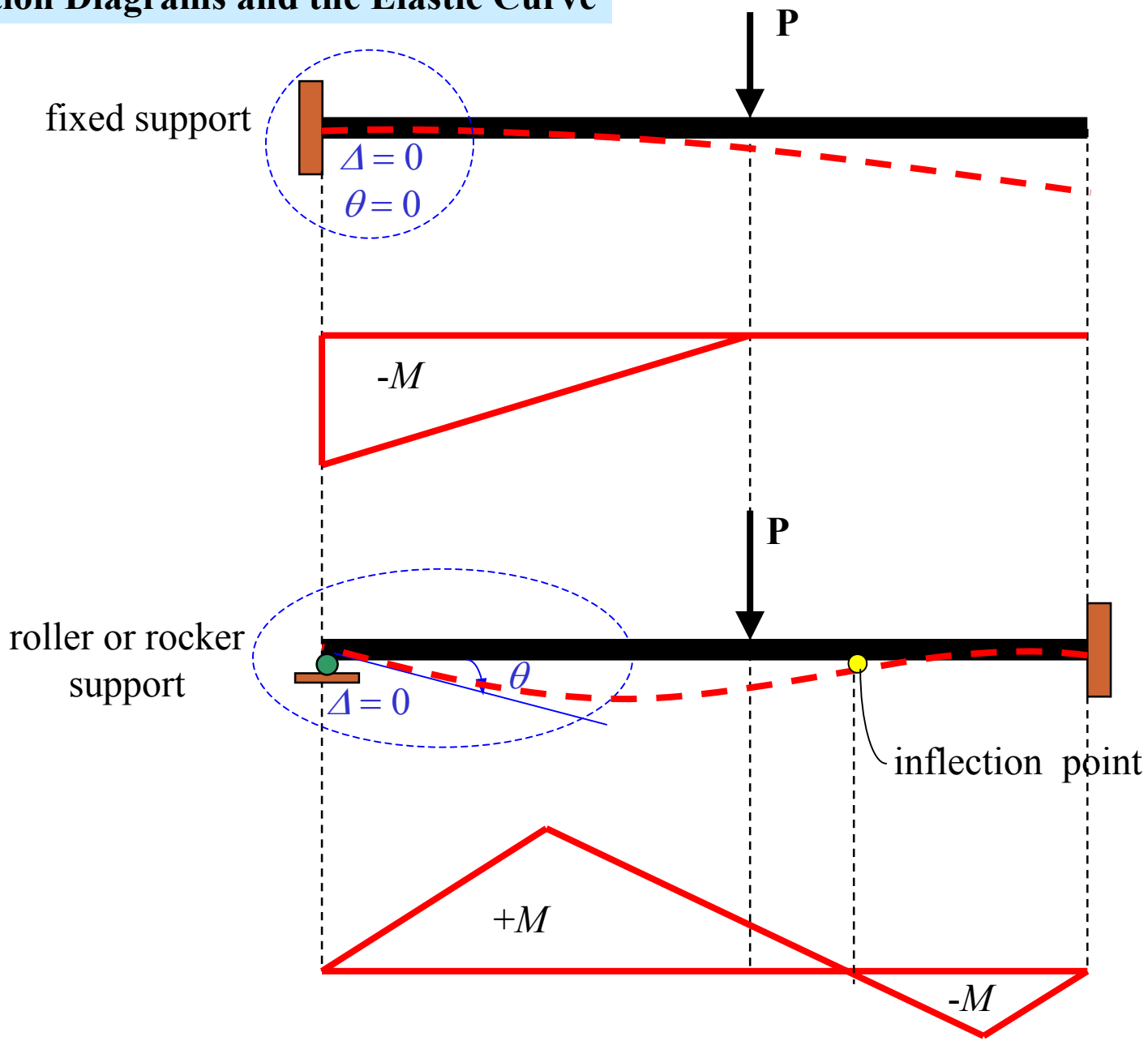
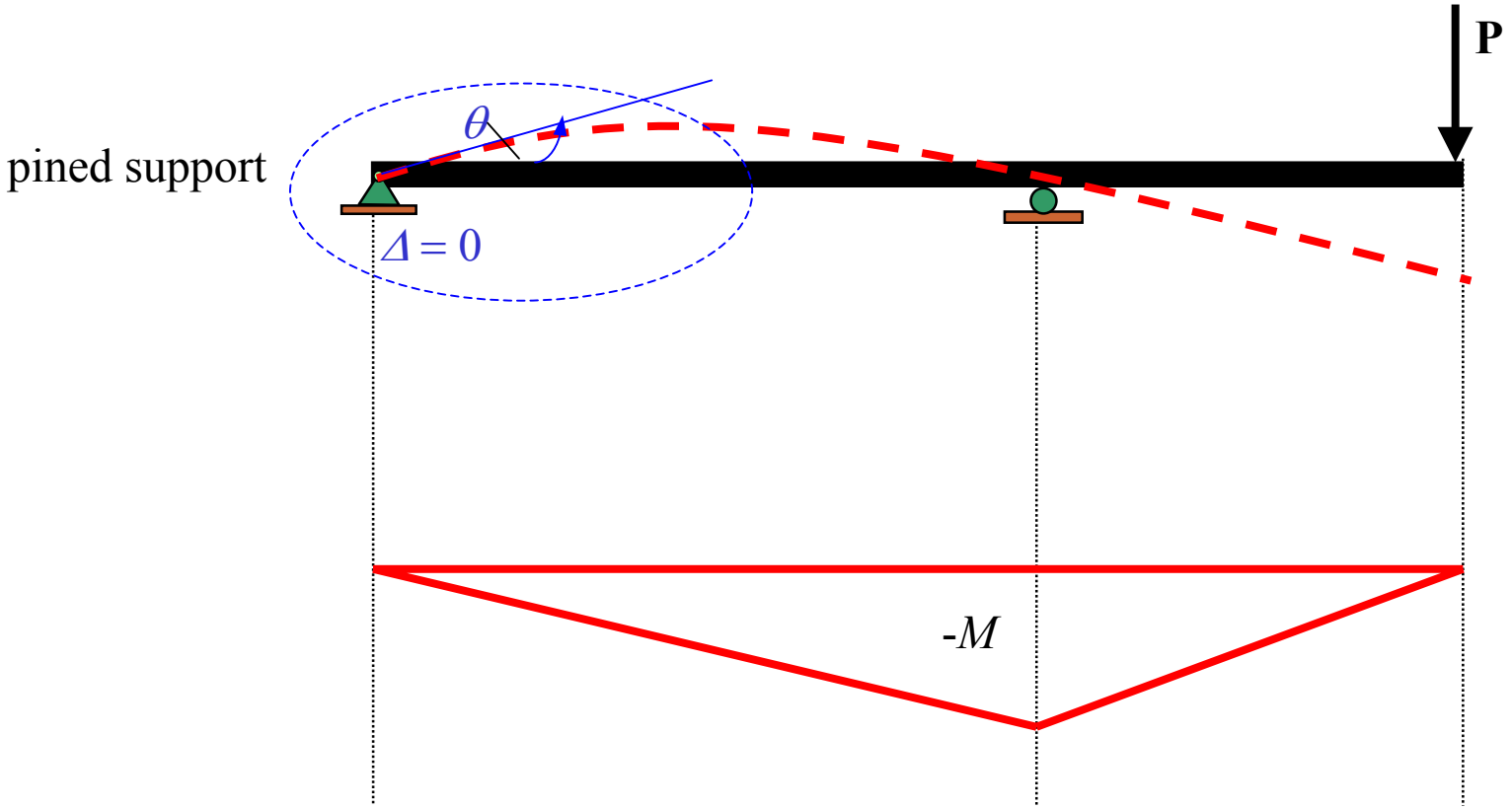
INTRODUCTION: STRUCTURAL ANALYSIS

