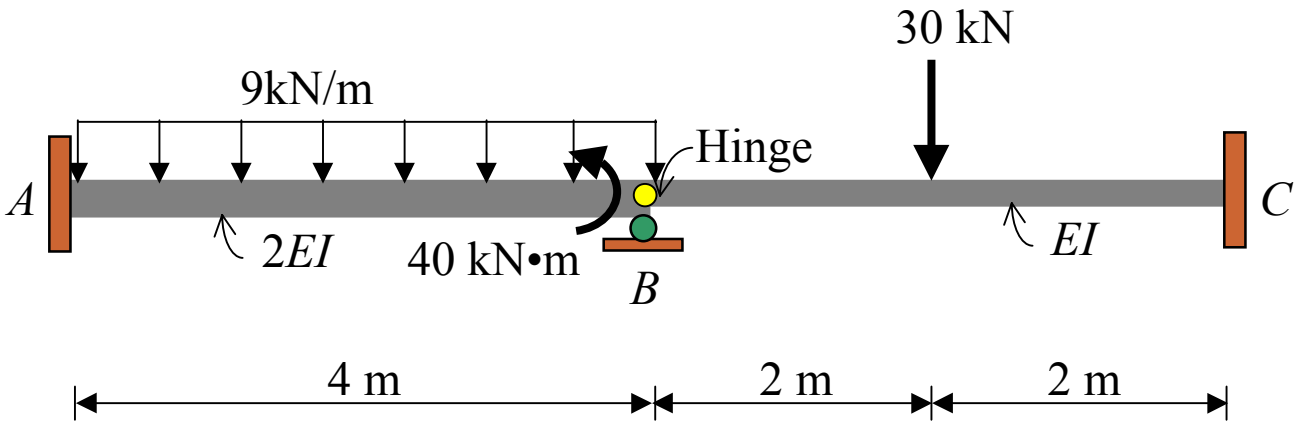
$$\begin{pmatrix} q_1 \\ q_3 \\ q_6 \\ q_7 \end{pmatrix} = \begin{matrix} 1 \\ 3 \\ 6 \\ 7 \end{matrix} \overset{\begin{matrix} 1 & 3 & 6 & 7 \end{matrix}}{\begin{pmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{pmatrix}} \begin{pmatrix} d_1 = -0.02382 \\ d_3 = 0.008933 \\ d_6 = 0.0 \\ d_7 = 0.0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -11.16 \\ 0.0 \\ 11.16 \\ -44.66 \end{pmatrix}$$

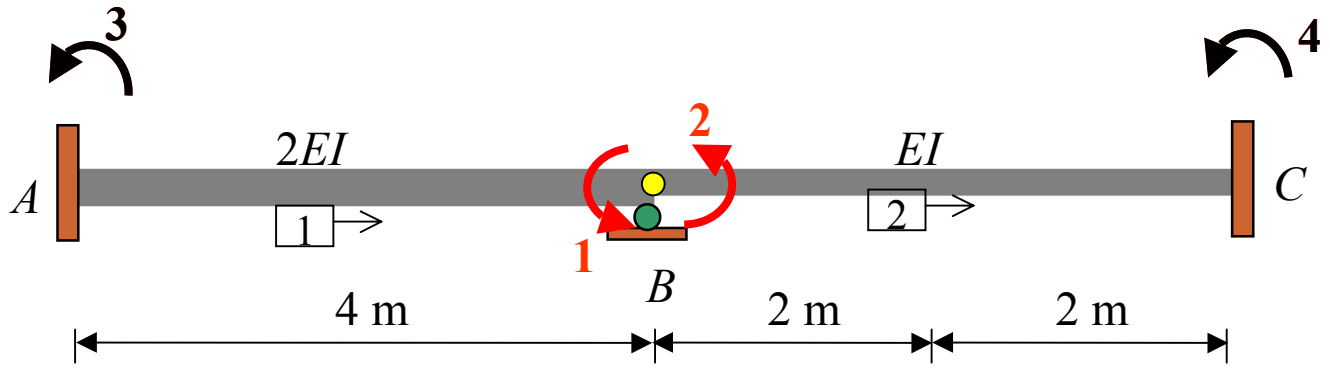



20 kN
9kN/m
θ_{BL}= -0.008333 rad
θ_{BR}= 0.008933 rad
A
B
C
4 m
4 m
107.32 kN•m
9kN/m
A
B
1
44.83 kN
8.83 kN
B
C
44.66 kN•m
2
11.16 kN
11.16 kN
44.83
V (kN)
+
8.83
x (m)
-
-11.16
-11.16
M
(kN•m)
x (m)
-
-
-44.66
-107.32

Example 8

For the beam shown, use the stiffness method to:
(a) Determine all the reactions at supports.
(b) Draw the **quantitative shear** and **bending moment diagrams** and **qualitative deflected shape.**
40 kN•m at the end of member *AB*. $E = 200$ GPa, $I = 50\text{x}10^{-6}$ m^4.





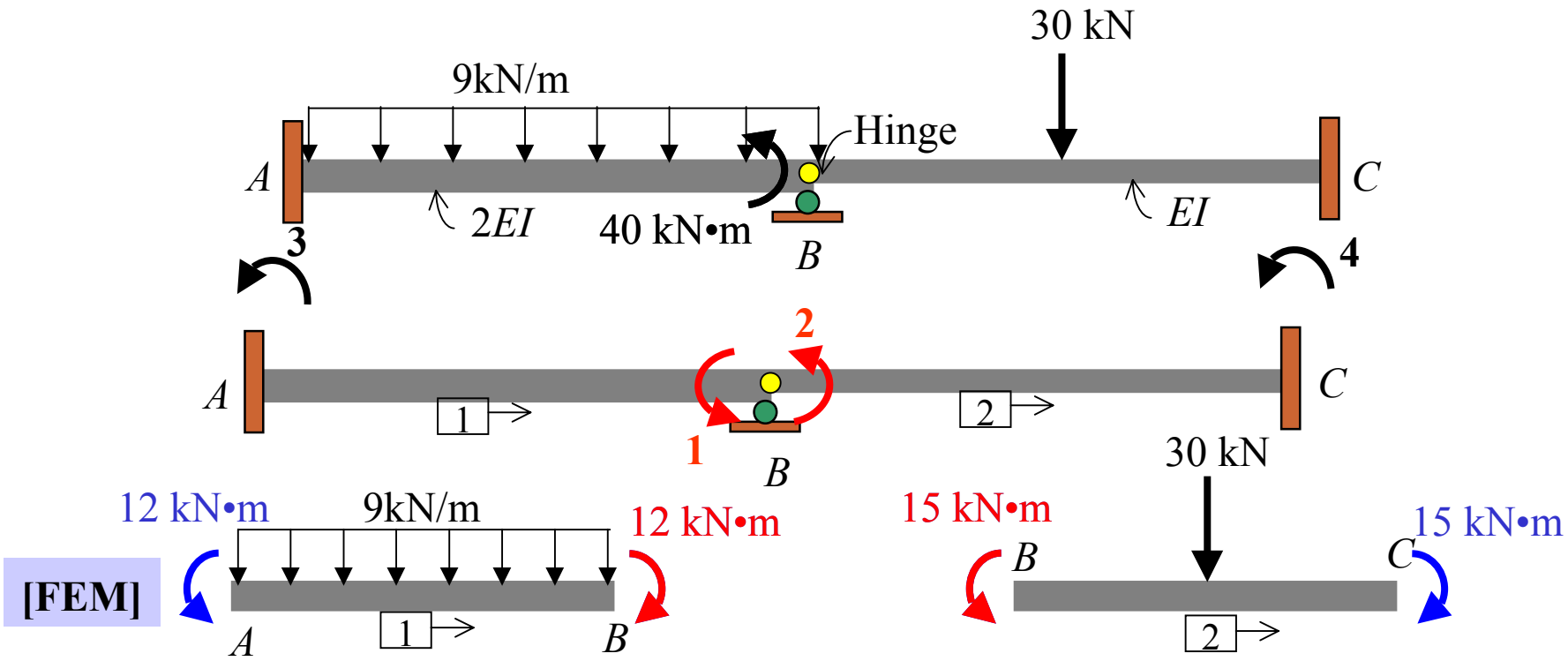
Use 2x2 stiffness matrix:

$$[k]_{2\times 2} = \begin{matrix} \\ M_i \\ M_j \end{matrix} \begin{matrix} \theta_i & \theta_j \\ \end{matrix} \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix}$$

$$[k]_1 = \begin{matrix} \\ 3 \\ 1 \end{matrix} \begin{matrix} 3 & 1 \\ \end{matrix} \begin{bmatrix} 2EI & EI \\ EI & 2EI \end{bmatrix}$$

$$[k]_2 = \begin{matrix} \\ 2 \\ 4 \end{matrix} \begin{matrix} 2 & 4 \\ \end{matrix} \begin{bmatrix} 1EI & 0.5EI \\ 0.5EI & 1EI \end{bmatrix}$$

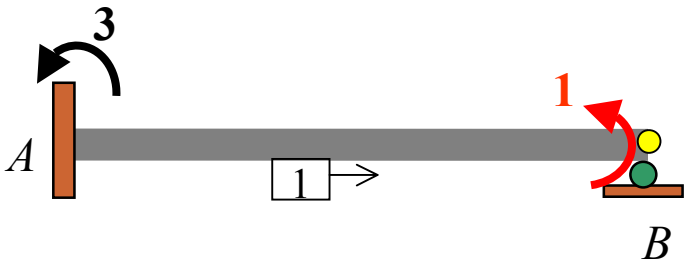
$$[\boldsymbol{K}] = EI \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} 1 & 2 \\ \end{matrix} \begin{bmatrix} 2.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$



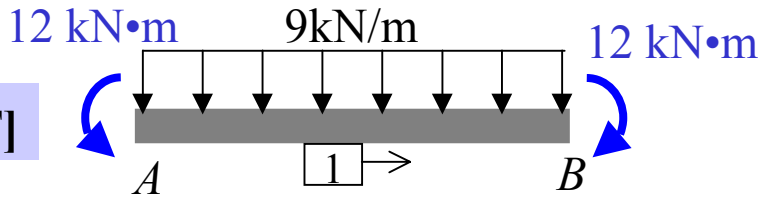
Global matrix:

$$\begin{pmatrix} Q_1 = 40 \\ Q_2 = 0.0 \end{pmatrix} = EI \begin{matrix} 1 \\ 2 \end{matrix} \begin{matrix} \overset{1}{\begin{pmatrix} 2.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}} \end{matrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} + \begin{pmatrix} -12 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \begin{pmatrix} 0.0026 & \text{rad} \\ -0.0015 & \text{rad} \end{pmatrix}$$

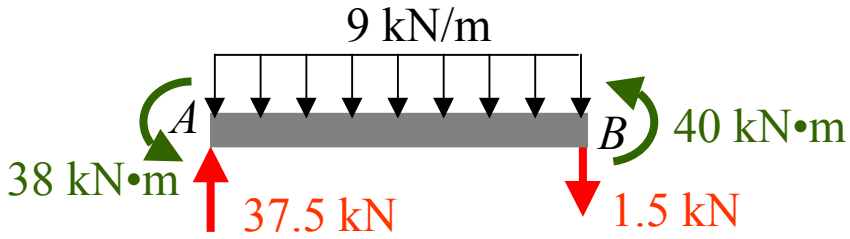


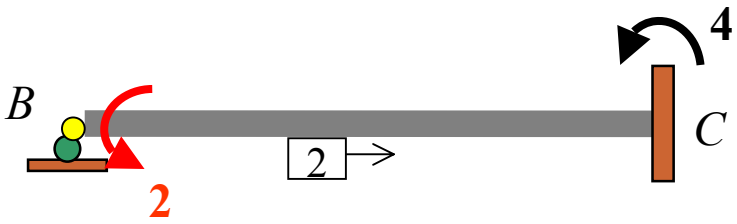
[FEF]

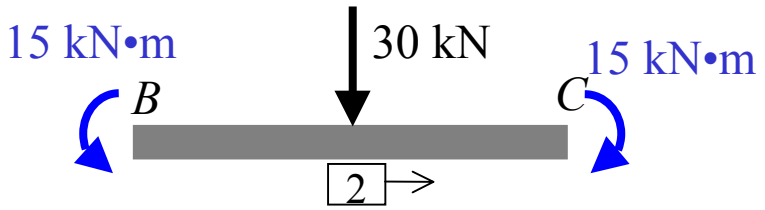


Member 1:

$$\begin{bmatrix} q_3 \\ q_1 \end{bmatrix} = \begin{matrix} 3 \\ 1 \end{matrix} \begin{bmatrix} 2EI & EI \\ EI & 2EI \end{bmatrix} \begin{bmatrix} d_3 = 0.0 \\ d_1 = 0.0026 \end{bmatrix} + \begin{bmatrix} 12 \\ -12 \end{bmatrix} = \begin{bmatrix} 38 \\ 40 \end{bmatrix}$$

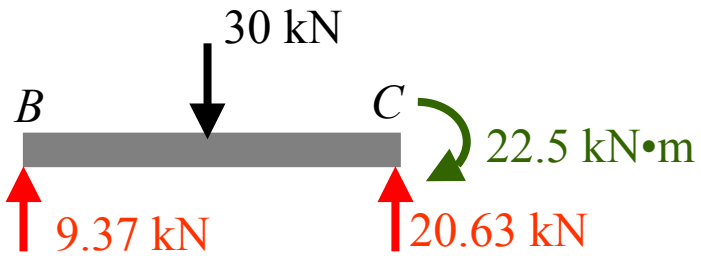


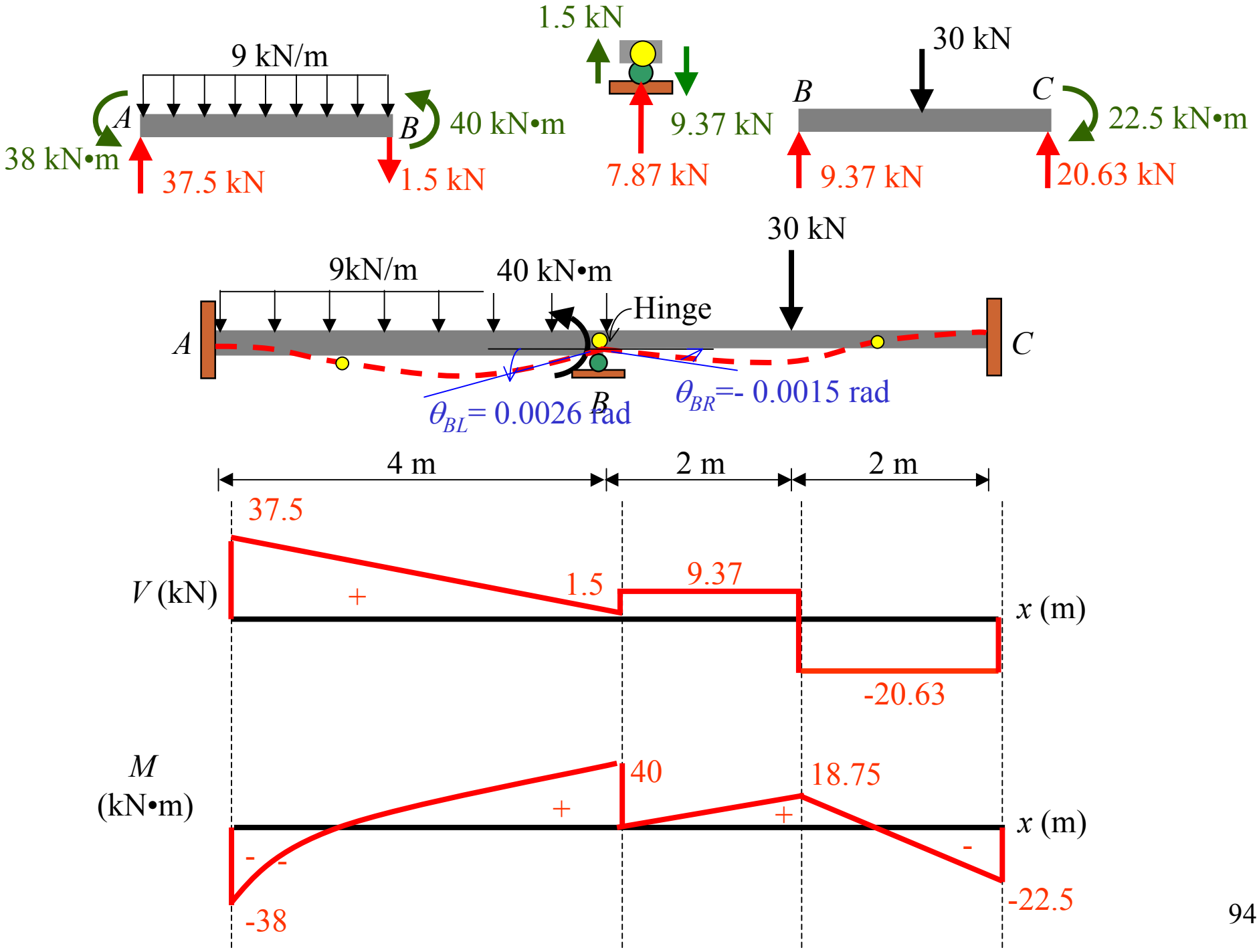




Member 2:

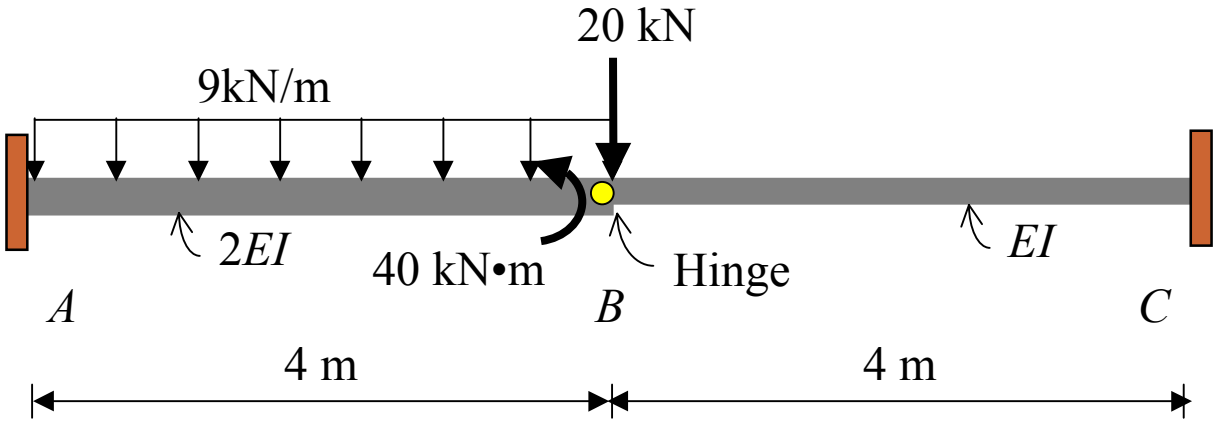
$$\begin{bmatrix} q_2 \\ q_4 \end{bmatrix} = \begin{matrix} 2 \\ 4 \end{matrix} \begin{bmatrix} 1EI & 0.5EI \\ 0.5EI & 1EI \end{bmatrix} \begin{matrix} 2 & 4 \end{matrix} \begin{bmatrix} d_2 = -0.0015 \\ d_4 = 0.0 \end{bmatrix} + \begin{bmatrix} 15 \\ -15 \end{bmatrix} = \begin{bmatrix} 0.0 \\ -22.5 \end{bmatrix}$$

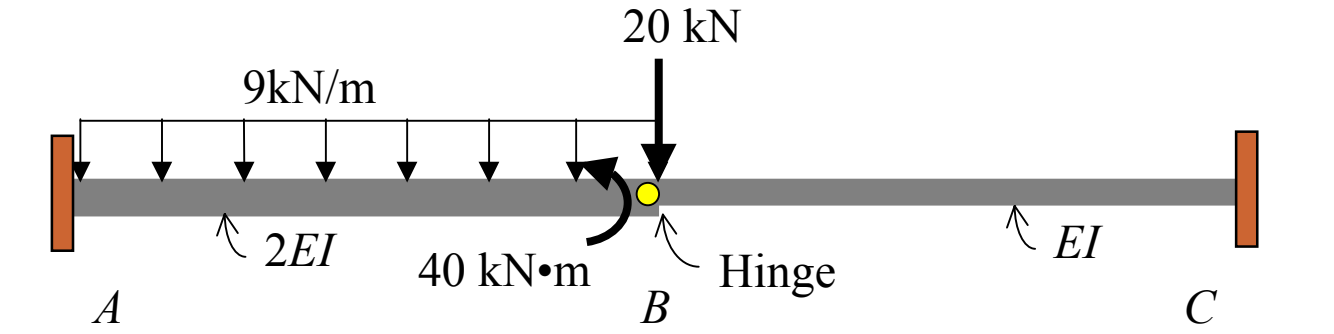


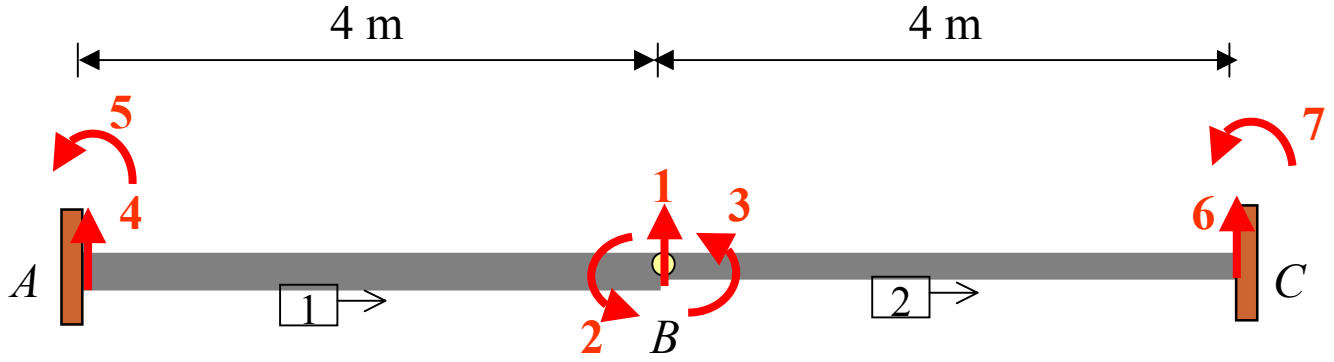

9 kN/m
A
B
38 kN•m
37.5 kN
1.5 kN
40 kN•m
1.5 kN
9.37 kN
7.87 kN
30 kN
B
C
9.37 kN
20.63 kN
22.5 kN•m
9kN/m
40 kN•m
Hinge
30 kN
A
B
C
θBL= 0.0026 rad
θBR=- 0.0015 rad
4 m
2 m
2 m
37.5
V (kN)
+
1.5
9.37
x (m)
-20.63
M
(kN•m)
40
18.75
+
+
x (m)
- -
-
-38
-22.5