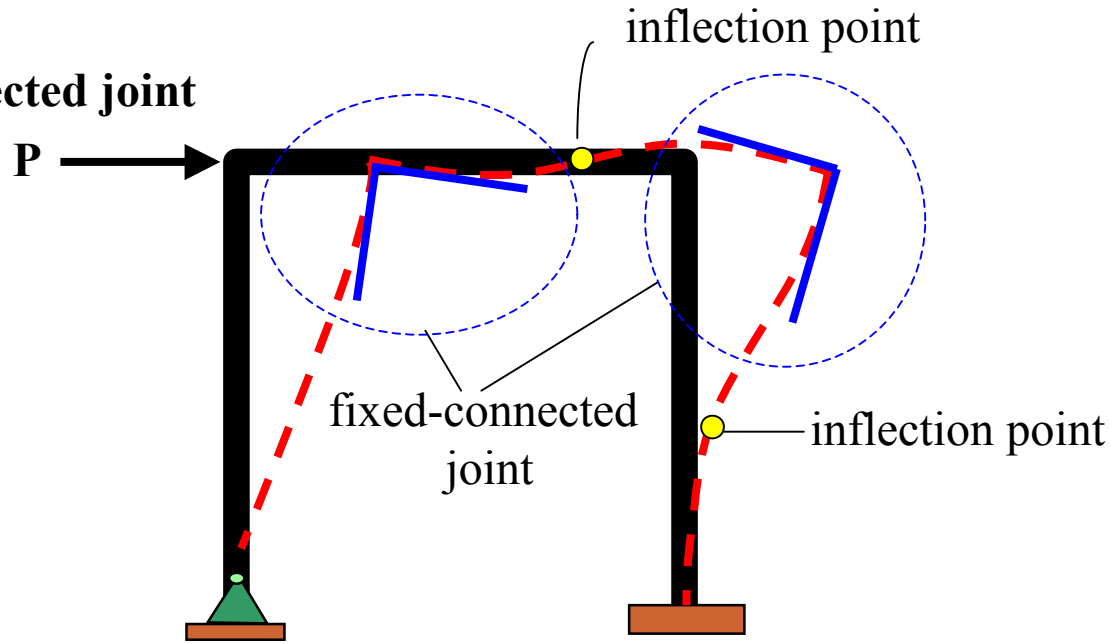
- Deflected Shape of Structures
- Method of Consistent Deformations
- Maxwell's Theorem of Reciprocal Displacement

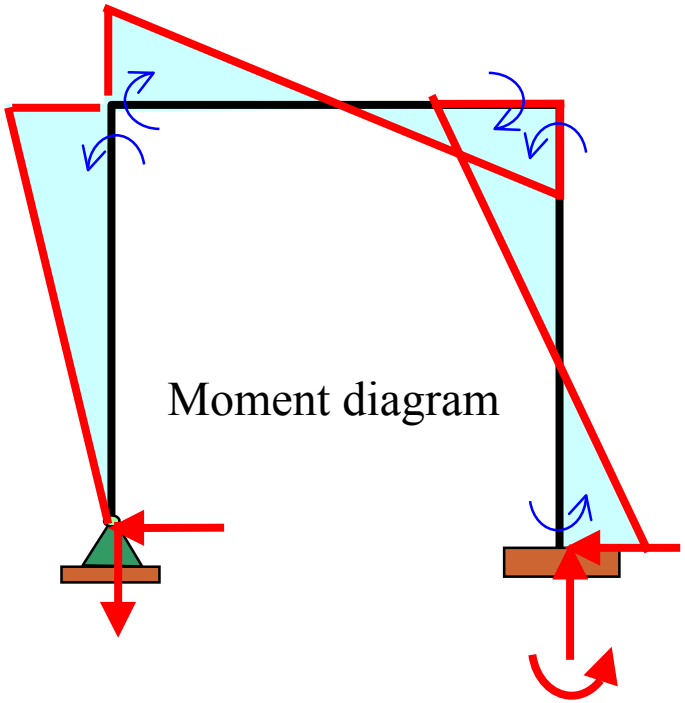
Deflection Diagrams and the Elastic Curve