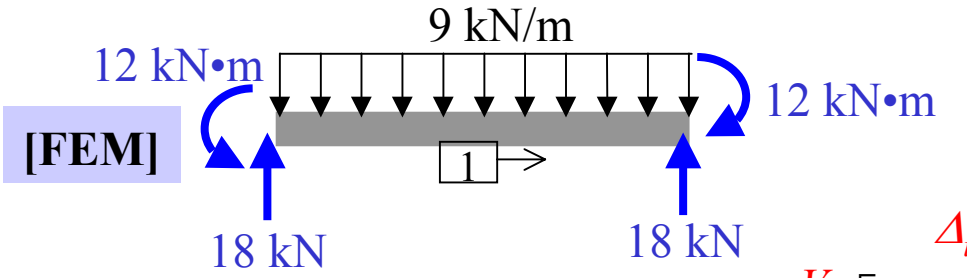
Example 9

For the beam shown, use the stiffness method to:
(a) Determine all the reactions at supports.
(b) Draw the **quantitative shear** and **bending moment diagrams** and **qualitative deflected shape.**
40 kN•m at the end of member *AB*. $E = 200$ GPa, $I = 50\text{x}10^{-6}$ m^4

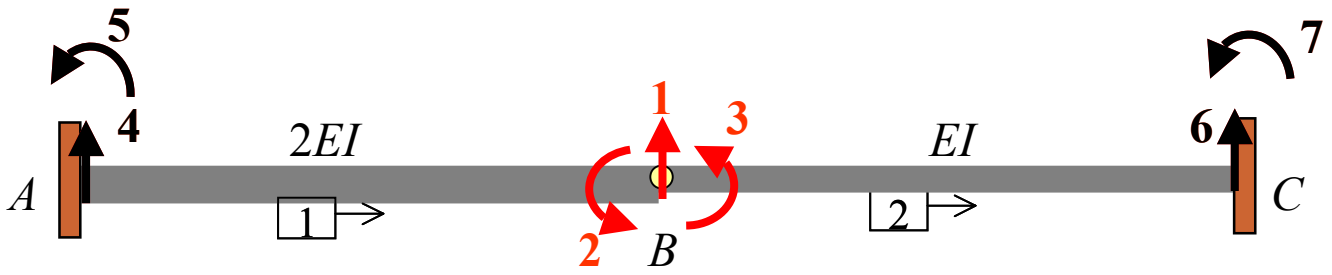








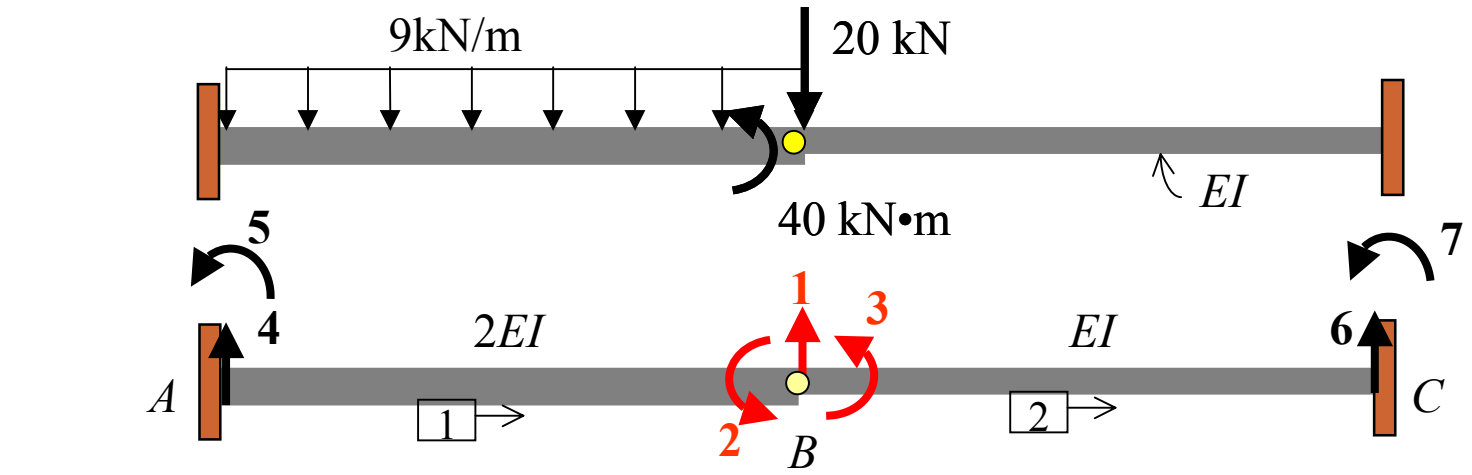
$$[k]_{4\times4} = \begin{array}{c} \\ V_i \\ M_i \\ V_j \\ M_j \end{array} \begin{array}{c} \begin{array}{cccc} \Delta_i & \theta_i & \Delta_j & \theta_j \end{array} \\ \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & -12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L & -6EI/L^2 & 2EI/L \\ -12EI/L^3 & -6EI/L^2 & 12EI/L^3 & -6EI/L^2 \\ 6EI/L^2 & 2EI/L & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

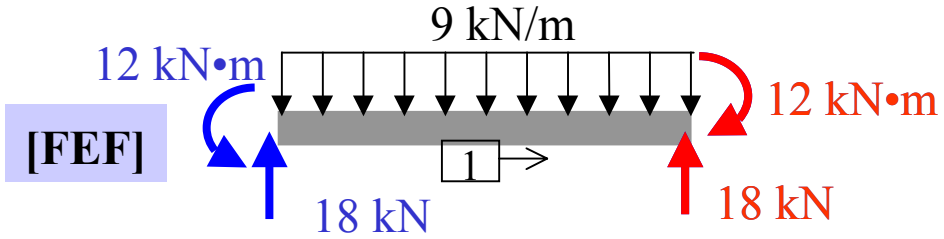


$$[k]_1 = \begin{array}{c} \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 4 & 5 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI \end{bmatrix} \end{array}$$

$$[K] = EI \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5625 & -0.75 & 0.375 \\ -0.75 & 2.0 & 0 \\ 0.375 & 0.0 & 1.0 \end{bmatrix} \end{array}$$

$$[k]_2 = \begin{array}{c} \\ 1 \\ 3 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 6 & 7 \end{array} \\ \begin{bmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{bmatrix} \end{array}$$

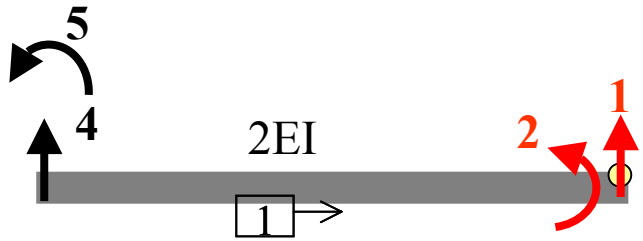




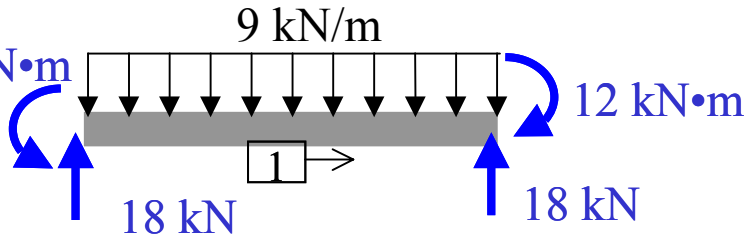
Global:

$$\begin{pmatrix} Q_1 = -20 \\ Q_2 = 40 \\ Q_3 = 0.0 \end{pmatrix} = EI \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.5625 & -0.75 & 0.375 \\ -0.75 & 2.0 & 0 \\ 0.375 & 0.0 & 1.0 \end{bmatrix} \end{array} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} + \begin{pmatrix} 18 \\ -12 \\ 0.0 \end{pmatrix}$$

$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} -0.01316 \text{ m} \\ -0.002333 \text{ rad} \\ 0.0049333 \text{ rad} \end{pmatrix}$$

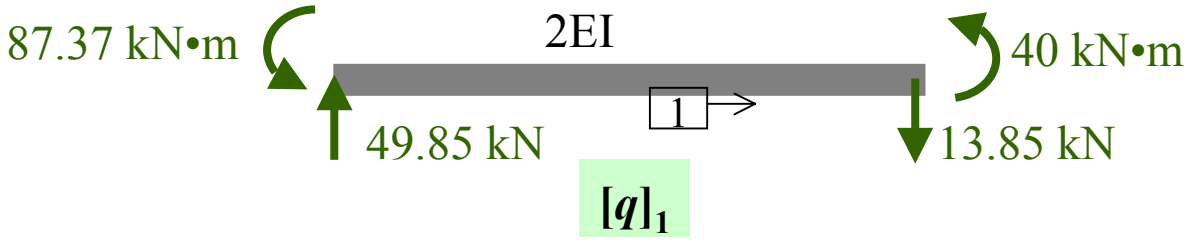


[FEF]

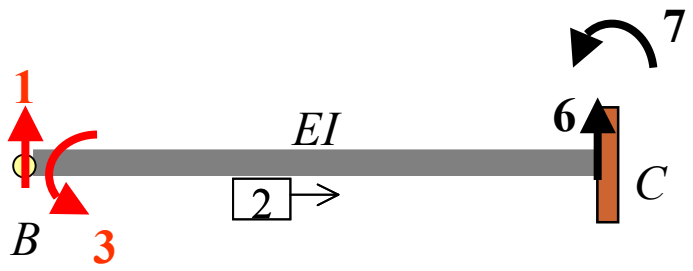


Member 1: $[q]_1 = [k]_1[d]_1 + [q^F]_1$

$$\begin{pmatrix} q_4 \\ q_5 \\ q_1 \\ q_2 \end{pmatrix} = \begin{matrix} \\ 4 \\ 5 \\ 1 \\ 2 \end{matrix} \begin{matrix} \begin{matrix} 4 & 5 & 1 & 2 \end{matrix} \\ \begin{pmatrix} 0.375EI & 0.75EI & -0.375EI & 0.75EI \\ 0.75EI & 2EI & -0.75EI & EI \\ -0.375EI & -0.75EI & 0.375EI & -0.75EI \\ 0.75EI & EI & -0.75EI & 2EI \end{pmatrix} \end{matrix} \begin{pmatrix} d_4 = 0.0 \\ d_5 = 0.0 \\ d_1 = -0.01316 \\ d_2 = -0.002333 \end{pmatrix} + \begin{pmatrix} 18 \\ 12 \\ 18 \\ -12 \end{pmatrix} = \begin{pmatrix} 49.85 \\ 87.37 \\ -13.85 \\ 40 \end{pmatrix}$$

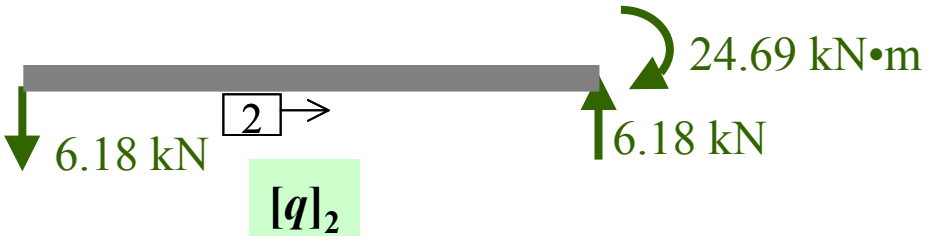


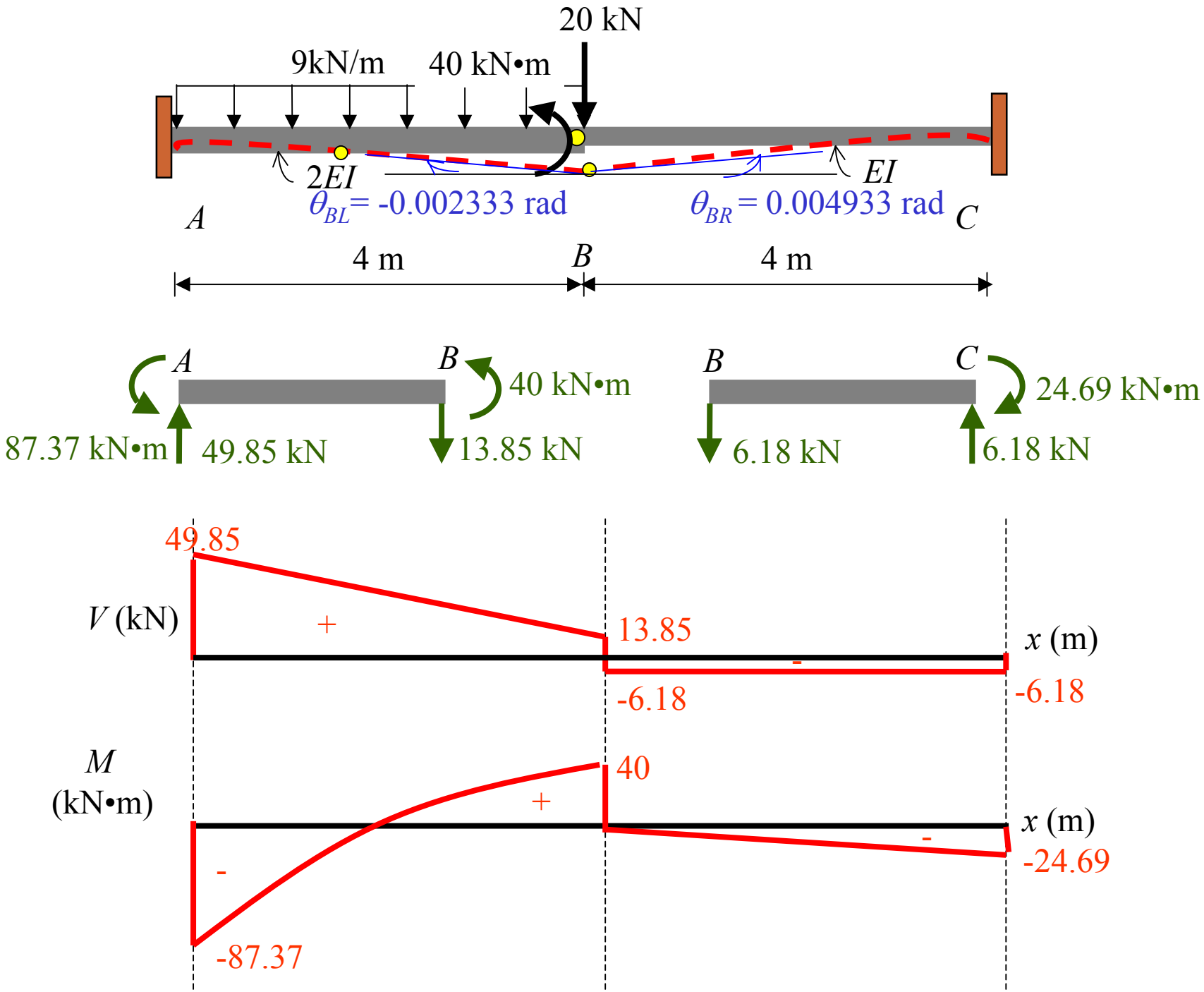
$[q]_1$



Member 2: $[q]_2 = [k]_2[d]_2 + [q^F]_2$

$$\begin{pmatrix} q_1 \\ q_3 \\ q_6 \\ q_7 \end{pmatrix} = \begin{matrix} 1 \\ 3 \\ 6 \\ 7 \end{matrix} \overset{\begin{matrix} 1 & 3 & 6 & 7 \end{matrix}}{\begin{pmatrix} 0.1875EI & 0.375EI & -0.1875EI & 0.375EI \\ 0.375EI & EI & -0.375EI & 0.5EI \\ -0.1875EI & -0.375EI & 0.1875EI & -0.375EI \\ 0.375EI & 0.5EI & -0.375EI & EI \end{pmatrix}} \begin{pmatrix} d_1 = -0.01316 \\ d_3 = 0.004933 \\ d_6 = 0.0 \\ d_7 = 0.0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -6.18 \\ 0.0 \\ 6.18 \\ -24.69 \end{pmatrix}$$



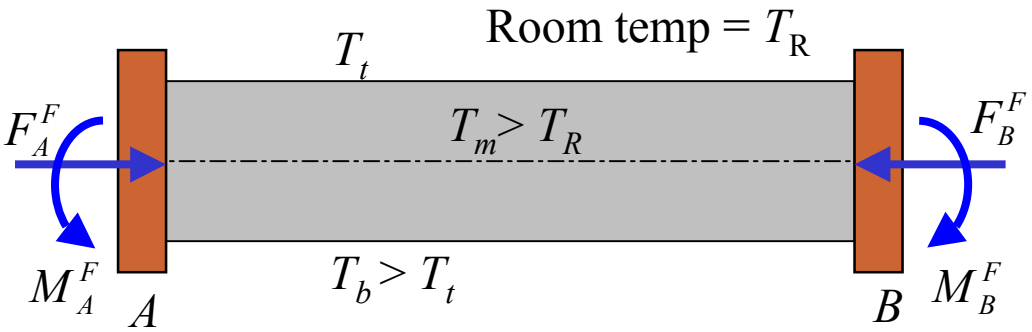

20 kN
9kN/m
40 kN•m
2EI
EI
θ_{BL}= -0.002333 rad
θ_{BR} = 0.004933 rad
A
B
C
4 m
4 m
87.37 kN•m
49.85 kN
13.85 kN
40 kN•m
6.18 kN
6.18 kN
24.69 kN•m
49.85
13.85
-6.18
-6.18
V (kN)
x (m)
40
-87.37
-24.69
M
(kN•m)
x (m)

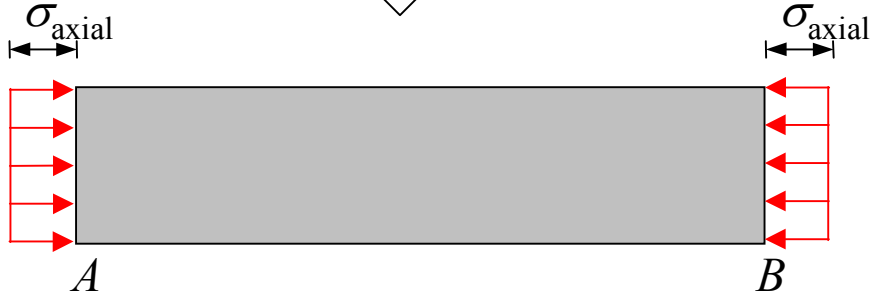
Temperature Effects

- **Fixed-End Forces (FEF)**
  - **Axial**
  - **Bending**

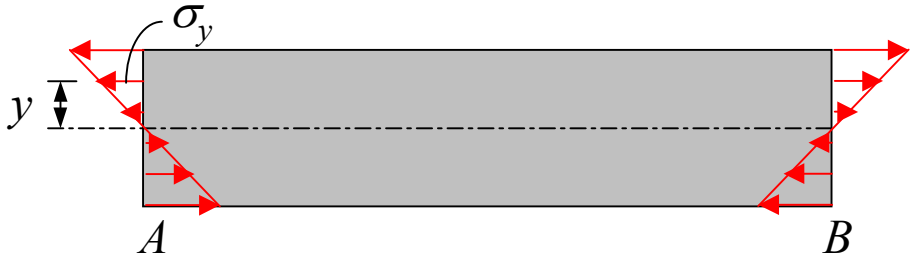
- **Curvature**

• **Thermal Fixed-End Forces (FEF)**

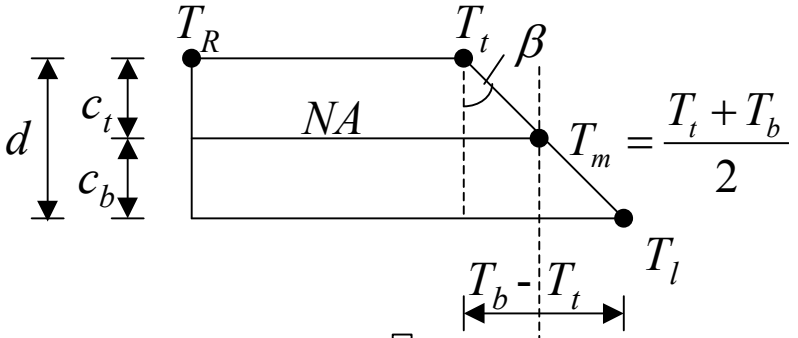


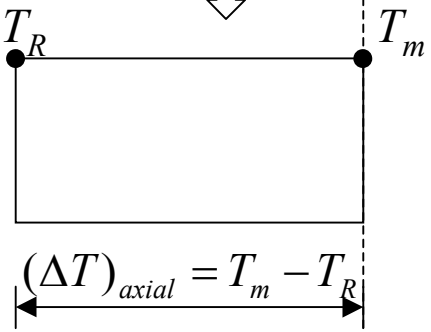


+

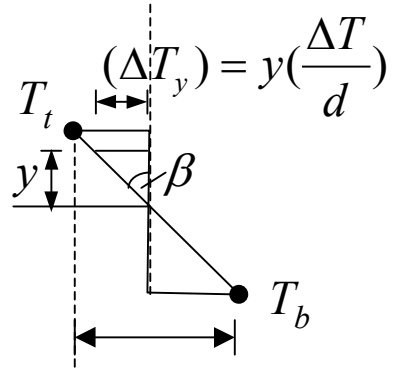


$$\tan\beta = \frac{T_b - T_t}{d} = \left(\frac{\Delta T}{d}\right)$$



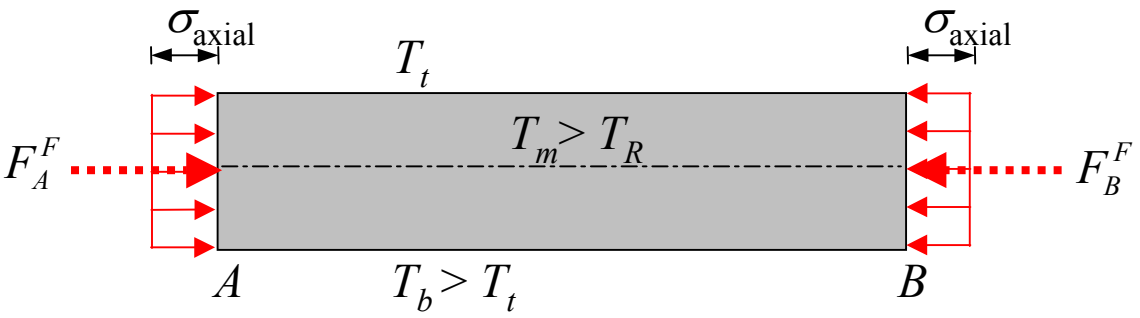


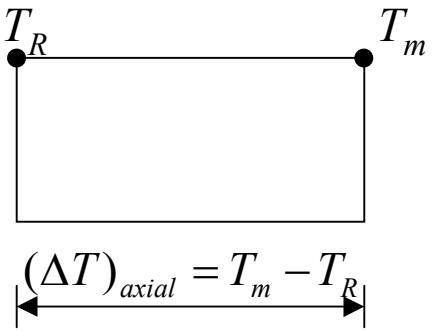
+



$$(\Delta T)_{bending} = T_b - T_t$$

- Axial





$$F_A^F = \int_A \sigma_{axial} dA$$

$$= \int E\varepsilon_{axial}\, dA$$

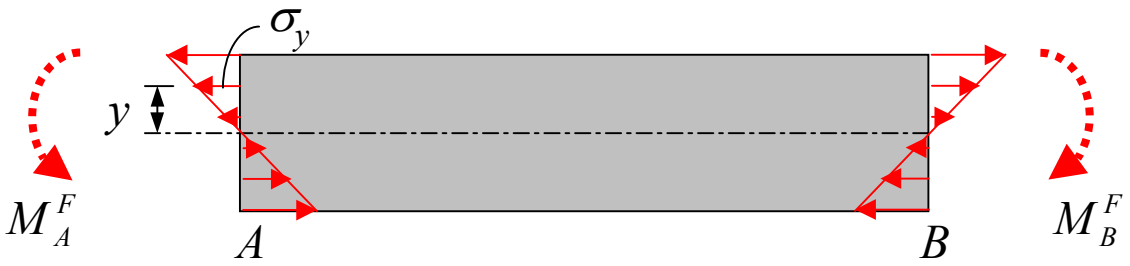
$$= \int E\alpha(\Delta T)_{axial}\, dA$$