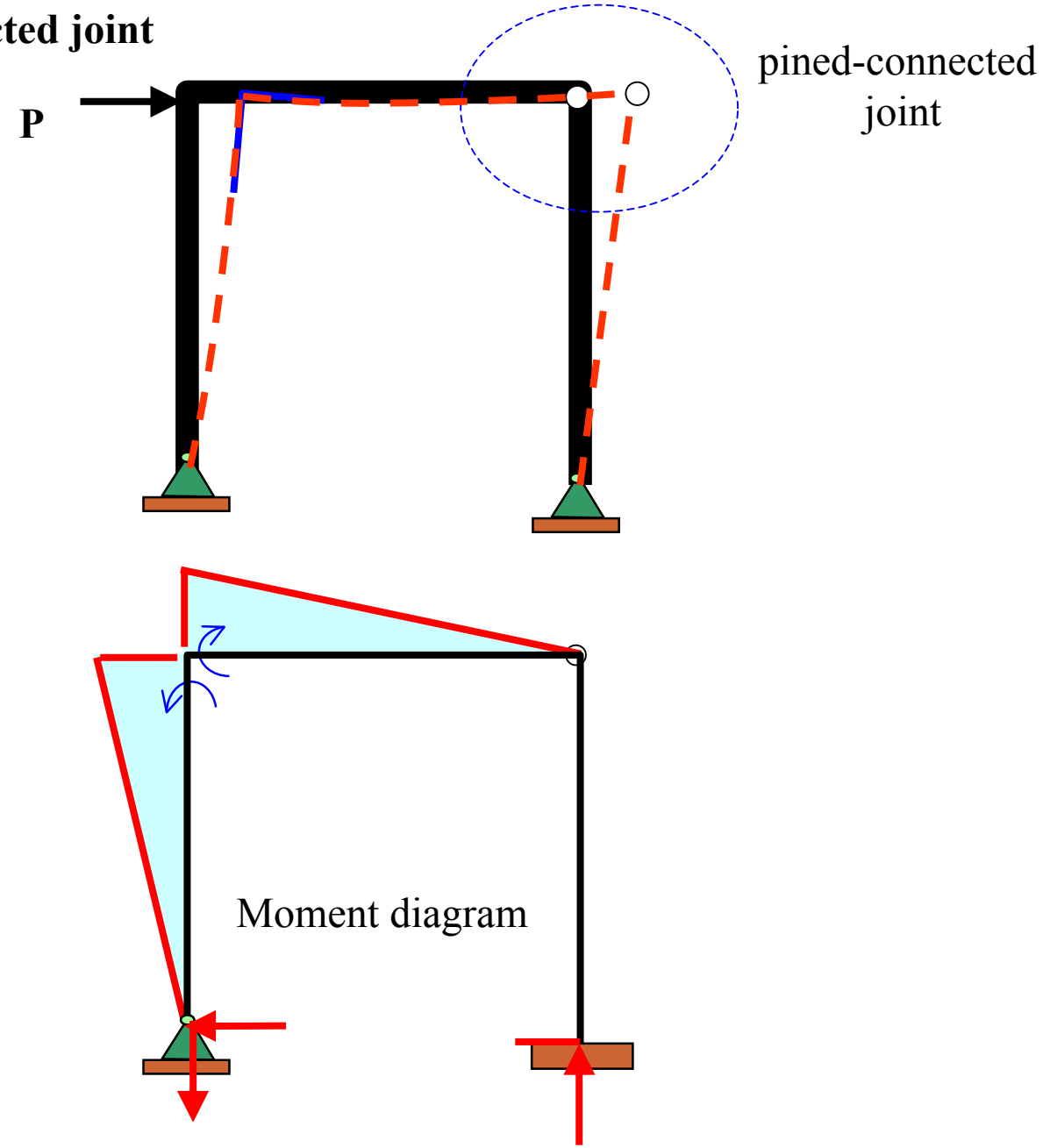

pined support
θ
Δ = 0
P
-M

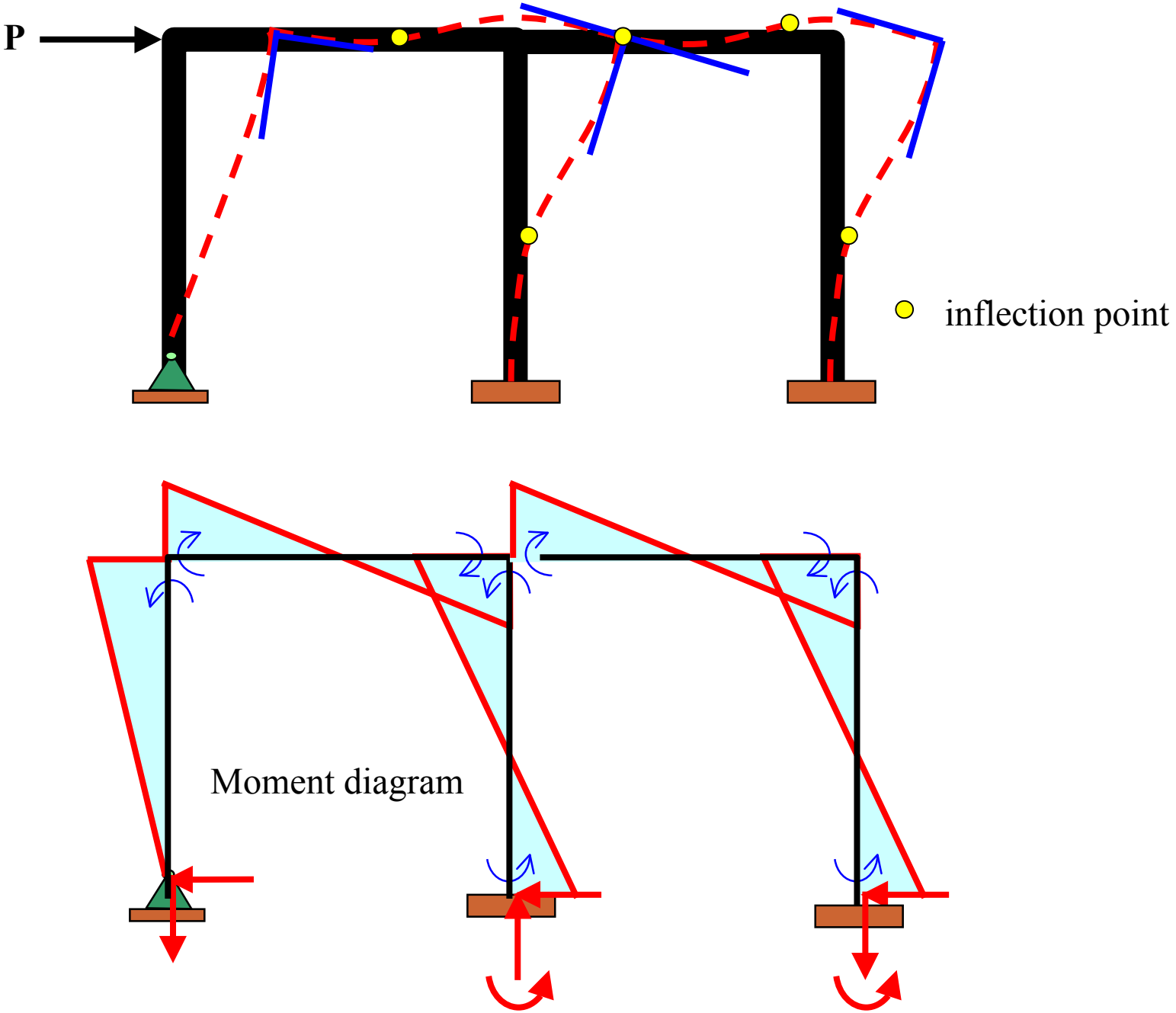
• **Fixed-connected joint**

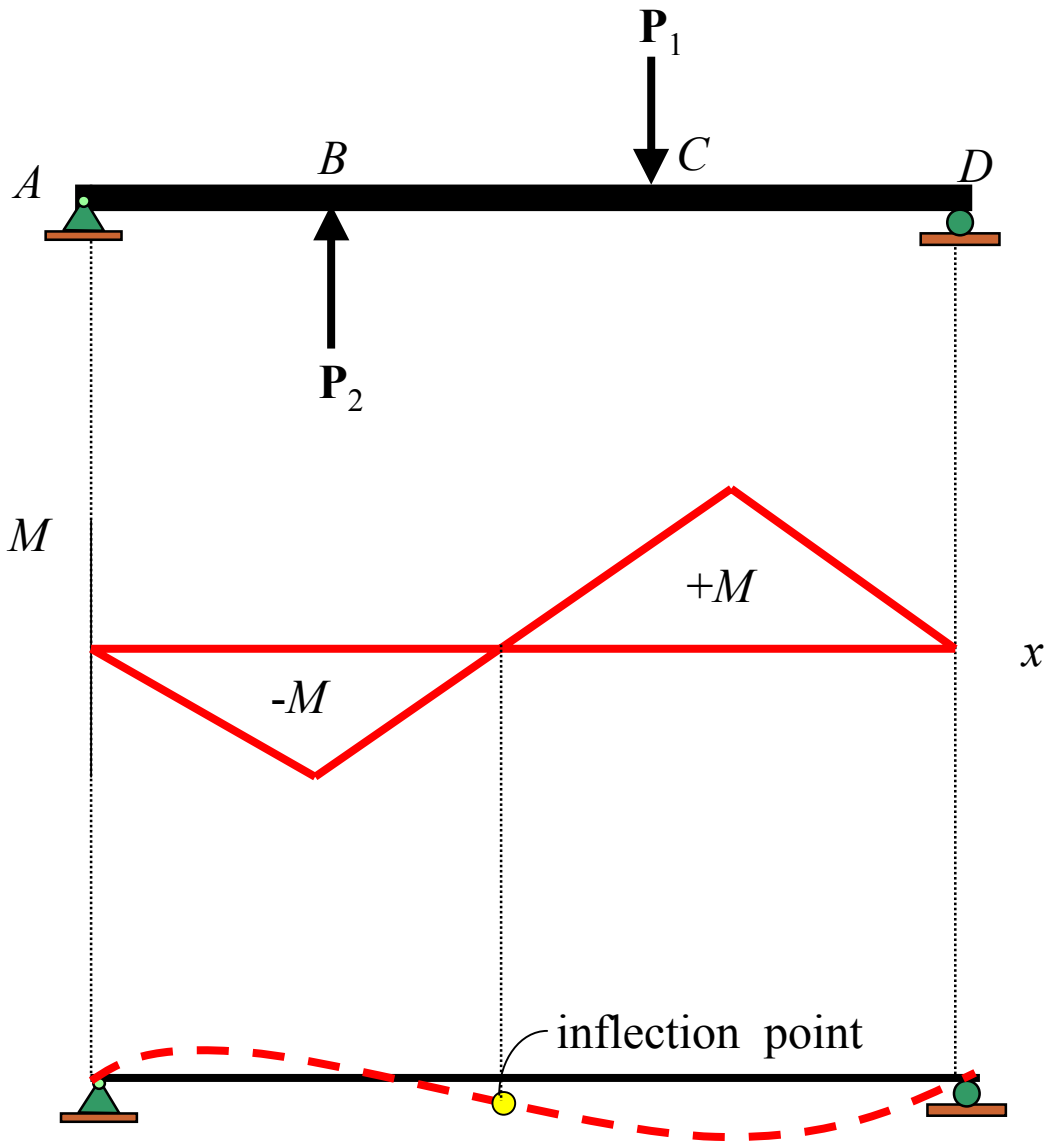


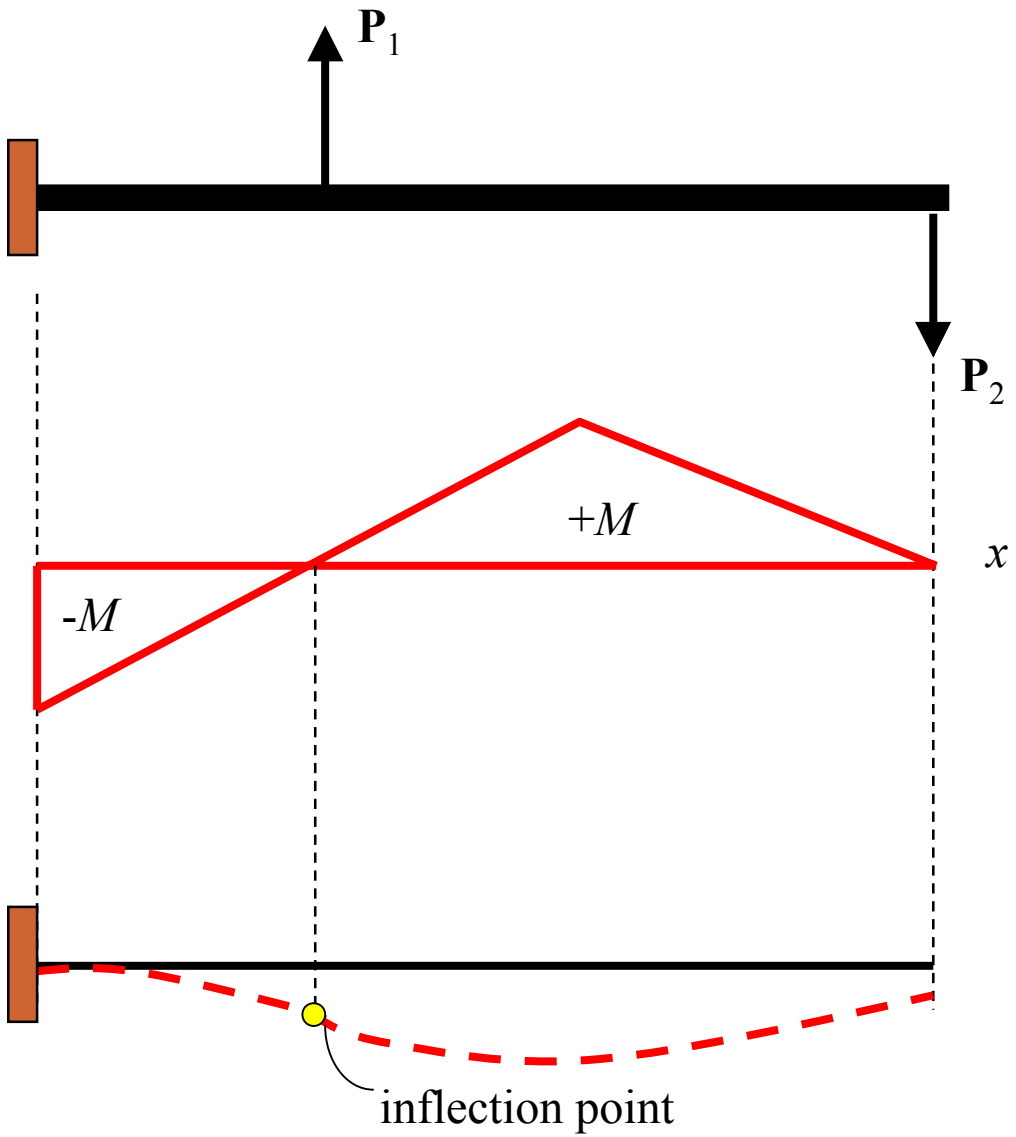


• **Pined-connected joint**



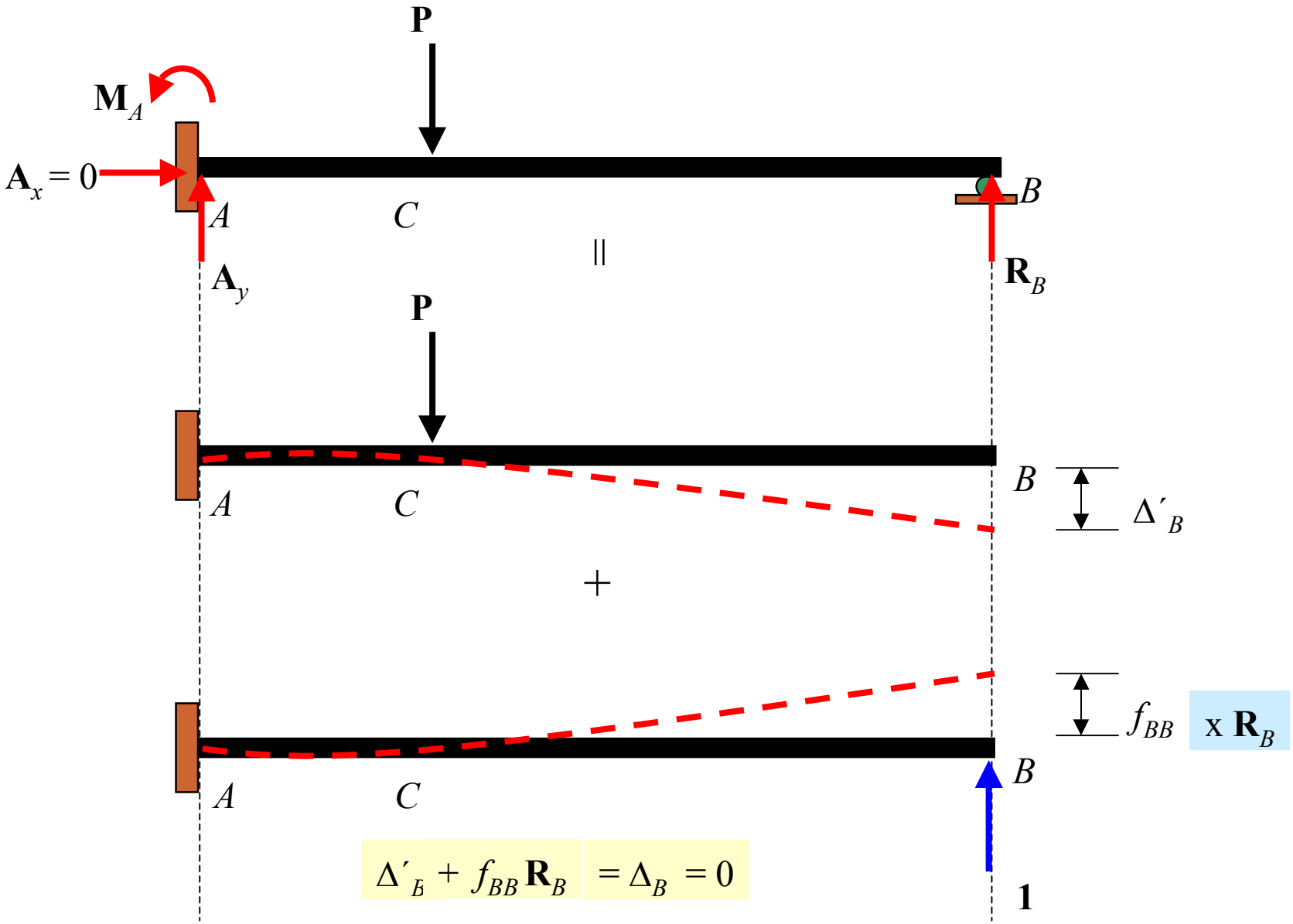

P
inflection point
Moment diagram


P1
A
B
C
D
P2
M
+M
-M
x
inflection point

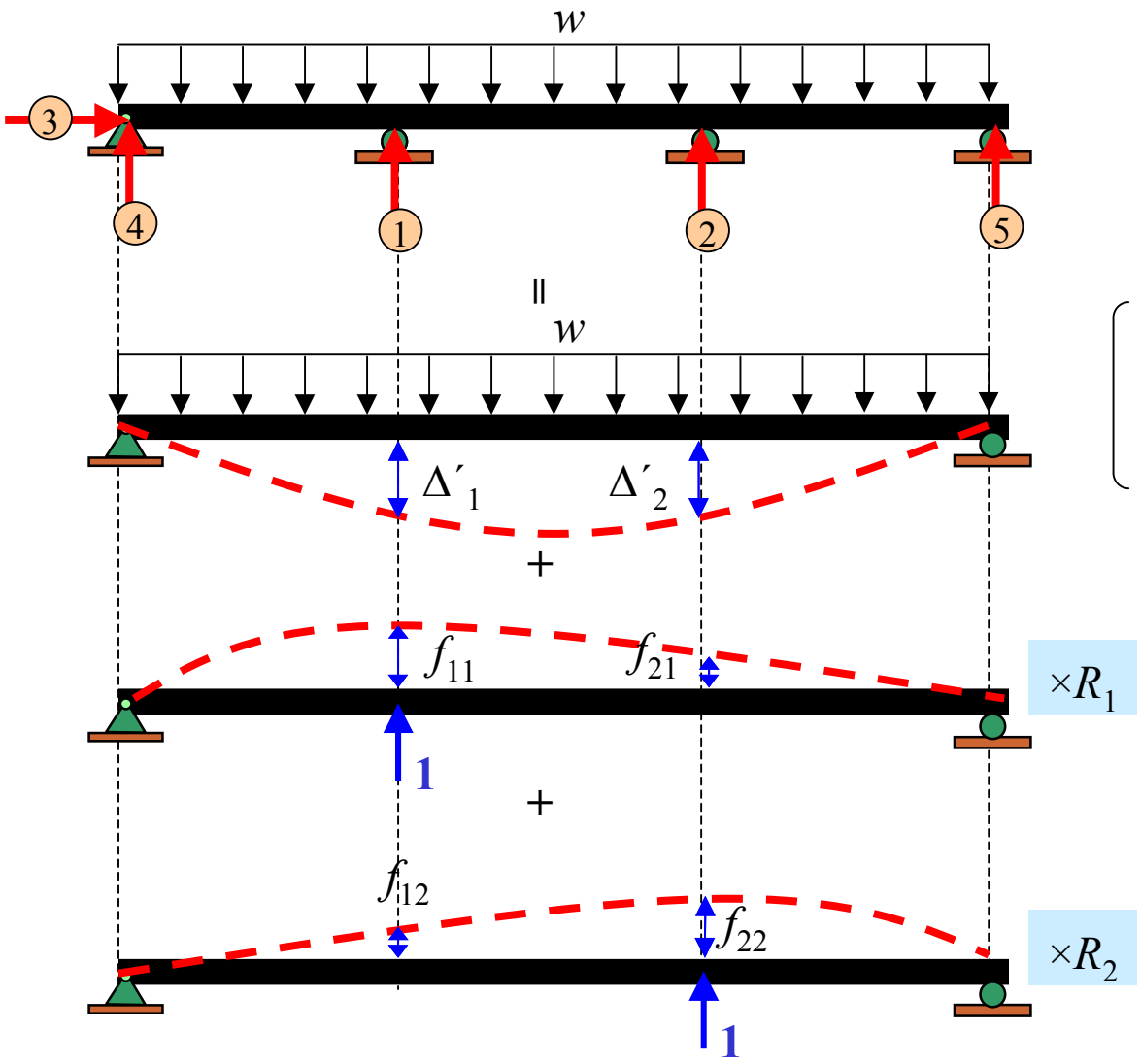