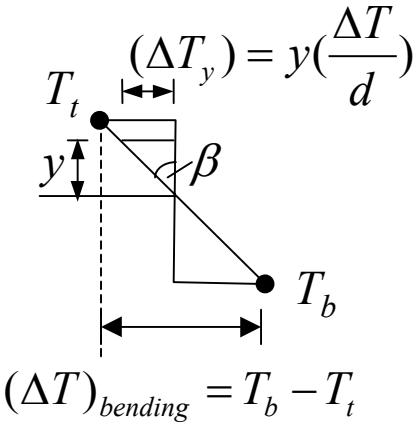
$$= E\alpha(\Delta T)_{axial} \int dA$$

$$= EA\alpha(\Delta T)_{axial}$$

$$(F_{axial}^F)_A = \alpha(T_m - T_R)AE$$

- Bending





$$M_A^F = \int_A y\sigma_y dA$$

$$= \int yE\varepsilon\, dA$$

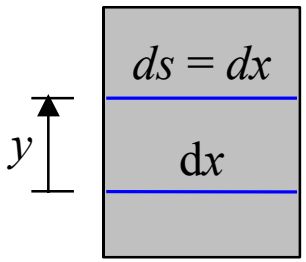
$$= \int yE\alpha(\Delta T_y)\, dA$$

$$= \int yE\alpha y(\frac{\Delta T}{d})\, dA$$

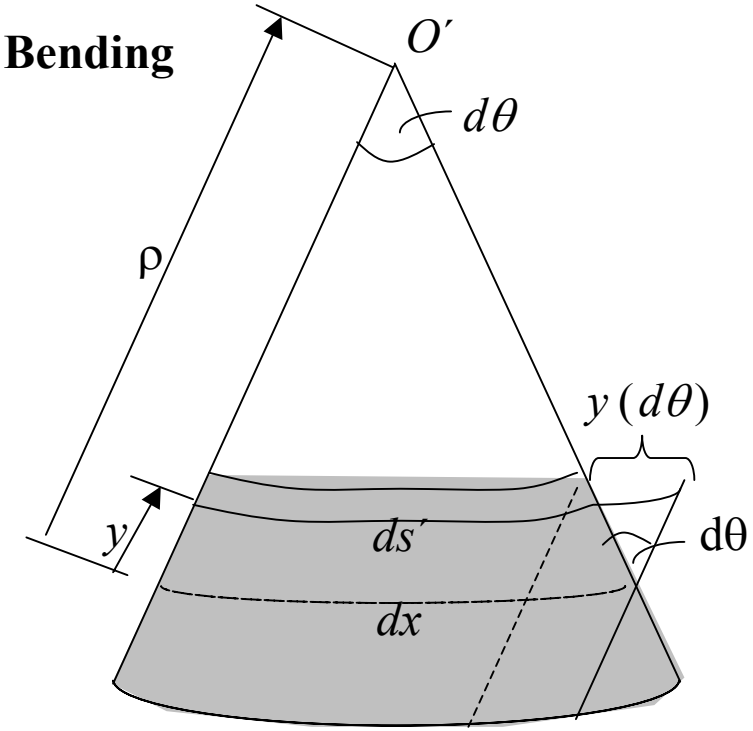
$$= E\alpha(\frac{\Delta T}{d})\int y^2 dA$$

$$(F_{bending}^F)_A = \alpha(\frac{T_l - T_u}{d})EI$$

• **Elastic Curve: Bending**



Before deformation

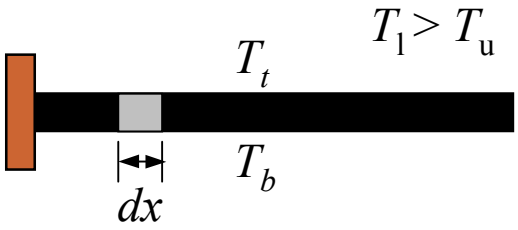


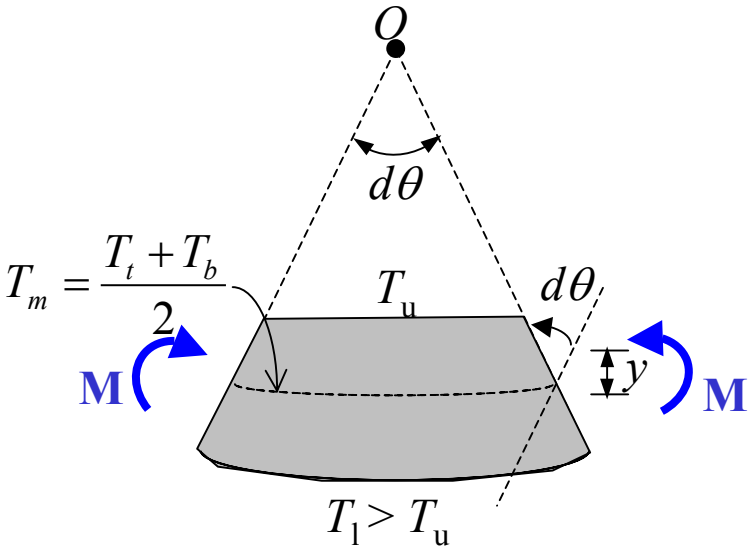
After deformation

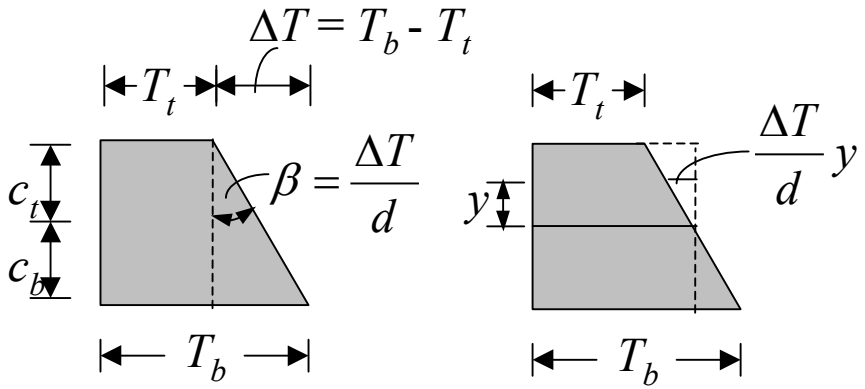
$$(dx) = \rho\,(d\theta)$$

$$\frac{1}{\rho} = \left(\frac{d\theta}{dx}\right)$$

- **Bending Temperature**







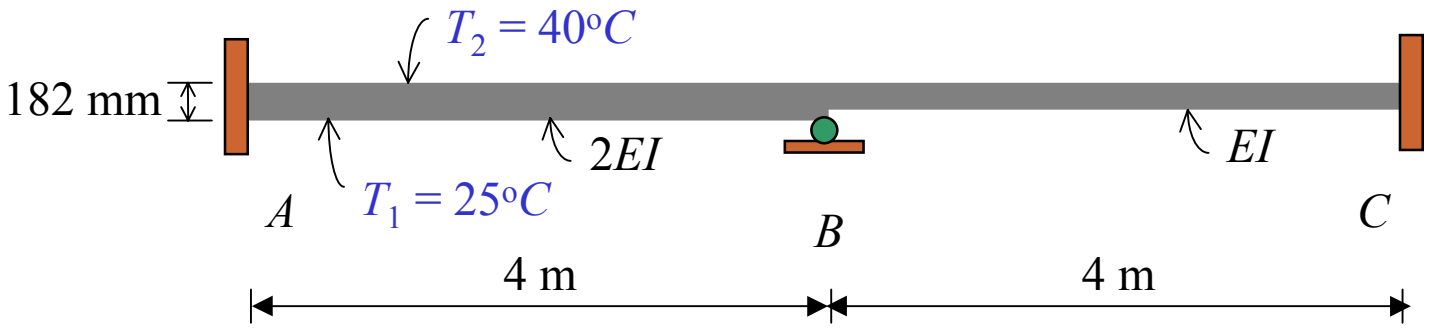
$$(d\theta)y = \alpha y(\frac{\Delta T}{d})dx$$

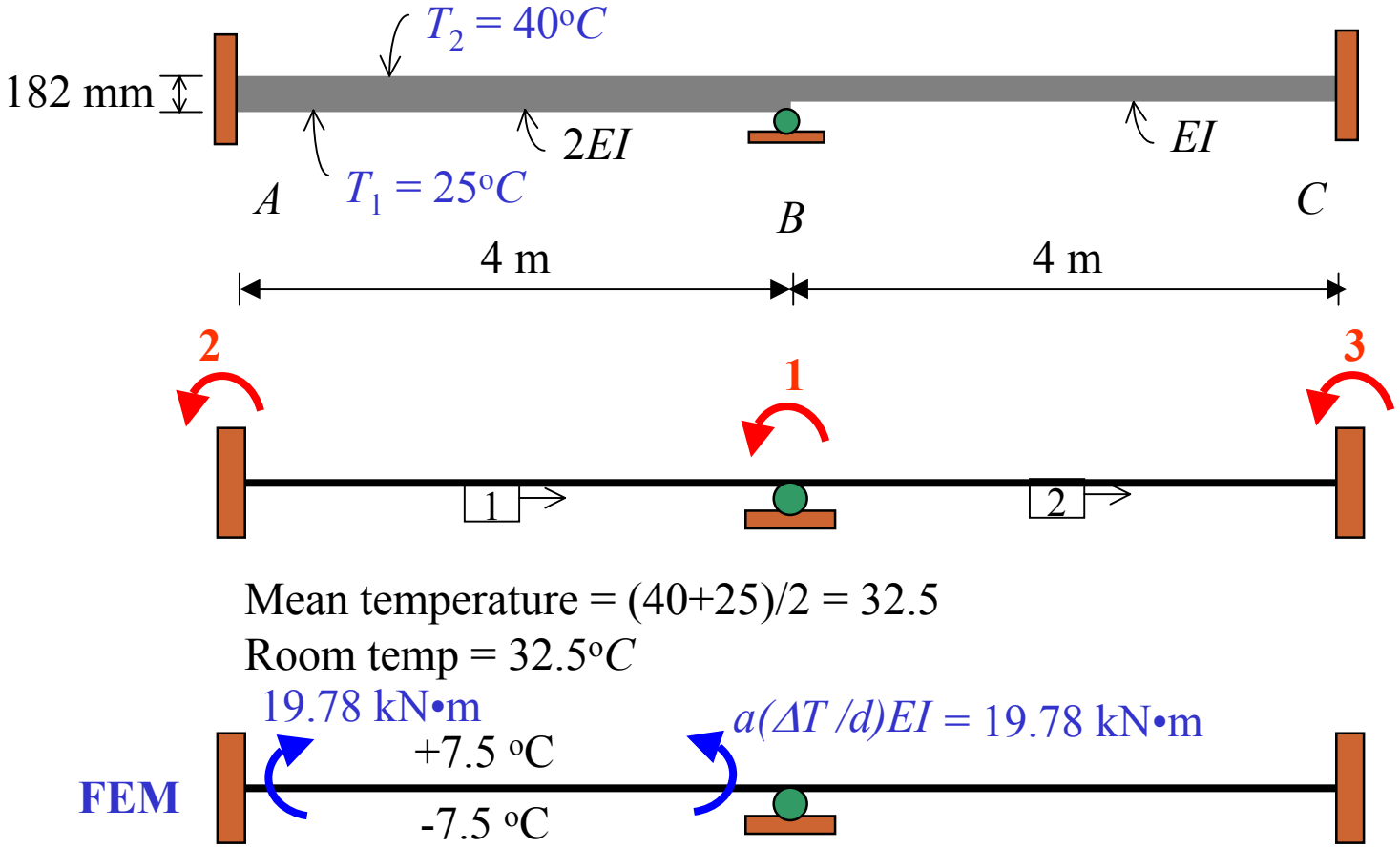
$$(d\theta) = \alpha(\frac{\Delta T}{d})dx$$

$$(\frac{d\theta}{dx}) = \frac{1}{\rho} = \alpha(\frac{\Delta T}{d}) = \frac{M}{EI}$$

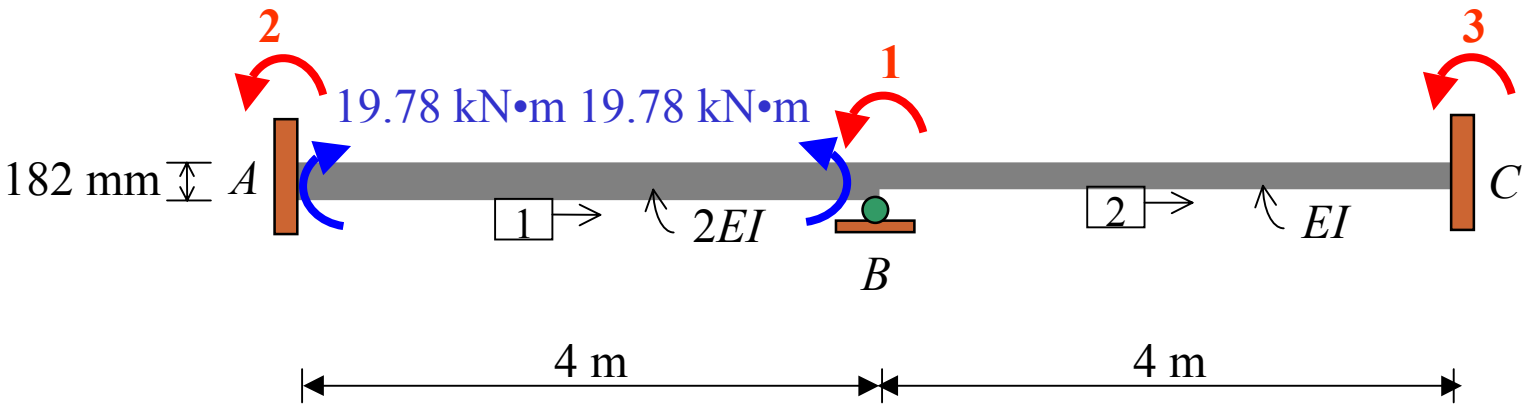
Example 10

For the beam shown, use the stiffness method to:
(a) Determine all the reactions at supports.
(b) Draw the **quantitative shear** and **bending moment diagrams** and **qualitative deflected shape.**
Room temp = 32.5^{o}C, α = 12x10^{-6} /^{o}C, E = 200 GPa, I = 50x10^{-6} m^{4}.





$$F^F_{bending} = \alpha(\frac{\Delta T}{d})(2EI) = (12\times10^{-6})(\frac{40-25}{0.182})(2\times200\times50) = 19.78\ kN\bullet m$$



$$[q] = [k][d] + [q^F]$$

Element 1:

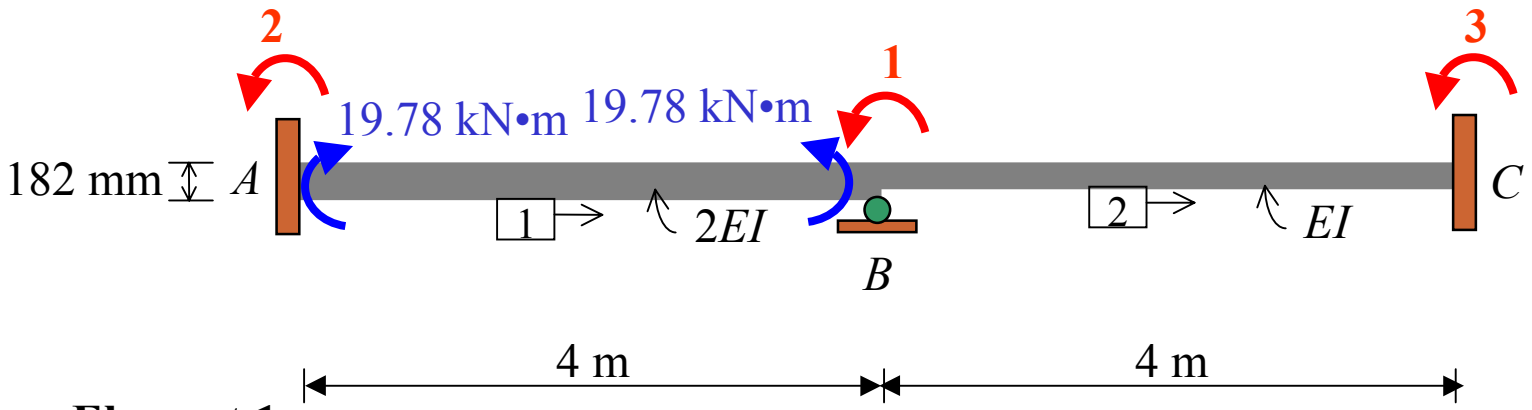
$$\begin{bmatrix} M_2 \\ M_1 \end{bmatrix} = EI \begin{matrix} 2 \\ 1 \end{matrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{matrix} 2 \\ 1 \end{matrix} \begin{bmatrix} q_2 \\ q_1 \end{bmatrix} + \begin{bmatrix} -1.978 \\ 1.978 \end{bmatrix} (10^{-3})\, EI$$

Element 2:

$$\begin{bmatrix} M_1 \\ M_3 \end{bmatrix} = EI \begin{matrix} 1 \\ 3 \end{matrix} \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{matrix} 1 \\ 3 \end{matrix} \begin{bmatrix} q_1 \\ q_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[\overset{0}{\not M_1}] = 3EI\theta_1 + (1.978\text{x}10^{-3})EI$$