P1
P2
+M
-M
x
inflection point

Method of Consistent Deformations

- **Beam 1 DOF**



▶ **Beam 2 DOF**



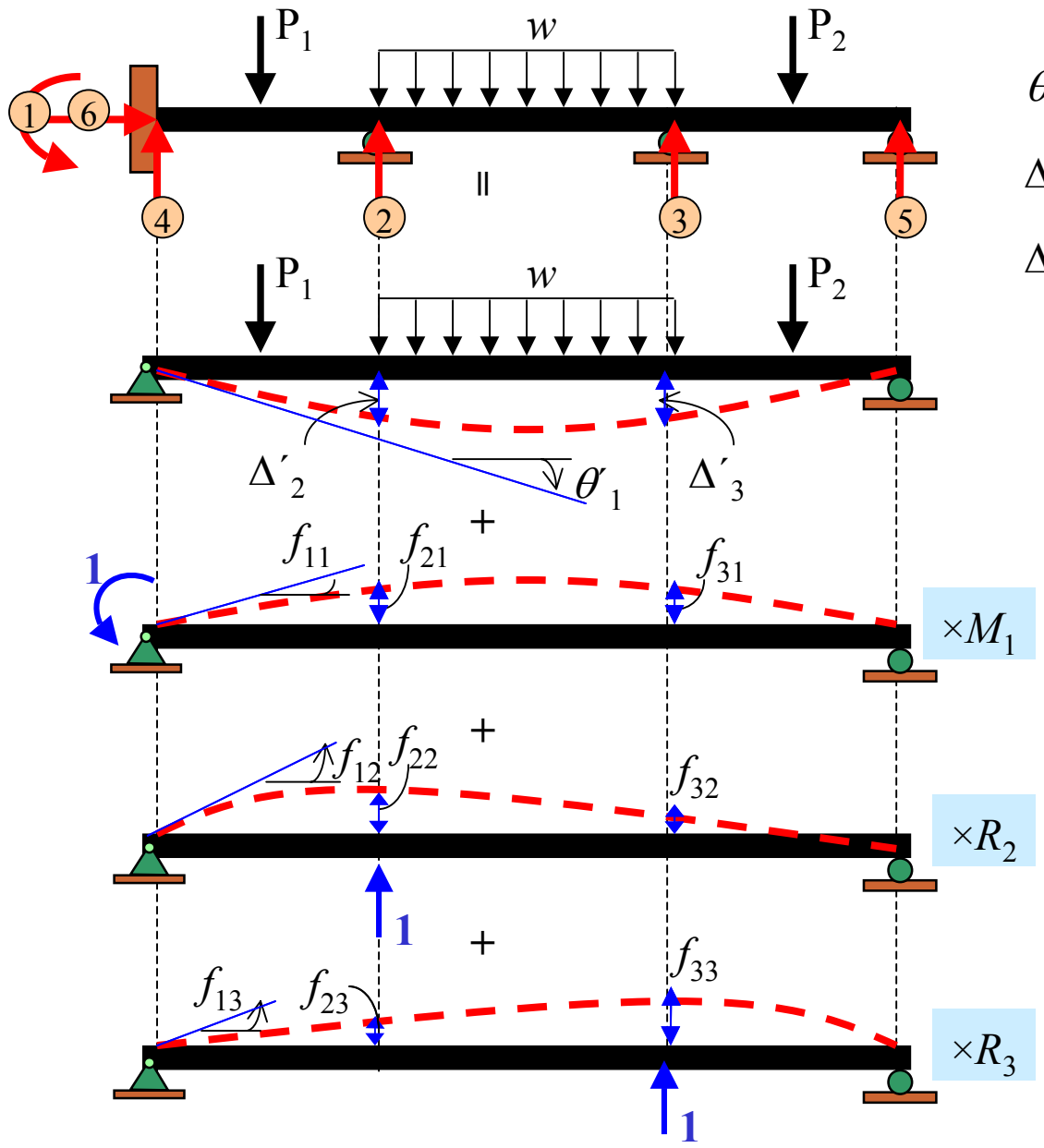
Compatibility Equations.

$$\Delta'_1 + f_{11}R_1 + f_{12}R_2 = \Delta_1 = 0$$

$$\Delta'_2 + f_{21}R_1 + f_{22}R_2 = \Delta_2 = 0$$

$$\begin{Bmatrix} \Delta'_1 \\ \Delta'_2 \end{Bmatrix} + \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} \begin{matrix} 0 \\ 0 \end{matrix}$$