$\theta_1 = -0.659\text{x}10^{-3}$ rad



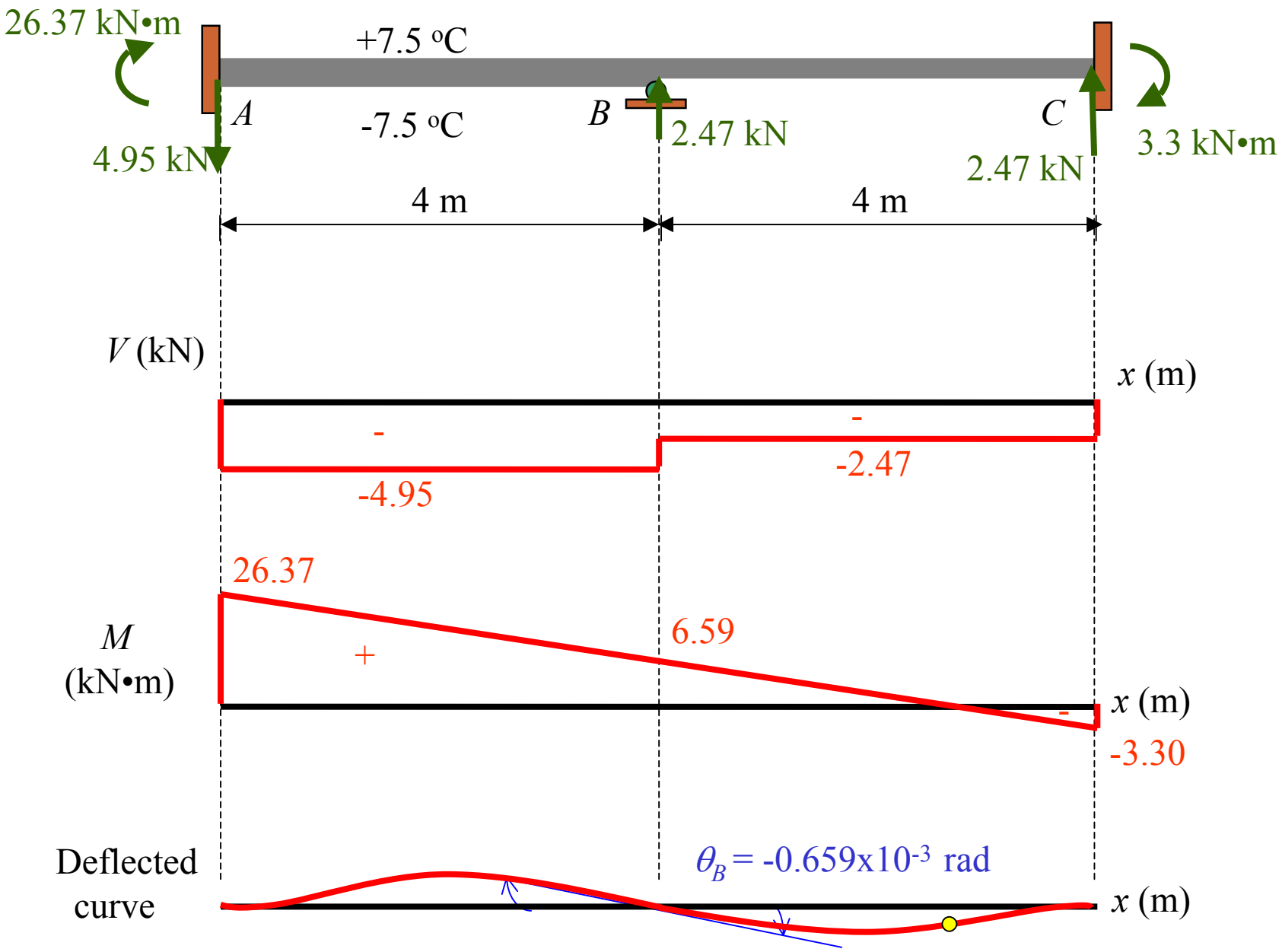
Element 1:

$$\begin{bmatrix} M_2 \\ M_1 \end{bmatrix} = EI \begin{matrix} 2 \\ 1 \end{matrix} \overset{\begin{matrix} 2 & 1 \end{matrix}}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \begin{bmatrix} q_2 = 0 \\ q_1 = -0.659\text{x}10^{-3} \end{bmatrix} + \begin{bmatrix} -19.78 \\ 19.78 \end{bmatrix} = \begin{bmatrix} -26.37 \text{ kN•m} \\ 6.59 \text{ kN•m} \end{bmatrix}$$

Element 2:

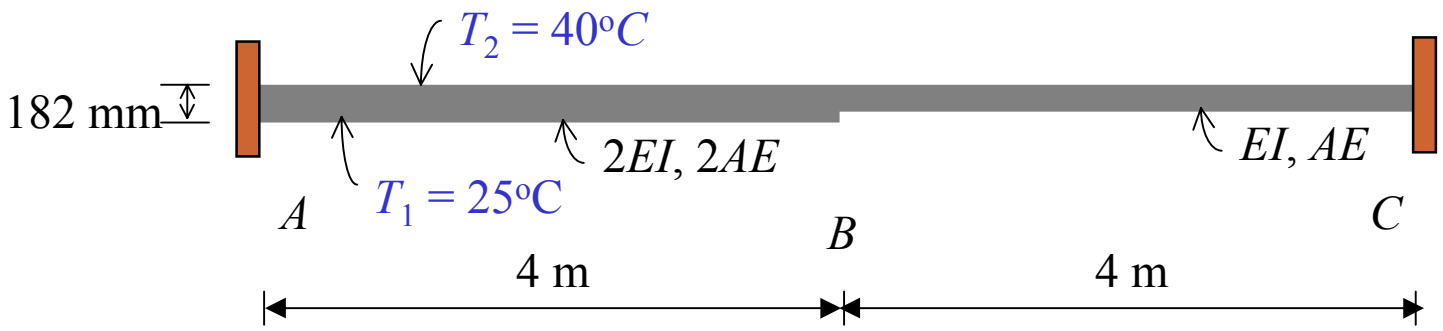
$$\begin{bmatrix} M_1 \\ M_3 \end{bmatrix} = EI \begin{matrix} 1 \\ 3 \end{matrix} \overset{\begin{matrix} 1 & 3 \end{matrix}}{\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}} \begin{bmatrix} q_1 = -0.659\text{x}10^{-3} \\ q_3 = 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -6.59 \text{ kN•m} \\ -3.30 \text{ kN•m} \end{bmatrix}$$

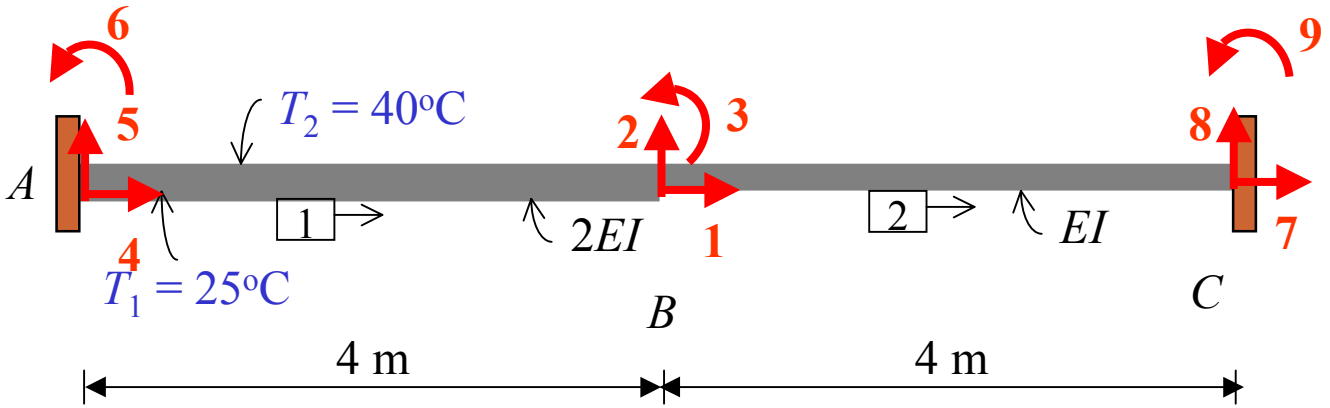



26.37 kN•m
+7.5 °C
A
-7.5 °C
B
C
2.47 kN
3.3 kN•m
4.95 kN
2.47 kN
4 m
4 m
V (kN)
x (m)
-
-
-2.47
-4.95
26.37
M
(kN•m)
6.59
+
-
x (m)
-3.30
Deflected
curve
θB = -0.659x10-3 rad
x (m)

Example 11

For the beam shown, use the stiffness method to:
(a) Determine all the reactions at supports.
(b) Draw the **quantitative shear** and **bending moment diagrams** and **qualitative deflected shape.**
Room temp = 28 ^{o}C, a = 12x10^{-6} /^{o}C, E = 200 GPa, I = 50x10^{-6} m^{4}, A = 20(10^{-3}) m^{2}





Element 1:

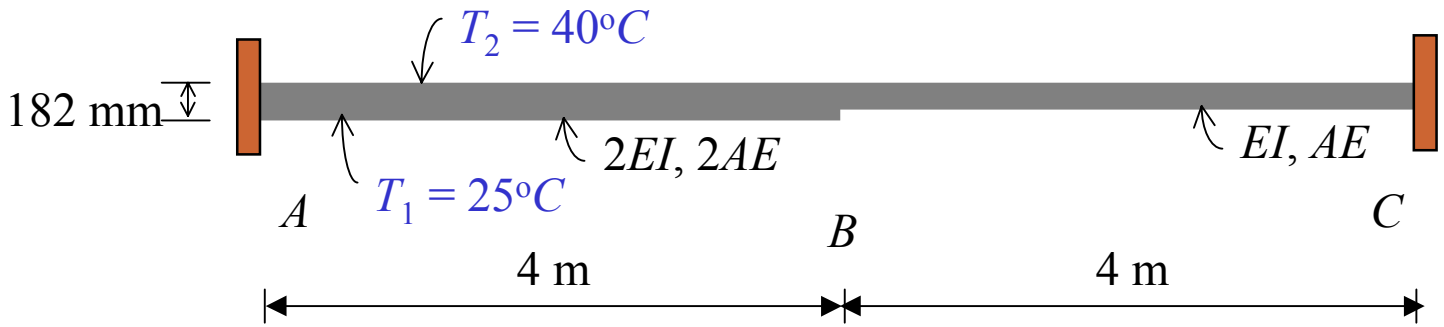
$$\frac{(2AE)}{L} = \frac{2(20\times10^{-3}\ m^2)(200\times10^{6}\ kN/m^2)}{(4\ m)} = 2(10^6)\ kN/m$$

$$\frac{4(2EI)}{L} = \frac{4\times2(200\times10^{6}\ kN/m^2)(50\times10^{-6}\ m^4)}{(4\ m)} = 20(10^3)\ kN\bullet m$$

$$\frac{2(2EI)}{L} = \frac{2\times2(200\times10^{6}\ kN/m^2)(50\times10^{-6}\ m^4)}{(4\ m)} = 10(10^3)\ kN\bullet m$$

$$\frac{6(2EI)}{L^2} = \frac{6\times2(200\times10^{6}\ kN/m^2)(50\times10^{-6}\ m^4)}{(4\ m)^2} = 7.5(10^3)\ kN$$

$$\frac{12(2EI)}{L^3} = \frac{12\times2(200\times10^{6}\ kN/m^2)(50\times10^{-6}\ m^4)}{(4\ m)^3} = 3.75(10^3)\ kN/m$$

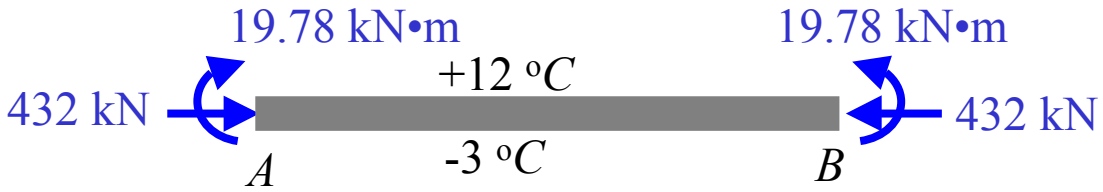


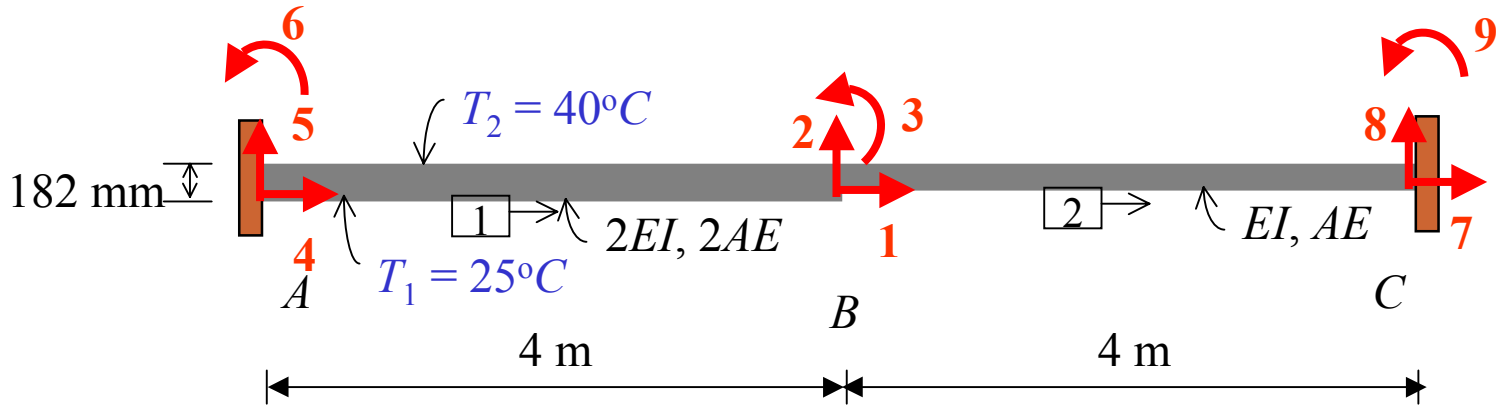
Fixed-end forces due to temperatures

$$F^F_{bending} = \alpha(\frac{\Delta T}{d})(2EI) = (12\times10^{-6})(\frac{40-25}{0.182})(2\times200\times50) = 19.78\ kN\bullet m$$

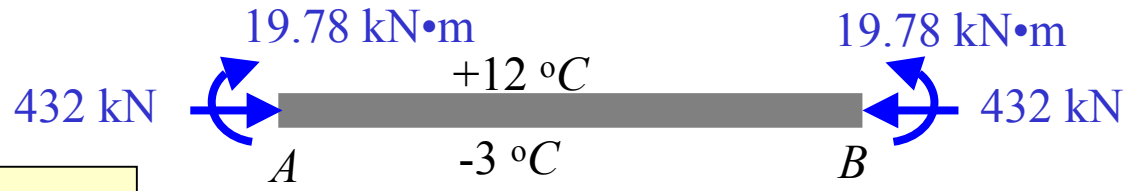
Mean temperature(T_m) = (40+25)/2 = 32.5 ^oC ,
$T_R = 28\ ^oC$

$$F^F_{axial} = \alpha(\Delta T)AE = (12\times10^{-6})(32.5-28)(2\times20\times10-3\ m^2)(200\times10^6\ kN/m^2) = 432\ kN$$



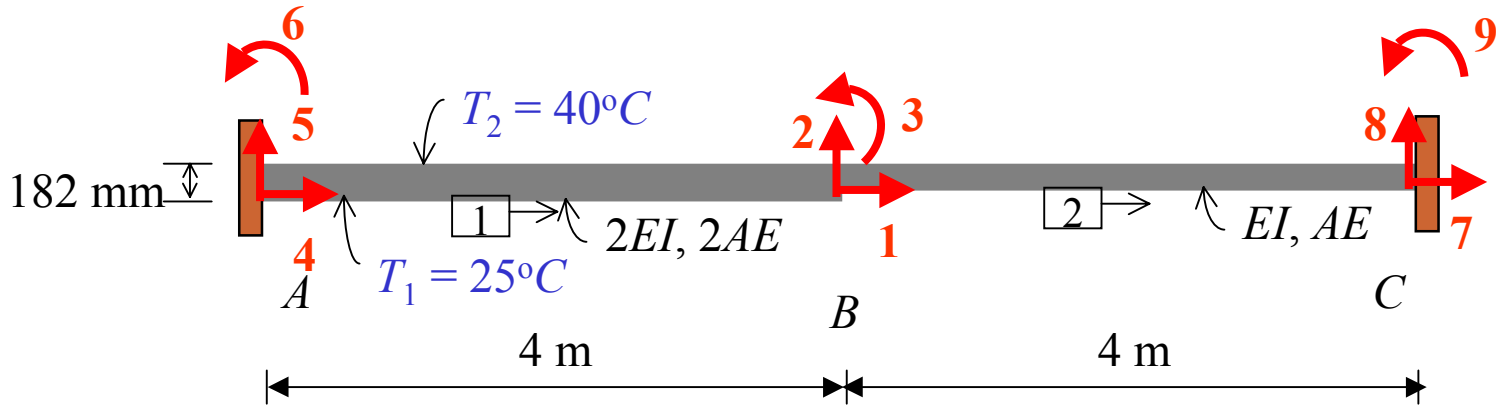


Element 1:



$[q] = [k][d] + [q^F]$

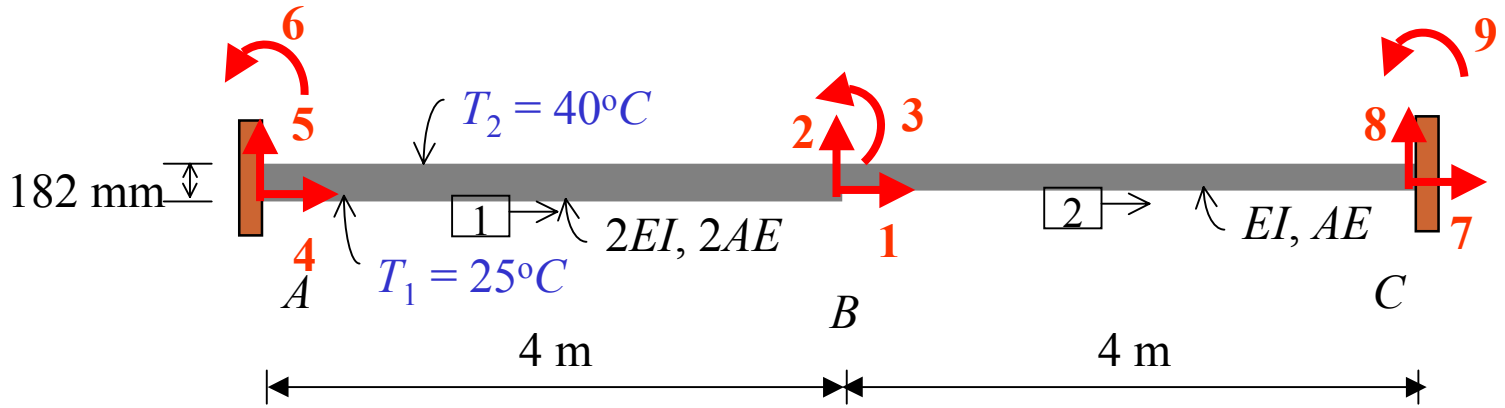
$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{matrix} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{matrix} \begin{pmatrix} 2\text{x}10^6 & 0.00 & 0.00 & -2\text{x}10^6 & 0.00 & 0.00 \\ 0.00 & 3750 & 7500 & 0.00 & -3750 & 7500 \\ 0.00 & 7500 & 20\text{x}10^3 & 0.00 & -7500 & 10\text{x}10^3 \\ -2\text{x}10^6 & 0.00 & 0.00 & 2\text{x}10^6 & 0.00 & 0.00 \\ 0.00 & -3750 & -7500 & 0.00 & 3750 & -7500 \\ 0.00 & 7500 & 10\text{x}10^3 & 0.00 & -7500 & 20\text{x}10^3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix} + \begin{pmatrix} 432 \\ 0.00 \\ -19.78 \\ -432 \\ 0.00 \\ 19.78 \end{pmatrix}$$



Element 2:

$$[q] = [k][d] + [q^F]$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \begin{pmatrix} 1\text{x}10^6 & 0.00 & 0.00 & -1\text{x}10^6 & 0.00 & 0.00 \\ 0.00 & 1875 & 3750 & 0.00 & -1875 & 3750 \\ 0.00 & 3750 & 10\text{x}10^3 & 0.00 & -3750 & 5\text{x}10^3 \\ -1\text{x}10^6 & 0.00 & 0.00 & 1\text{x}10^6 & 0.00 & 0.00 \\ 0.00 & -1875 & -3750 & 0.00 & 1875 & -3750 \\ 0.00 & 3750 & 5\text{x}10^3 & 0.00 & -3750 & 10\text{x}10^3 \end{pmatrix} \end{array} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



Global:

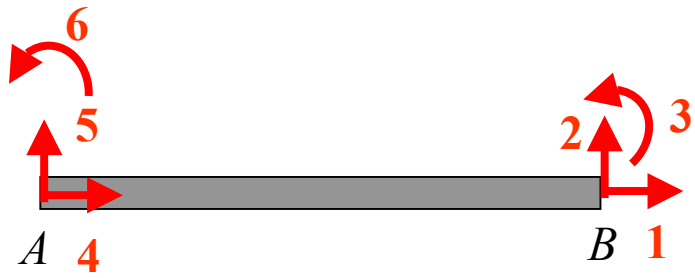
$$\begin{pmatrix} Q_1 = 0.0 \\ Q_2 = 0.0 \\ Q_3 = 0.0 \end{pmatrix} = \begin{matrix} \\ 1 \\ 2 \\ 3 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} 3\text{x}10^6 & 0.0 & 0.0 \\ 0.0 & 5625 & -3750 \\ 0.0 & -3750 & 30\text{x}10^3 \end{pmatrix} \end{matrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} + \begin{pmatrix} -432 \\ 0.0 \\ 19.78 \end{pmatrix}$$

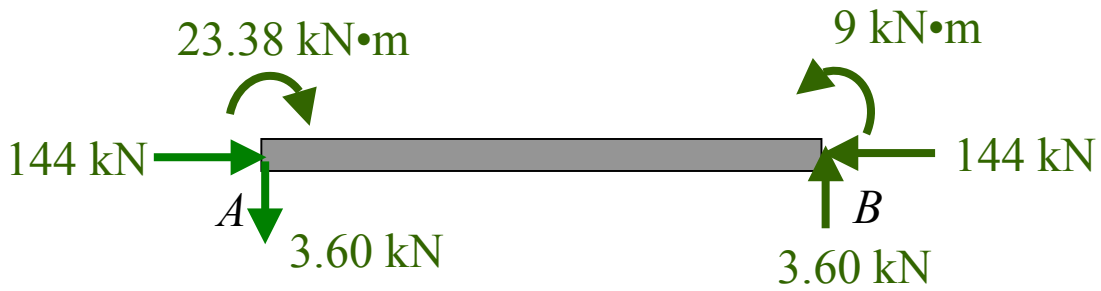
$$\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0.000144 & \text{m} \\ -0.0004795 & \text{m} \\ -719.3\text{x}10^{-6} & \text{rad} \end{pmatrix}$$

Element 1:

$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{matrix} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{matrix} \begin{matrix} \begin{matrix} 4 & 5 & 6 & 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} 2\text{x}10^6 & 0.00 & 0.00 & -2\text{x}10^6 & 0.00 & 0.00 \\ 0.00 & 3750 & 7500 & 0.00 & -3750 & 7500 \\ 0.00 & 7500 & 20\text{x}10^3 & 0.00 & -7500 & 10\text{x}10^3 \\ -2\text{x}10^6 & 0.00 & 0.00 & 2\text{x}10^6 & 0.00 & 0.00 \\ 0.00 & -3750 & -7500 & 0.00 & 3750 & -7500 \\ 0.00 & 7500 & 10\text{x}10^3 & 0.00 & -7500 & 20\text{x}10^3 \end{pmatrix} \end{matrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ d_1 = 144\text{x}10^{-6} \\ d_2 = -479.5\text{x}10^{-6} \\ d_3 = -719.3\text{x}10^{-6} \end{pmatrix} + \begin{pmatrix} 432 \\ 0.00 \\ -19.78 \\ -432 \\ 0.00 \\ 19.78 \end{pmatrix}$$

$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 144.0 & \text{kN} \\ -3.60 & \text{kN} \\ -23.38 & \text{kN•m} \\ -144.0 & \text{kN} \\ 3.60 & \text{kN} \\ 9.00 & \text{kN•m} \end{pmatrix}$$

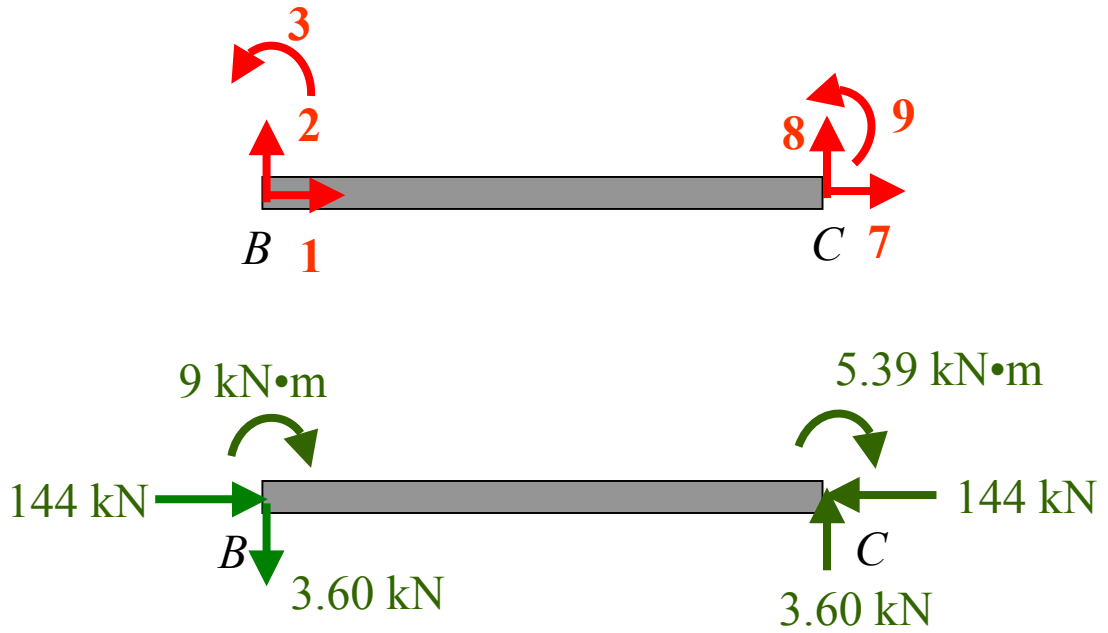




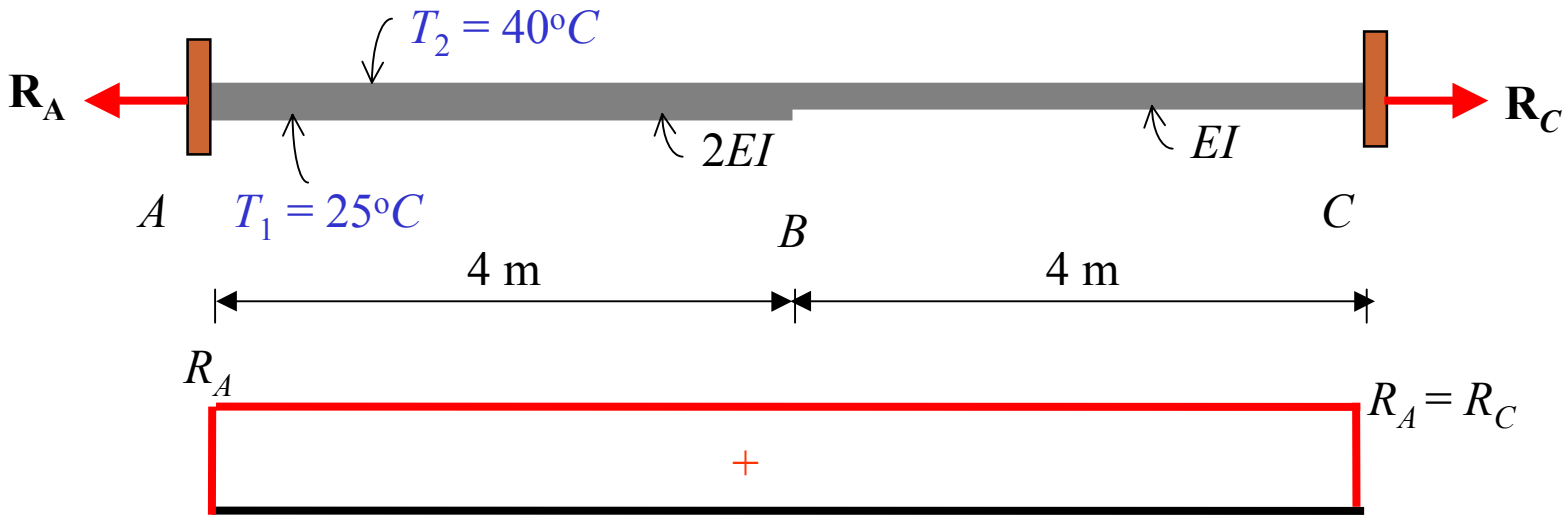
Element 2:

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 7 & 8 & 9 \end{matrix} \\ \begin{pmatrix} 1\text{x}10^{6} & 0.00 & 0.00 & -1\text{x}10^{6} & 0.00 & 0.00 \\ 0.00 & 1875 & 3750 & 0.00 & -1875 & 3750 \\ 0.00 & 37500 & 10\text{x}10^{3} & 0.00 & -3750 & 5\text{x}10^{3} \\ -1\text{x}10^{6} & 0.00 & 0.00 & 1\text{x}10^{6} & 0.00 & 0.00 \\ 0.00 & -1875 & -3750 & 0.00 & 1875 & -3750 \\ 0.00 & 3750 & 5\text{x}10^{3} & 0.00 & -3750 & 10\text{x}10^{3} \end{pmatrix} \end{matrix} \begin{pmatrix} d_1 = 144\text{x}10^{-6} \\ d_2 = -479.5\text{x}10^{-6} \\ d_3 = -719.3\text{x}10^{-6} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_7 \\ q_8 \\ q_9 \end{pmatrix} = \begin{pmatrix} 144 & \text{kN} \\ -3.6 & \text{kN} \\ -9 & \text{kN•m} \\ -144 & \text{kN} \\ 3.6 & \text{kN} \\ -5.39 & \text{kN•m} \end{pmatrix}$$



Isolate axial part from the system

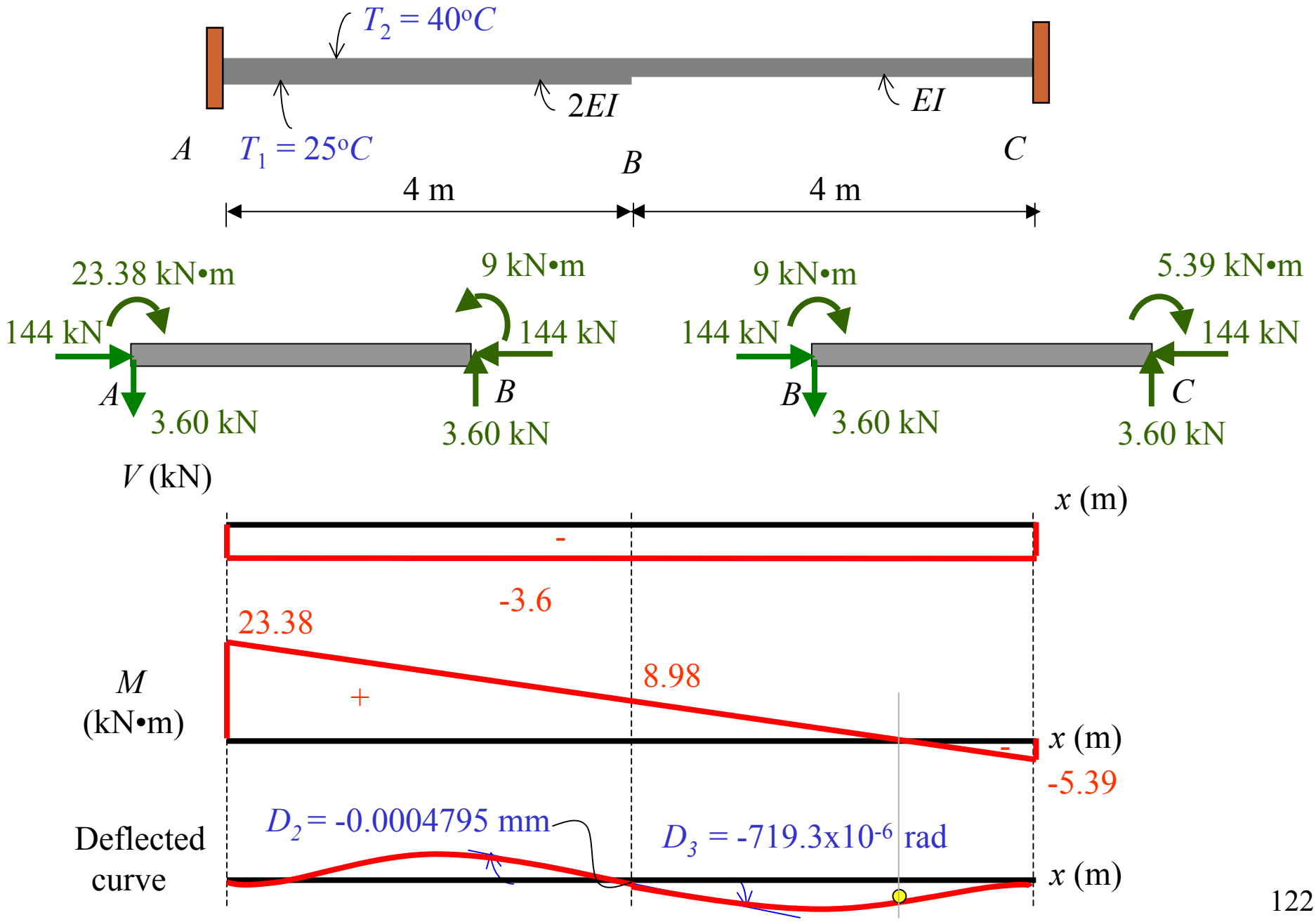


Compatibility equation: $d_{C/A} = 0$

$$\frac{R_A(4)}{2AE} + \frac{R_A(4)}{AE} + 12\times10^{-6}(32.5-28)(4) = 0$$

$$R_A = 144 \text{ kN}$$

$$R_C = \text{-}144 \text{ kN}$$

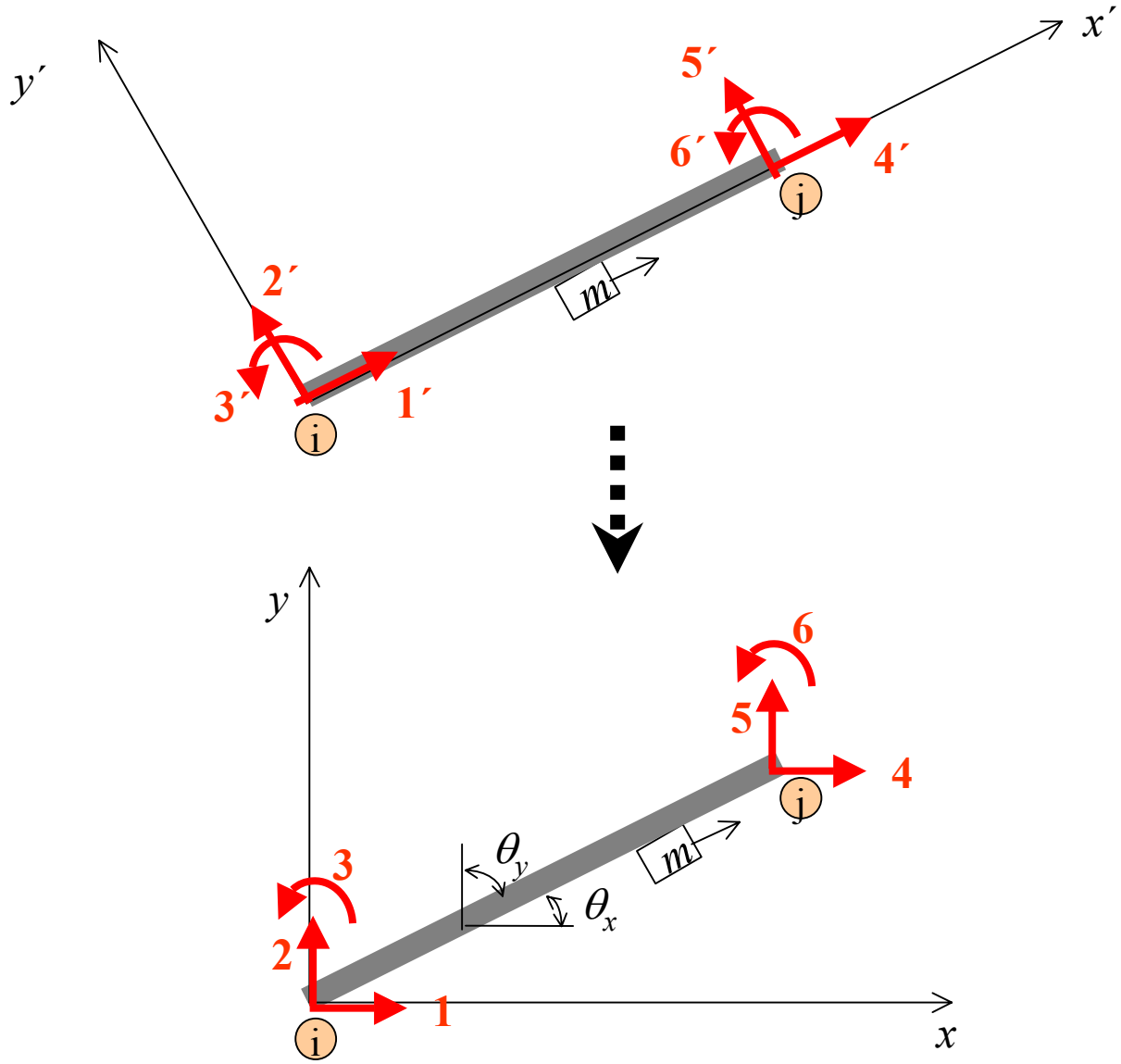

$T_2 = 40^oC$
$T_1 = 25^oC$
A
B
C
2EI
EI
4 m
4 m
23.38 kN•m
144 kN
3.60 kN
9 kN•m
144 kN
3.60 kN
9 kN•m
144 kN
3.60 kN
5.39 kN•m
144 kN
3.60 kN
V (kN)
x (m)
-3.6
23.38
8.98
M (kN•m)
-5.39
$D_2 = -0.0004795$ mm
$D_3 = -719.3x10^{-6}$ rad
Deflected curve

Skew Roller Support

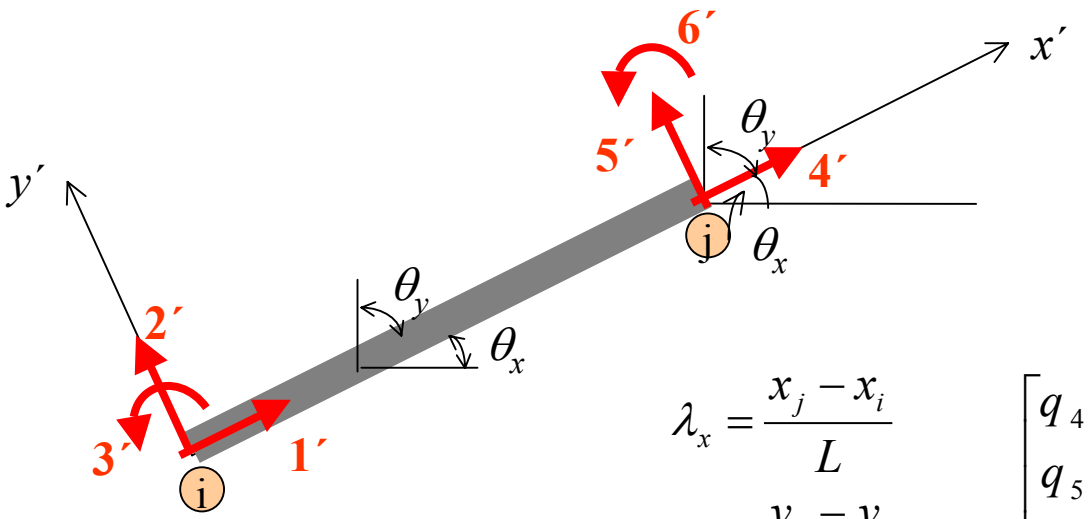
- Force Transformation
- Displacement Transformation
- Stiffness Matrix



• **Displacement and Force Transformation Matrices**



Force Transformation



$$\lambda_x = \frac{x_j - x_i}{L}$$

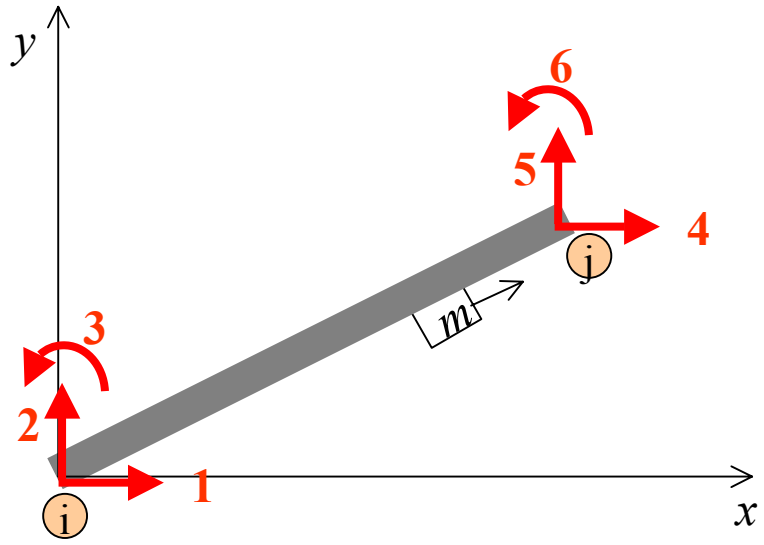
$$\lambda_y = \frac{y_j - y_i}{L}$$

$$q_4 = q_{4'}\cos\theta_x - q_{5'}\cos\theta_y \qquad (\lambda_x,\ \lambda_y)$$

$$q_5 = q_{4'}\cos\theta_y - q_{5'}\cos\theta_x$$

$$q_6 = q_{6'}$$

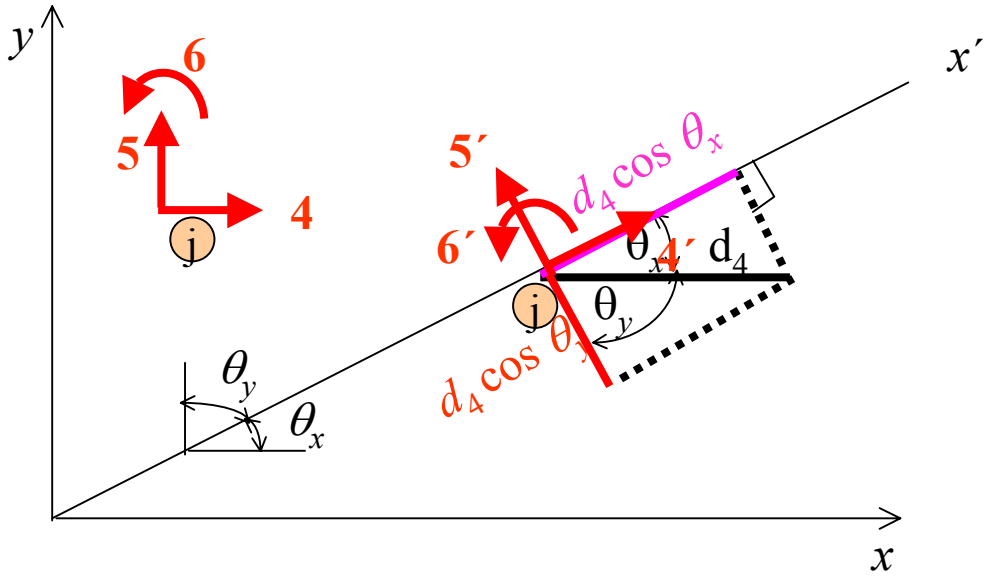
$$\begin{bmatrix} q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} \lambda_x & -\lambda_y & 0 \\ \lambda_y & \lambda_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$



$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} \lambda_x & -\lambda_y & 0 & 0 & 0 & 0 \\ \lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_x & -\lambda_y & 0 \\ 0 & 0 & 0 & \lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$

$$[q] = [T][q']$$

Displacement Transformation

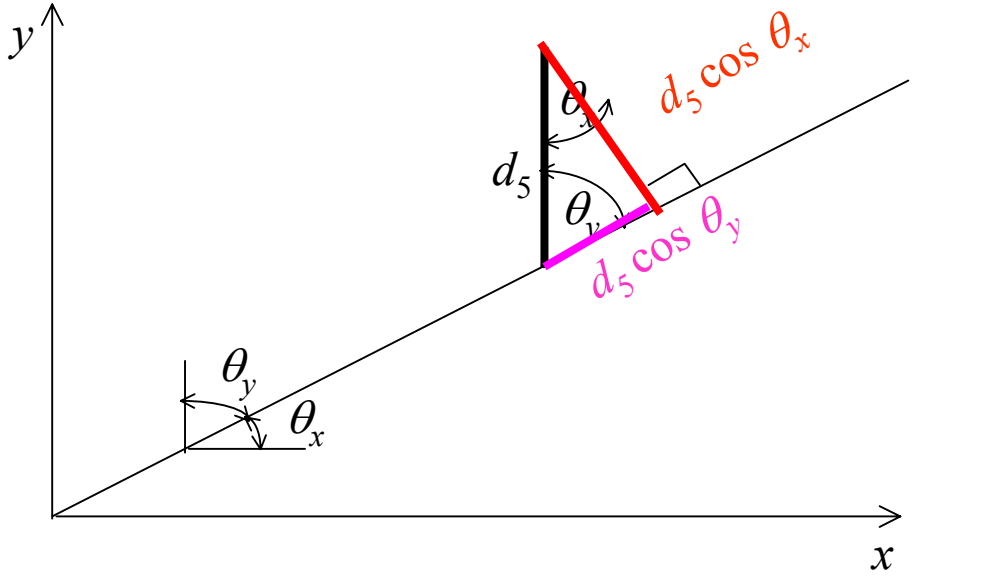


$$d'_4 = d_4 \cos\theta_x + d_5 \cos\theta_y$$

$$d'_5 = -d_4 \cos\theta_y + d_5 \cos\theta_x$$

$$d'_6 = d_6$$

$$\begin{bmatrix} d_{4'} \\ d_{5'} \\ d_{6'} \end{bmatrix} = \begin{bmatrix} \lambda_x & \lambda_y & 0 \\ -\lambda_y & \lambda_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_4 \\ d_5 \\ d_6 \end{bmatrix}$$



$$\begin{bmatrix} d_{1'} \\ d_{2'} \\ d_{3'} \\ d_{4'} \\ d_{5'} \\ d_{6'} \end{bmatrix} = \left[\begin{array}{ccc:ccc} \lambda_x & \lambda_y & 0 & 0 & 0 & 0 \\ -\lambda_y & \lambda_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \lambda_x & \lambda_y & 0 \\ 0 & 0 & 0 & -\lambda_y & \lambda_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{bmatrix}$$

$$[d'] = [T][d]$$

$$[q] = [T]^T [q']$$

$$= [T]^T ([k'][d'] + [q'^F])$$

$$= [T]^T [k'][d'] + [T]^T [q'^F]$$

$$[q] = [T]^T [k'][T][d] + [T]^T [q'^F] = [k][d] + [q^F]$$

Therefore, $[k] = [T]^T [k'][T]$

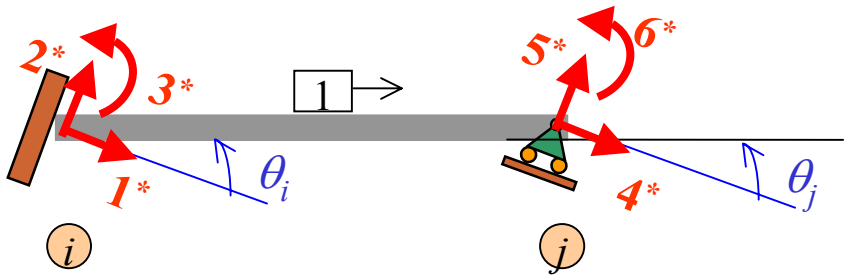
$$[q^F] = [T]^T [q'^F]$$

$$[q] = [T]^T [q']$$

$$[d'] = [T][d]$$

$$[k] = [T]^T [k'][T]$$

Stiffness matrix





$\lambda_{ix} = \cos\ \theta_i$ $\qquad$ $\lambda_{jx} = \cos\ \theta_j$

$\lambda_{iy} = \sin\ \theta_i$ $\qquad$ $\lambda_{jy} = \sin\ \theta_j$

$$[q^*] = [T]^T[q']$$

$$\begin{bmatrix} q_{1*} \\ q_{2*} \\ q_{3*} \\ q_{4*} \\ q_{5*} \\ q_{6*} \end{bmatrix} = \begin{array}{c} \\ 1* \\ 2* \\ 3* \\ 4* \\ 5* \\ 6* \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} \lambda_{ix} & -\lambda_{iy} & 0 & 0 & 0 & 0 \\ \lambda_{iy} & \lambda_{ix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{jx} & -\lambda_{jy} & 0 \\ 0 & 0 & 0 & \lambda_{jy} & \lambda_{jx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$

$[T]^T$

$$
[T] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
\lambda_{ix} & \lambda_{iy} & 0 & 0 & 0 & 0 \\
-\lambda_{iy} & \lambda_{ix} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda_{jx} & \lambda_{jy} & 0 \\
0 & 0 & 0 & -\lambda_{jy} & \lambda_{jx} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

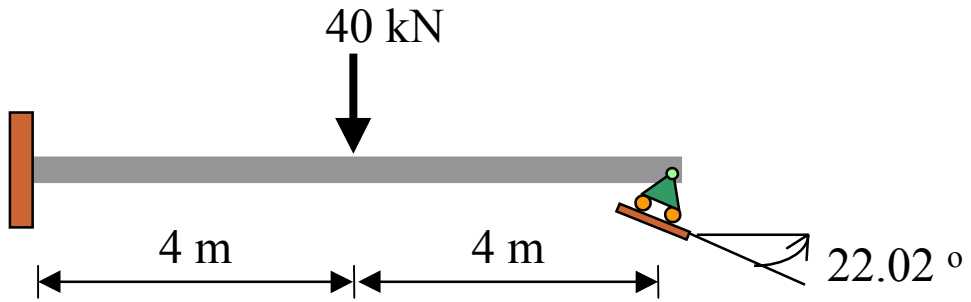
$$
[k'] = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
AE/L & 0 & 0 & -AE/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-AE/L & 0 & 0 & AE/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
\end{array}
$$

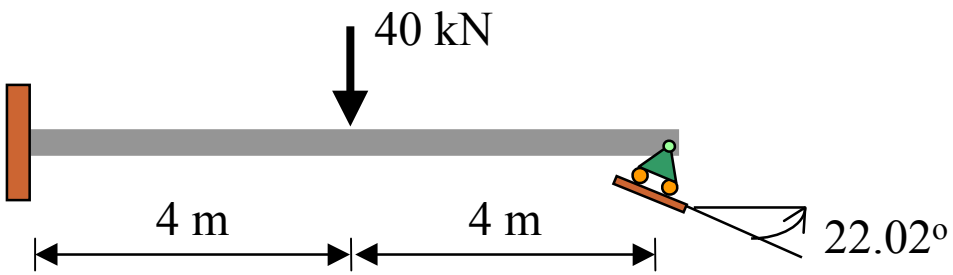
$$[\,\boldsymbol{k}\,] = [\,\boldsymbol{T}\,]^{\mathrm{T}}[\,\boldsymbol{k}'\,][\boldsymbol{T}] =$$

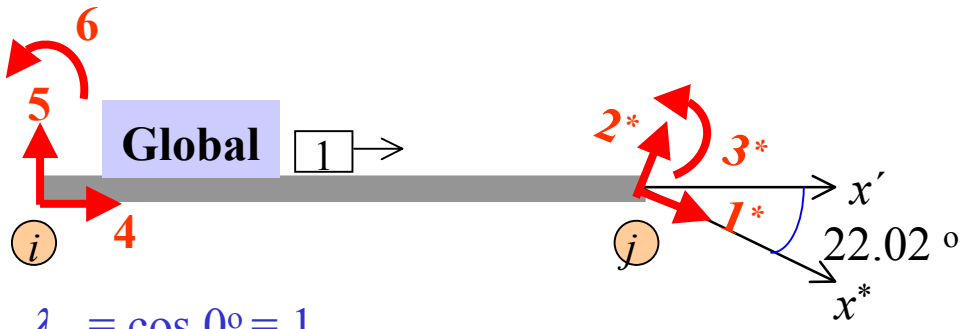
	U_i	V_i	M_i	U_j	V_j	M_j
U_i	$\left(\frac{AE}{L}\lambda_{ix}^2 + \frac{12EI}{L^3}\lambda_{iy}^2\right)$	$\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{ix}\lambda_{iy}$	$-\frac{6EI}{L^2}\lambda_{iy}$	$-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}\right)$	$-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}\right)$	$-\frac{6EI}{L^2}\lambda_{iy}$
V_i	$\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{ix}\lambda_{iy}$	$\left(\frac{AE}{L}\lambda_{iy}^2 + \frac{12EI}{L^3}\lambda_{ix}^2\right)$	$\frac{6EI}{L^2}\lambda_{ix}$	$-\left(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}\right)$	$-\left(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}\right)$	$\frac{6EI}{L^2}\lambda_{ix}$
M_i	$-\frac{6EI}{L^2}\lambda_{iy}$	$\frac{6EI}{L^2}\lambda_{ix}$	$\frac{4EI}{L}$	$\frac{6EI}{L^2}\lambda_{jy}$	$-\frac{6EI}{L^2}\lambda_{jx}$	$\frac{2EI}{L}$
U_j	$-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jx} + \frac{12EI}{L^3}\lambda_{iy}\lambda_{jy}\right)$	$-\left(\frac{AE}{L}\lambda_{iy}\lambda_{jx} - \frac{12EI}{L^3}\lambda_{ix}\lambda_{jy}\right)$	$\frac{6EI}{L^2}\lambda_{jy}$	$\left(\frac{AE}{L}\lambda_{jx}^2 + \frac{12EI}{L^3}\lambda_{jy}^2\right)$	$\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{jx}\lambda_{jy}$	$\frac{6EI}{L^2}\lambda_{jy}$
V_j	$-\left(\frac{AE}{L}\lambda_{ix}\lambda_{jy} - \frac{12EI}{L^3}\lambda_{iy}\lambda_{jx}\right)$	$-\left(\frac{AE}{L}\lambda_{iy}\lambda_{jy} + \frac{12EI}{L^3}\lambda_{ix}\lambda_{jx}\right)$	$-\frac{6EI}{L^2}\lambda_{jx}$	$\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_{jx}\lambda_{jy}$	$\left(\frac{AE}{L}\lambda_{jy}^2 + \frac{12EI}{L^3}\lambda_{jx}^2\right)$	$-\frac{6EI}{L^2}\lambda_{jx}$
M_j	$-\frac{6EI}{L^2}\lambda_{iy}$	$\frac{6EI}{L^2}\lambda_{ix}$	$\frac{2EI}{L}$	$\frac{6EI}{L^2}\lambda_{jy}$	$-\frac{6EI}{L^2}\lambda_{jx}$	$\frac{4EI}{L}$

Example 12

For the beam shown:
(a) Use the stiffness method to determine all the **reactions** at supports.
(b) Draw the **quantitative free-body diagram** of member.
(c) Draw the **quantitative bending moment diagrams**
and **qualitative deflected shape.**
Take $I = 200(10^6)$ mm^4, $A = 6(10^3)$ mm^2, and $E = 200$ GPa for all members.
Include axial deformation in the stiffness matrix.

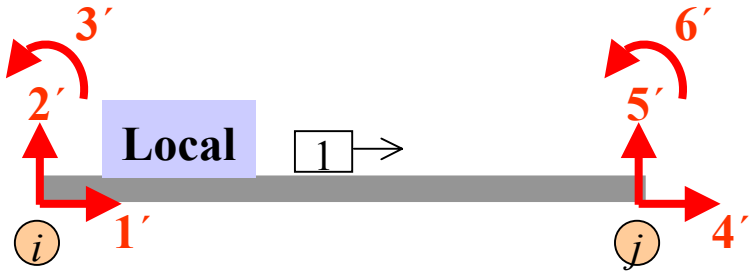


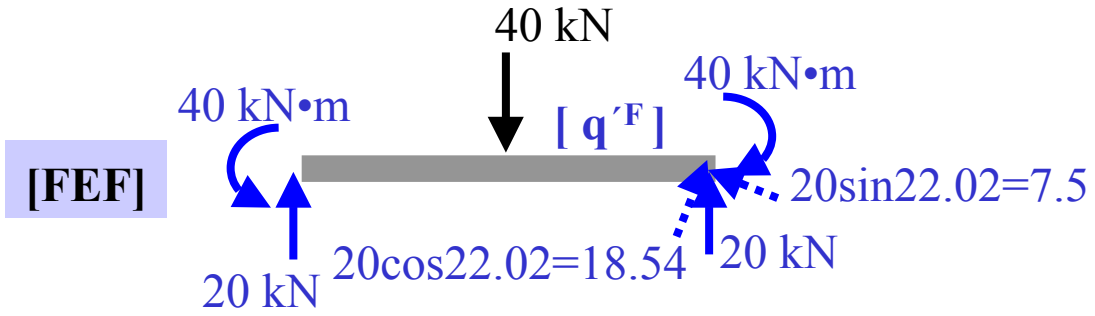




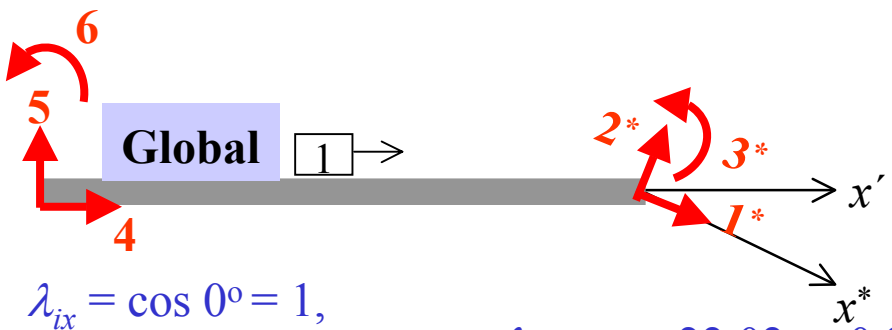
$\lambda_{ix} = \cos 0^o = 1,$
$\lambda_{iy} = \cos 90^o = 0$

$\lambda_{jx} = \cos 22.02^o = 0.9271,$
$\lambda_{jy} = \cos 67.98^o = 0.3749$



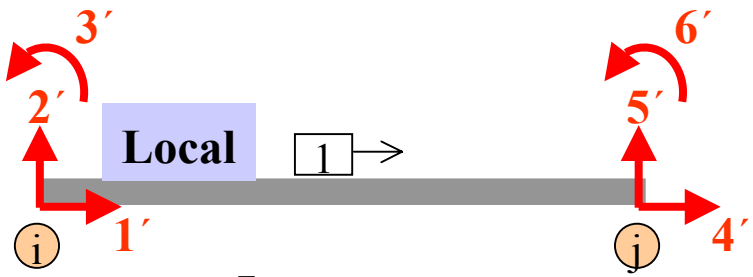


• **Transformation matrix**



$\lambda_{ix} = \cos 0^{o} = 1,$
$\lambda_{iy} = \cos 90^{o} = 0$

$\lambda_{jx} = \cos 22.02^{o} = 0.9271,$
$\lambda_{jy} = \cos 67.98^{o} = 0.3749$

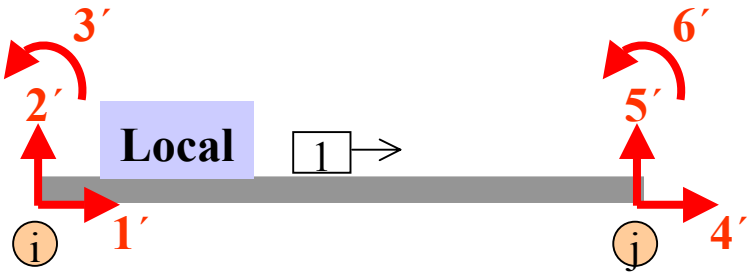


$$[T]^{T} = \begin{bmatrix} \lambda_{ix} & -\lambda_{iy} & 0 & 0 & 0 & 0 \\ \lambda_{iy} & \lambda_{ix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{jx} & -\lambda_{jy} & 0 \\ 0 & 0 & 0 & \lambda_{jy} & \lambda_{jx} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Member 1: $[q] = [T]^{T}[q']$

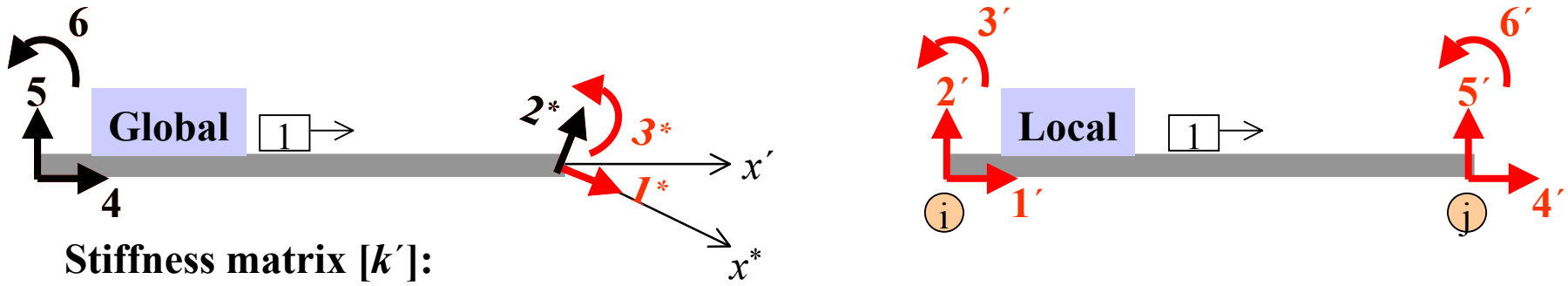
$$\begin{bmatrix} q_{4} \\ q_{5} \\ q_{6} \\ q_{1^*} \\ q_{2^*} \\ q_{3^*} \end{bmatrix} = \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1^* \\ 2^* \\ 3^* \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9271 & -0.3749 & 0 \\ 0 & 0 & 0 & 0.3749 & 0.9271 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{bmatrix}$$

• **Local stiffness matrix**



$$[k]_{6\times 6} = \begin{array}{c} \\ N_i \\ V_i \\ M_i \\ N_j \\ V_j \\ M_j \end{array} \begin{array}{c} \begin{array}{cccccc} \delta_i & \Delta_i & \theta_i & \delta_j & \Delta_j & \theta_j \end{array} \\ \begin{bmatrix} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix} \end{array}$$

$$[k']_1 = 10^3 \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{pmatrix} 150.0 & 0.000 & 0.000 & -150.0 & 0.000 & 0.000 \\ 0.000 & 0.9375 & 3.750 & 0.000 & -0.9375 & 3.750 \\ 0.000 & 3.750 & 20.00 & 0.000 & -3.750 & 10.00 \\ -150.0 & 0.000 & 0.000 & 150.0 & 0.000 & 0.000 \\ 0.000 & -0.9375 & -3.750 & 0.000 & 0.9375 & -3.750 \\ 0.000 & 3.750 & 10.00 & 0.000 & -3.750 & 20.00 \end{pmatrix} \end{array}$$

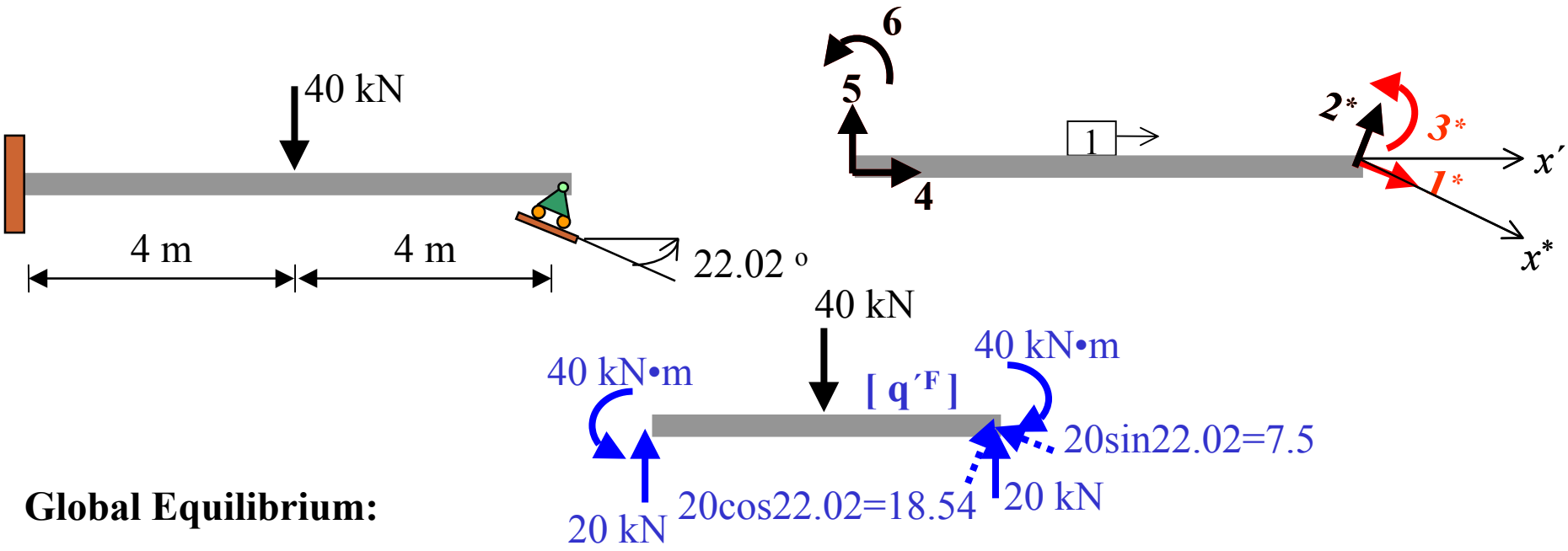


Stiffness matrix [k']:

$$[k']_1 = 10^3 \begin{array}{c} \\ 1' \\ 2' \\ 3' \\ 4' \\ 5' \\ 6' \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{pmatrix} 150.0 & 0.000 & 0.000 & -150.0 & 0.000 & 0.000 \\ 0.000 & 0.9375 & 3.750 & 0.000 & -0.9375 & 3.750 \\ 0.000 & 3.750 & 20.00 & 0.000 & -3.750 & 10.00 \\ -150.0 & 0.000 & 0.000 & 150.0 & 0.000 & 0.000 \\ 0.000 & -0.9375 & -3.750 & 0.000 & 0.9375 & -3.750 \\ 0.000 & 3.750 & 10.00 & 0.000 & -3.750 & 20.00 \end{pmatrix} \end{array}$$

Stiffness matrix [k^*]: $[k^*] = [T]^T[k'][T]$

$$[k^*]_1 = 10^3 \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1^* \\ 2^* \\ 3^* \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1^* & 2^* & 3^* \end{array} \\ \begin{pmatrix} 150.0 & 0.000 & 0.000 & -139.0 & -56.25 & 0.000 \\ 0.000 & 0.9375 & 3.750 & 0.351 & -0.869 & 3.750 \\ 0.000 & 3.750 & 20.00 & 1.406 & -3.750 & 10.00 \\ -139.0 & 0.351 & 1.406 & 129.0 & 51.82 & 1.406 \\ -56.25 & -0.869 & -3.750 & 51.82 & 21.90 & -3.476 \\ 0.000 & 3.750 & 10.00 & 1.406 & -3.476 & 20.00 \end{pmatrix} \end{array}$$

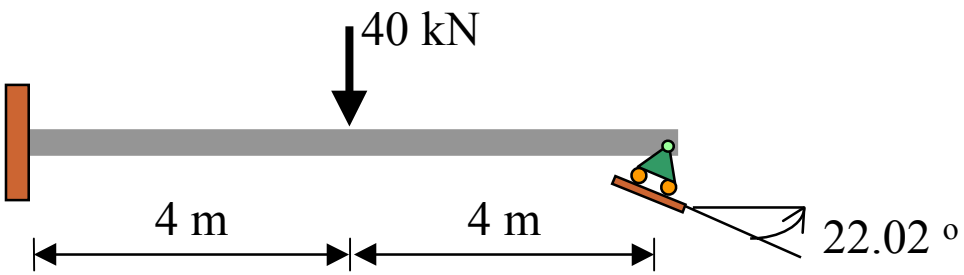


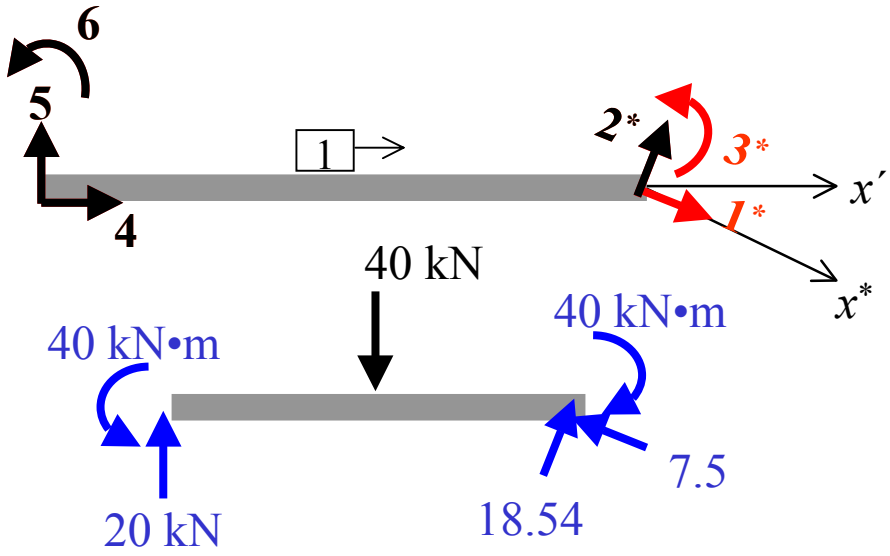
Global Equilibrium:

$$[Q] = [K][D] + [Q^F]$$

$$\begin{pmatrix} Q_1 = 0.0 \\ Q_3 = 0.0 \end{pmatrix} = 10^3 \begin{matrix} 1^* \\ 3^* \end{matrix} \begin{pmatrix} 129 & 1.406 \\ 1.406 & 20.0 \end{pmatrix} \begin{pmatrix} D_{1^*} \\ D_{3^*} \end{pmatrix} + \begin{pmatrix} -7.5 \\ -40 \end{pmatrix}$$

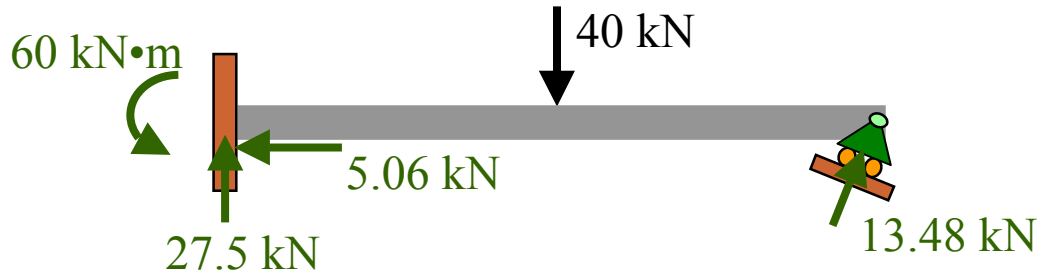
$$\begin{pmatrix} D_{1^*} \\ D_{3^*} \end{pmatrix} = \begin{pmatrix} 36.37\text{x}10^{-6} & \text{m} \\ 0.002 & \text{rad} \end{pmatrix}$$

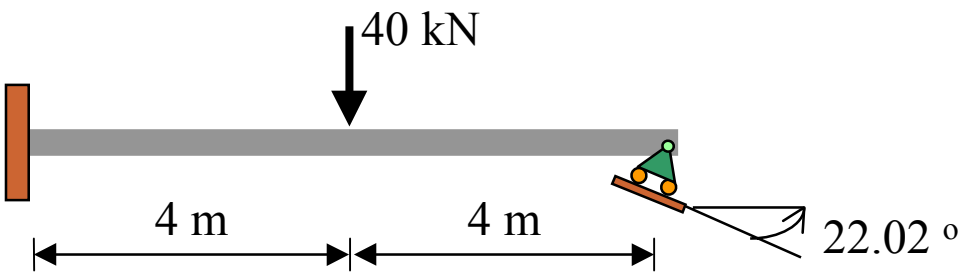


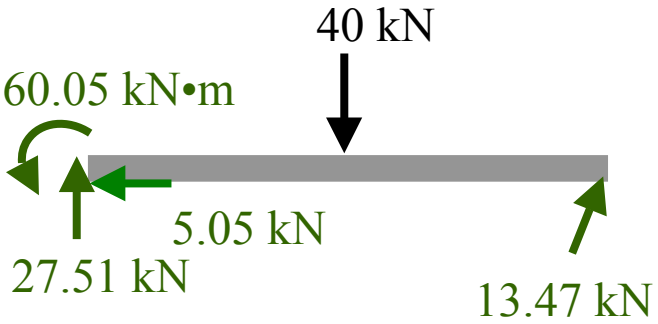


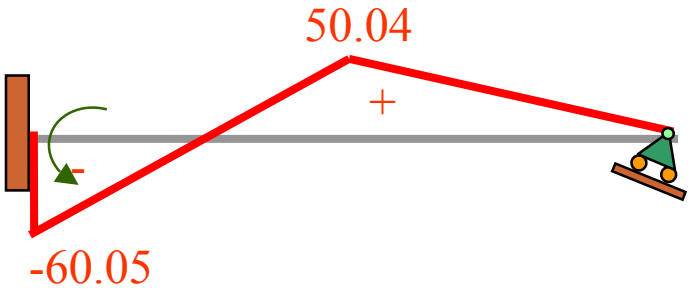
Member Force : $[q] = [k^*][D] + [q^F]$

$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_{1^*} \\ q_{2^*} \\ q_{3^*} \end{pmatrix} = 10^3 \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1^* \\ 2^* \\ 3^* \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1^* & 2^* & 3^* \end{array} \\ \begin{pmatrix} 150.0 & 0.000 & 0.000 & -139.0 & -56.25 & 0.000 \\ 0.000 & 0.9375 & 3.750 & 0.351 & -0.869 & 3.750 \\ 0.000 & 3.750 & 20.00 & 1.406 & -3.750 & 10.00 \\ -139.0 & 0.351 & 1.406 & 129.0 & 51.82 & 1.406 \\ -56.25 & -0.869 & -3.750 & 51.82 & 21.90 & -3.476 \\ 0.000 & 3.750 & 10.00 & 1.406 & -3.476 & 20.00 \end{pmatrix} \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 36.4\text{x}10^{-6} \\ 0 \\ 2\text{x}10^{-3} \end{pmatrix} + \begin{pmatrix} 0 \\ 20 \\ 40.0 \\ -7.54 \\ 18.54 \\ -40.0 \end{pmatrix} = \begin{pmatrix} -5.06 \\ 27.5 \\ 60 \\ 0 \\ 13.48 \\ 0 \end{pmatrix}$$

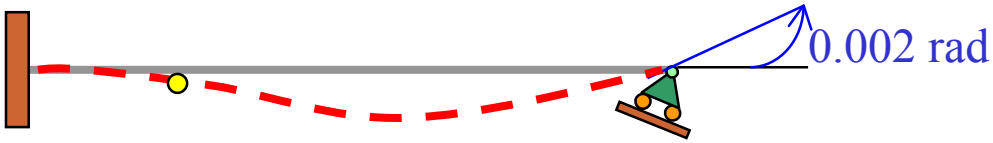








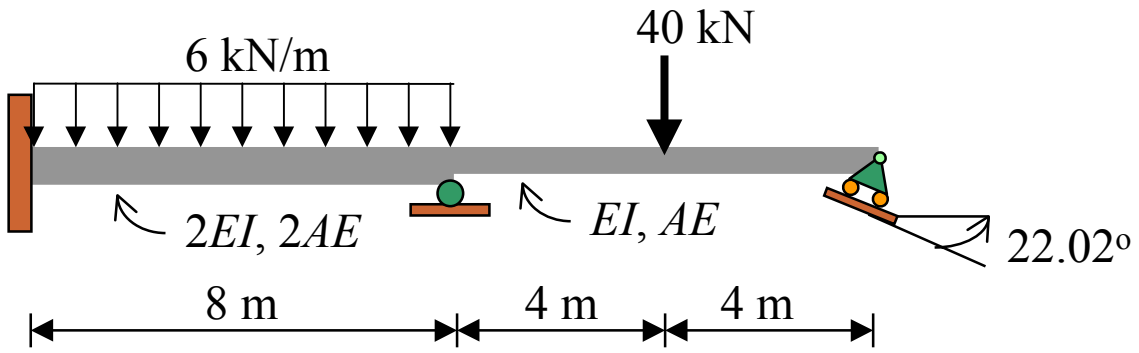
Bending moment diagram (kN•m)

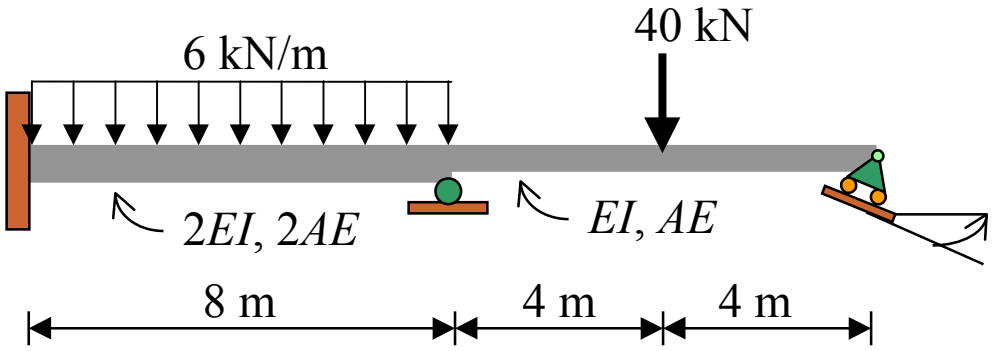


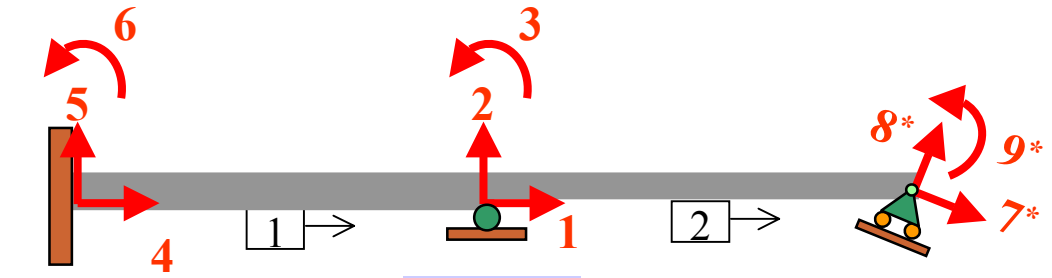
Deflected shape

Example 13

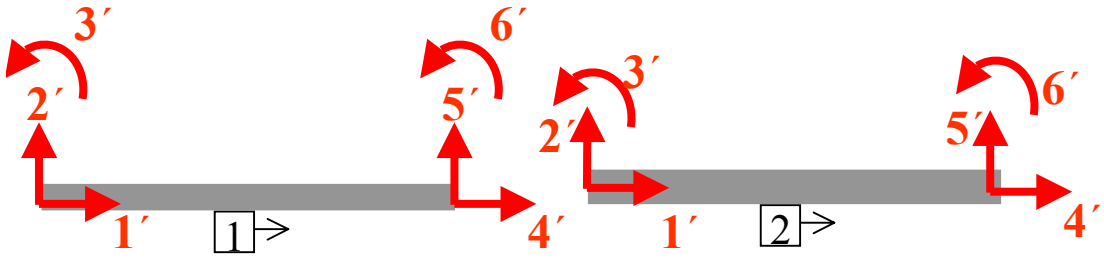
For the beam shown:
(a) Use the stiffness method to determine all the **reactions** at supports.
(b) Draw the **quantitative free-body diagram** of member.
(c) Draw the **quantitative bending moment diagrams**
and **qualitative deflected shape.**
Take $I = 200(10^6)$ mm^4, $A = 6(10^3)$ mm^2, and $E = 200$ GPa for all members.
Include axial deformation in the stiffness matrix.







Global



Local

$$\frac{AE}{L} = \frac{(0.006\ m^2)(200\times10^6\ kN/m^2)}{(8\ m)}$$

$$= 150\times10^3\ kN \bullet m$$

$$\frac{4EI}{L} = \frac{4(200\times10^6\ kN/m^2)(0.0002\ m^4)}{(8\ m)}$$

$$= 20\times10^3\ kN \bullet m$$

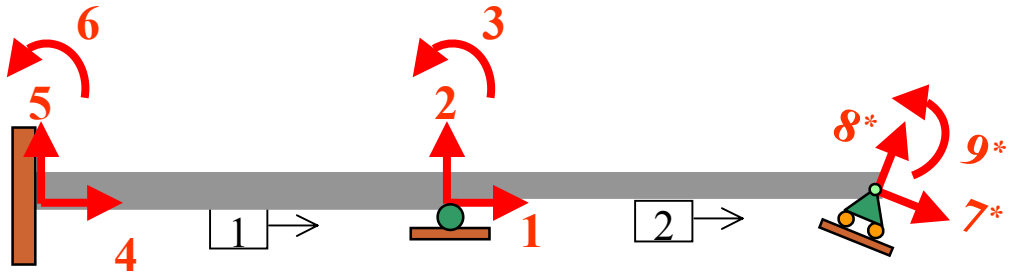
$$\frac{2EI}{L} = 10\times10^3\ kN \bullet m$$

$$\frac{6EI}{L^2} = \frac{6(200\times10^6\ kN/m^2)(0.0002\ m^4)}{(8\ m)^2}$$

$$= 3.75\times10^3\ kN$$

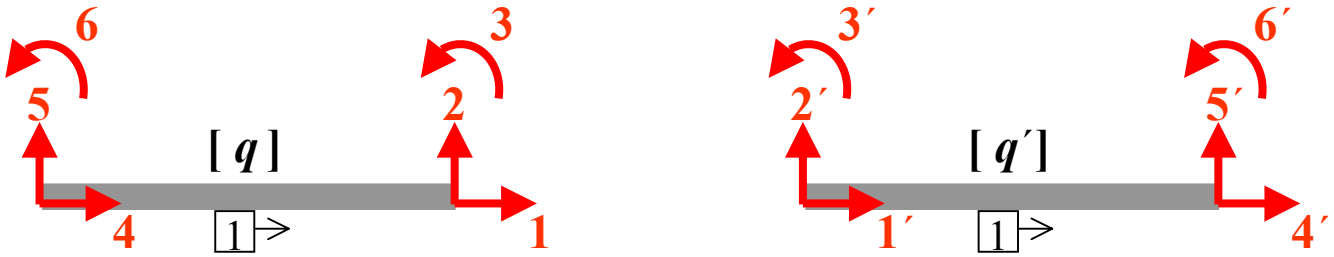
$$\frac{12EI}{L^2} = \frac{12(200\times10^6\ kN/m^2)(0.0002\ m^4)}{(8\ m)^3}$$

$$= 0.9375\times10^3\ kN/m$$



Global

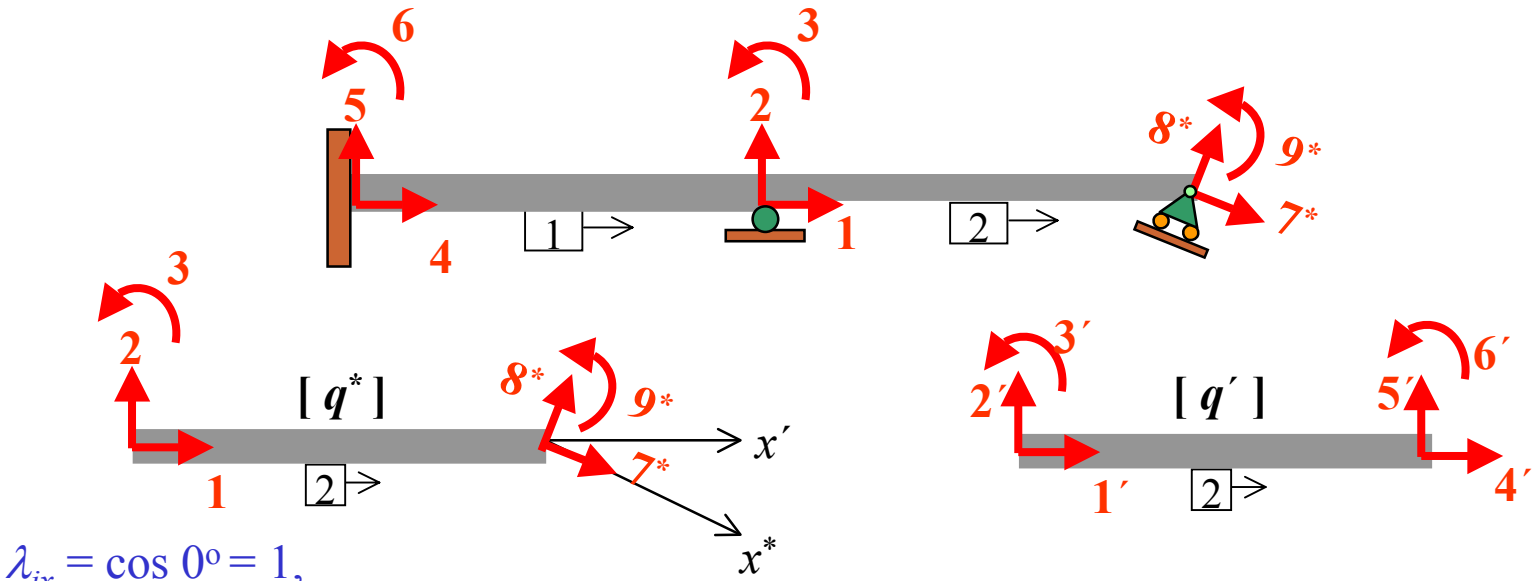
Local : Member 1



$[q] = [q']$

Thus, $[k] = [k']$

$$[k]_1 = [k']_1 = 2\text{x}10^3 \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 150.0 & 0.000 & 0.000 & -150.0 & 0.000 & 0.000 \\ 0.000 & 0.9375 & 3.750 & 0.000 & -0.9375 & 3.750 \\ 0.000 & 3.750 & 20.00 & 0.000 & -3.750 & 10.00 \\ -150.0 & 0.000 & 0.000 & 150.0 & 0.000 & 0.000 \\ 0.000 & -0.9375 & -3.750 & 0.000 & 0.9375 & -3.750 \\ 0.000 & 3.750 & 10.00 & 0.000 & -3.750 & 20.00 \end{pmatrix} \end{array}$$

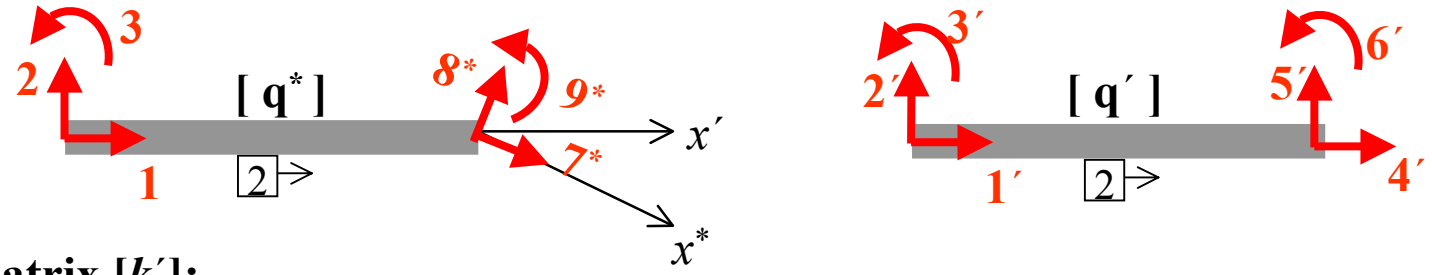


$\lambda_{ix} = \cos 0^{o} = 1,$
$\lambda_{iy} = \cos 90^{o} = 0$

$\lambda_{jx} = \cos 22.02^{o} = 0.9271,$
$\lambda_{jy} = \cos 67.98^{o} = 0.3749$

Member 2: Use transformation matrix, $[q^*] = [T]^T[q']$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_{7*} \\ q_{8*} \\ q_{9*} \end{pmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7^* \\ 8^* \\ 9^* \end{array} \begin{array}{c} \begin{array}{cccccc} 1' & 2' & 3' & 4' & 5' & 6' \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9271 & -0.3749 & 0 \\ 0 & 0 & 0 & 0.3749 & 0.9271 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} q_{1'} \\ q_{2'} \\ q_{3'} \\ q_{4'} \\ q_{5'} \\ q_{6'} \end{pmatrix}$$



Stiffness matrix [k']:

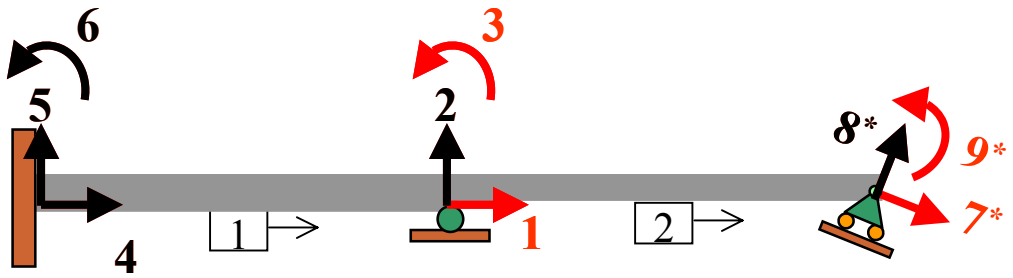
$$[k']_2 = 10^3$$

	1´	2´	3´	4´	5´	6´
1´	150.0	0.000	0.000	-150.0	0.000	0.000
2´	0.000	0.9375	3.750	0.000	-0.9375	3.750
3´	0.000	3.750	20.00	0.000	-3.750	10.00
4´	-150.0	0.000	0.000	150.0	0.000	0.000
5´	0.000	-0.9375	-3.750	0.000	0.9375	-3.750
6´	0.000	3.750	10.00	0.000	-3.750	20.00

Stiffness matrix [k^*]: [k^*] = [T]^T[k'][T]

$$[k^*]_2 = 10^3$$

	1	2	3	7*	8*	9*
1	150.0	0.000	0.000	-139.0	-56.25	0.000
2	0.000	0.9375	3.750	0.351	-0.869	3.750
3	0.000	3.750	20.00	1.406	-3.477	10.00
7*	-139.0	0.351	1.406	129.0	51.82	1.406
8*	-56.25	-0.869	-3.477	51.82	21.90	-3.476
9*	0.000	3.750	10.00	1.406	-3.476	20.00

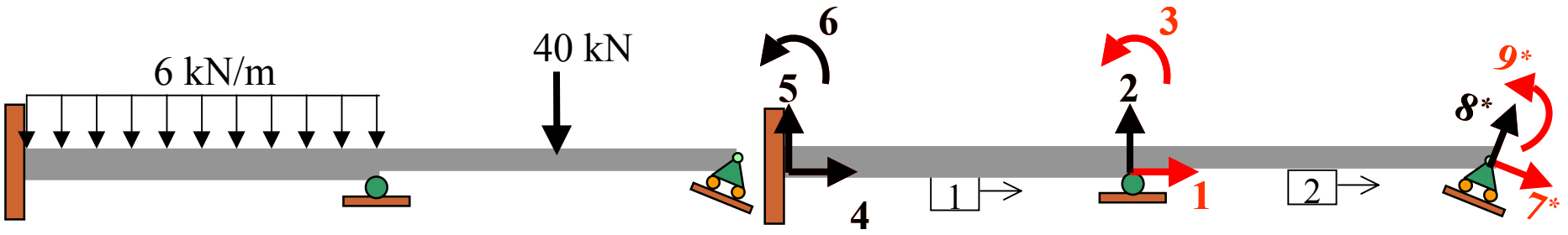


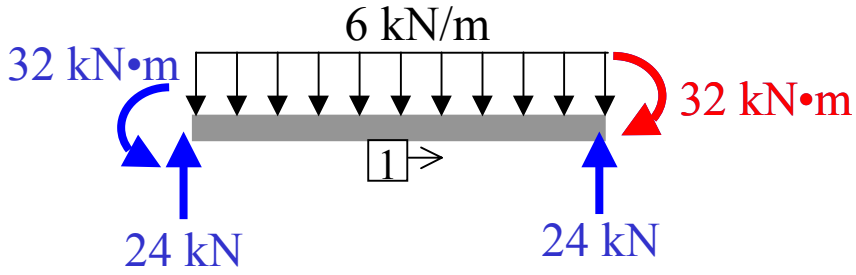
$[k]_1 = 2\text{x}10^3$

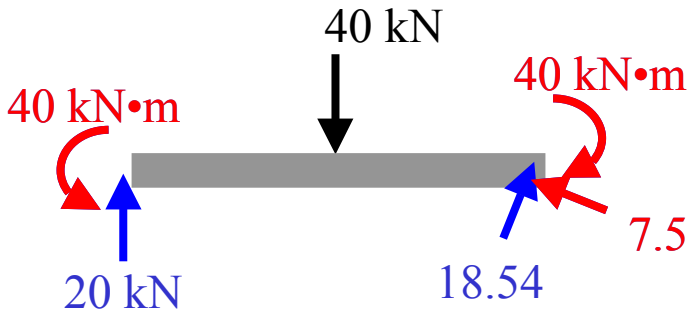
	4	5	6	1	2	3
4	150.0	0.000	0.000	-150.0	0.000	0.000
5	0.000	0.9375	3.750	0.000	-0.9375	3.750
6	0.000	3.750	20.00	0.000	-3.750	10.00
1	-150.0	0.000	0.000	150.0	0.000	0.000
2	0.000	-0.9375	-3.750	0.000	0.9375	-3.750
3	0.000	3.750	10.00	0.000	-3.750	20.00

$[k^*]_2 = 10^3$

	1	2	3	7*	8*	9*
1	150.0	0.000	0.000	-139.0	-56.25	0.000
2	0.000	0.9375	3.750	0.351	-0.869	3.750
3	0.000	3.750	20.00	1.406	-3.477	10.00
7*	-139.0	0.351	1.406	129.0	51.82	1.406
8*	-56.25	-0.869	-3.477	51.82	21.90	-3.476
9*	0.000	3.750	10.00	1.406	-3.476	20.00



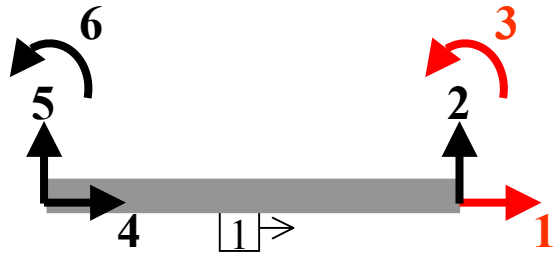


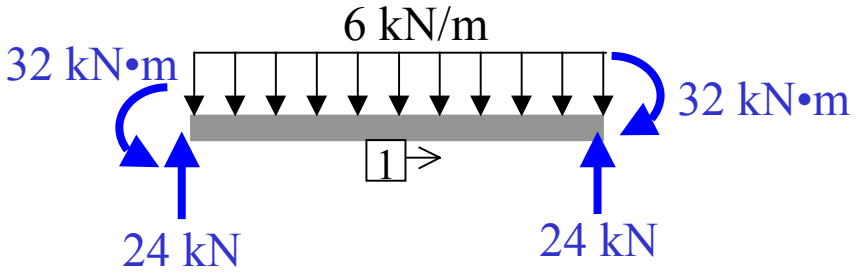


Global:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 10^3 \begin{array}{c} \\ 1 \\ 3 \\ 7^* \\ 9^* \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 3 & 7^* & 9^* \end{array} \\ \begin{pmatrix} 450 & 0 & -139 & 0 \\ 0 & 60 & 1.406 & 10 \\ -139 & 1.406 & 129 & 1.406 \\ 0 & 10 & 1.406 & 20 \end{pmatrix} \end{array} \begin{pmatrix} D_1 \\ D_3 \\ D_{7*} \\ D_{9*} \end{pmatrix} + \begin{pmatrix} 0 \\ -32 + 40 = 8 \\ -7.5 \\ -40 \end{pmatrix}$$

$$\begin{pmatrix} D_1 \\ D_3 \\ D_{7*} \\ D_{9*} \end{pmatrix} = \begin{pmatrix} 18.15\text{x}10^{-6} & \text{m} \\ -509.84\text{x}10^{-6} & \text{rad} \\ 58.73\text{x}10^{-6} & \text{m} \\ 0.00225 & \text{rad} \end{pmatrix}$$

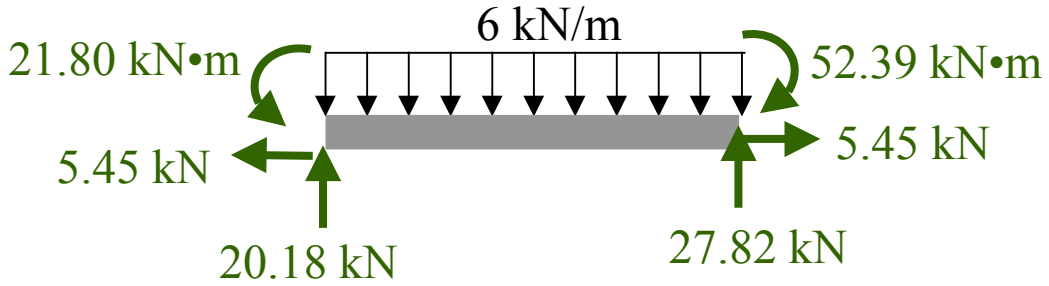


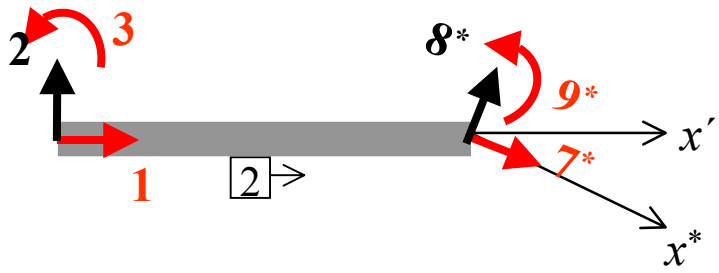


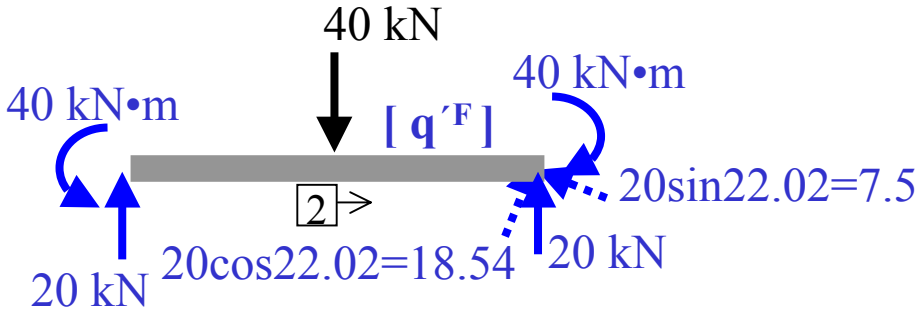
Member 1: $[q] = [k][d] + [q^F]$

$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = 2\text{x}10^3 \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 150.0 & 0.000 & 0.000 & -150.0 & 0.000 & 0.000 \\ 0.000 & 0.9375 & 3.750 & 0.000 & -0.9375 & 3.750 \\ 0.000 & 3.750 & 20.00 & 0.000 & -3.750 & 10.00 \\ -150.0 & 0.000 & 0.000 & 150.0 & 0.000 & 0.000 \\ 0.000 & -0.9375 & -3.750 & 0.000 & 0.9375 & -3.750 \\ 0.000 & 3.750 & 10.00 & 0.000 & -3.750 & 20.00 \end{pmatrix} \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ d_1=18.15\text{x}10^{-6} \\ 0 \\ d_3=-509.84\ \text{x}10^{-6} \end{pmatrix} + \begin{pmatrix} 0 \\ 24 \\ 32 \\ 0 \\ 24 \\ -32 \end{pmatrix}$$

$$\begin{pmatrix} q_4 \\ q_5 \\ q_6 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} -5.45 & \text{kN} \\ 20.18 & \text{kN} \\ 21.80 & \text{kN•m} \\ 5.45 & \text{kN} \\ 27.82 & \text{kN} \\ -52.39 & \text{kN•m} \end{pmatrix}$$



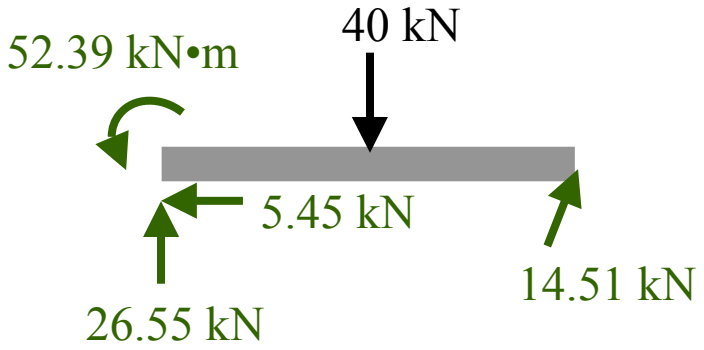


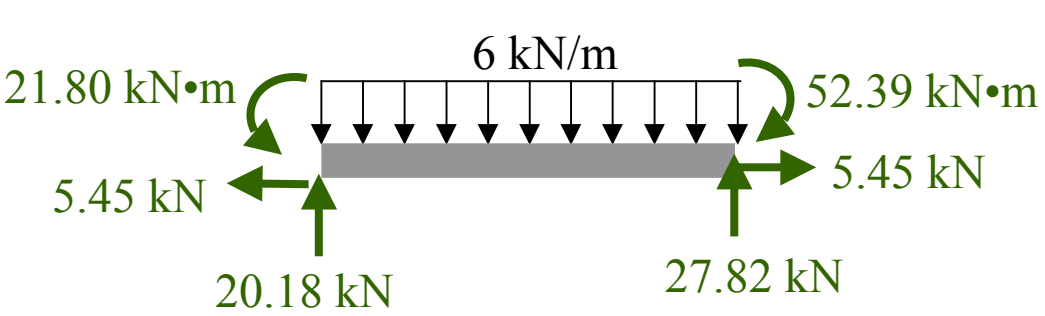


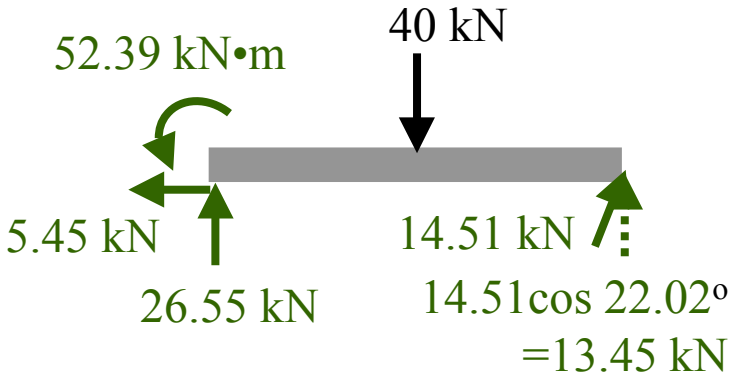
Member 2: $[q] = [k][d] + [q^F]$

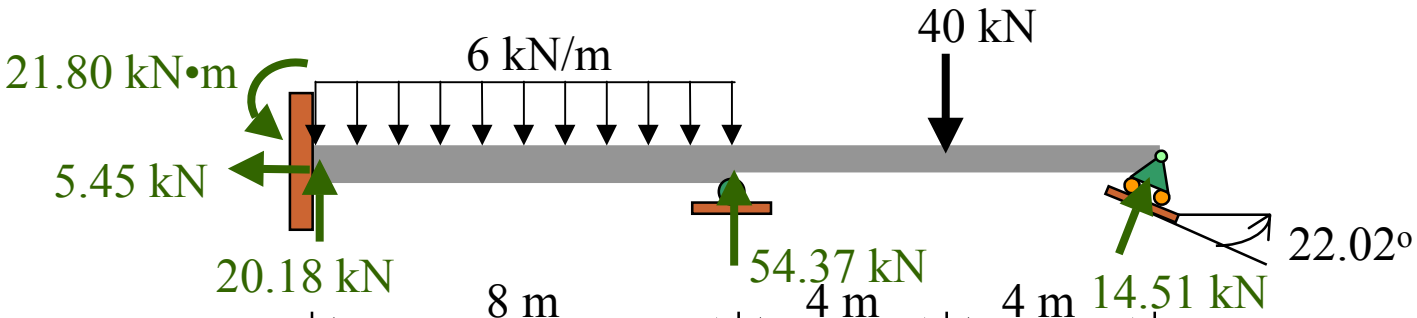
$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_{7*} \\ q_{8*} \\ q_{9*} \end{pmatrix} = 10^3 \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7^* \\ 8^* \\ 9^* \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7^* & 8^* & 9^* \end{array} \\ \begin{pmatrix} 150.0 & 0.000 & 0.000 & -139.0 & -56.25 & 0.000 \\ 0.000 & 0.9375 & 3.750 & 0.351 & -0.869 & 3.750 \\ 0.000 & 3.750 & 20.00 & 1.406 & -3.750 & 10.00 \\ -139.0 & 0.351 & 1.406 & 129.0 & 51.82 & 1.406 \\ -56.25 & -0.869 & -3.750 & 51.82 & 21.90 & -3.476 \\ 0.000 & 3.750 & 10.00 & 1.406 & -3.476 & 20.00 \end{pmatrix} \end{array} \begin{pmatrix} 18.15\text{x}10^{-6} \\ 0 \\ -509.84\text{x}10^{-6} \\ 58.73\text{x}10^{-6} \\ 0 \\ 0.00225 \end{pmatrix} + \begin{pmatrix} 0 \\ 20 \\ 40 \\ -7.5 \\ 18.54 \\ -40 \end{pmatrix}$$

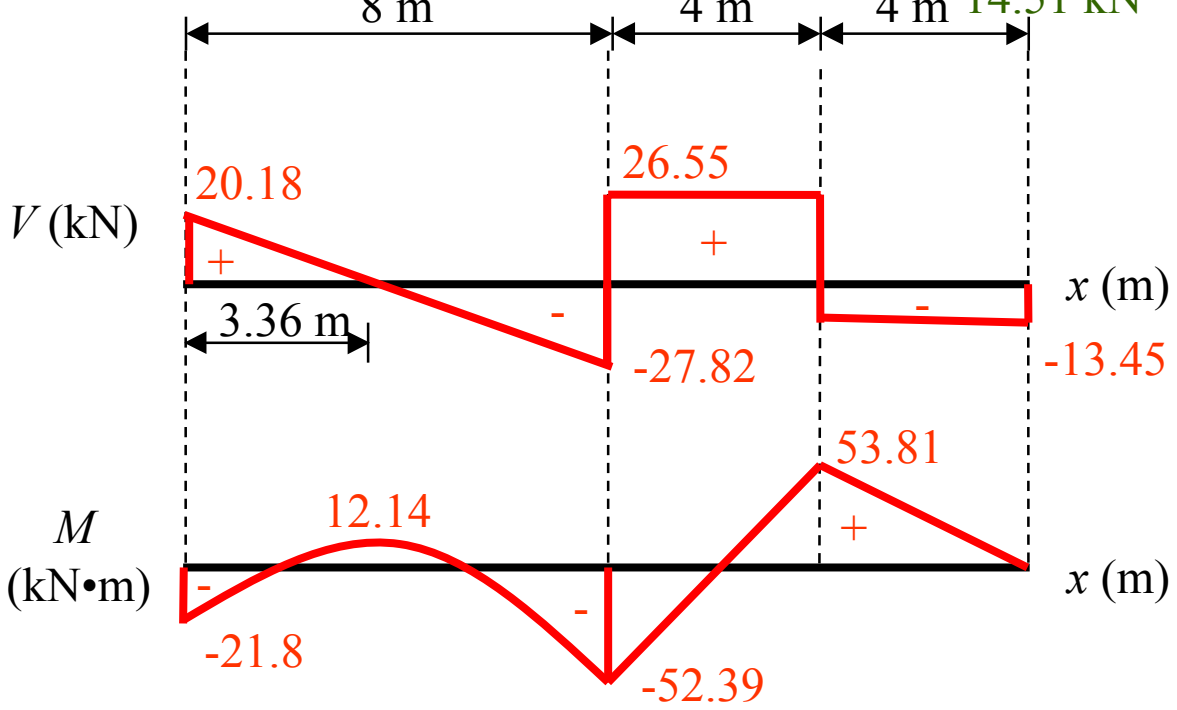
$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_{7*} \\ q_{8*} \\ q_{9*} \end{pmatrix} = \begin{pmatrix} -5.45 & \text{kN} \\ 26.55 & \text{kN} \\ 52.39 & \text{kN•m} \\ 0 & \text{kN} \\ 14.51 & \text{kN} \\ 0 & \text{kN•m} \end{pmatrix}$$




6 kN/m
21.80 kN•m
52.39 kN•m
5.45 kN
5.45 kN
20.18 kN
27.82 kN


40 kN
52.39 kN•m
5.45 kN
14.51 kN
26.55 kN
14.51cos 22.02°
=13.45 kN


40 kN
6 kN/m
21.80 kN•m
5.45 kN
20.18 kN
54.37 kN
22.02°
14.51 kN


8 m
4 m
4 m
V (kN)
20.18
26.55
+
+
-
-
3.36 m
-27.82
-13.45
x (m)
M
(kN•m)
12.14
53.81
+
-
-
-21.8
-52.39
x (m)