▶ **Beam 3 DOF**



Compatibility Equations.

$$\theta'_1 + f_{11}M_1 + f_{12}R_2 + f_{13}R_3 = \theta_1 = 0$$

$$\Delta'_2 + f_{21}M_1 + f_{22}R_2 + f_{23}R_3 = \Delta_2 = 0$$

$$\Delta'_3 + f_{31}M_1 + f_{32}R_2 + f_{33}R_3 = \Delta_3 = 0$$

$$\begin{pmatrix} \theta'_1 \\ \Delta'_2 \\ \Delta'_3 \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \begin{pmatrix} M_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Compatibility Equation for *n* span.

$$\Delta'_1 + f_{11}R_1 + f_{12}R_2 + f_{13}R_3 = \Delta_1$$
$$\Delta'_2 + f_{21}R_1 + f_{22}R_2 + f_{23}R_3 = \Delta_2$$
$$\vdots$$
$$\Delta'_n + f_{n1}R_1 + f_{n2}R_2 + f_{nn}R_n = \Delta_n$$

If $\Delta_1 = \Delta_2 = \ldots\ldots = \Delta_n = 0$;

$$\begin{pmatrix} \Delta'_1 \\ \Delta'_2 \\ \vdots \\ \Delta'_n \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{13} \\ f_{21} & f_{22} & \cdots & f_{23} \\ & \vdots & & \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \\ R_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

or

$$[\Delta'] + [f][R] = 0$$

$$[R] = -[f]^{-1}[\Delta']$$

$[f_{ij}]$ = Flexibility matrix dependent on

$$\frac{1}{EI}, \quad \frac{1}{EA}, \quad \frac{1}{GJ}$$

Equilibrium Equations

Fixed-end force matrix

$$[Q] = [K][D] + [Q^f]$$

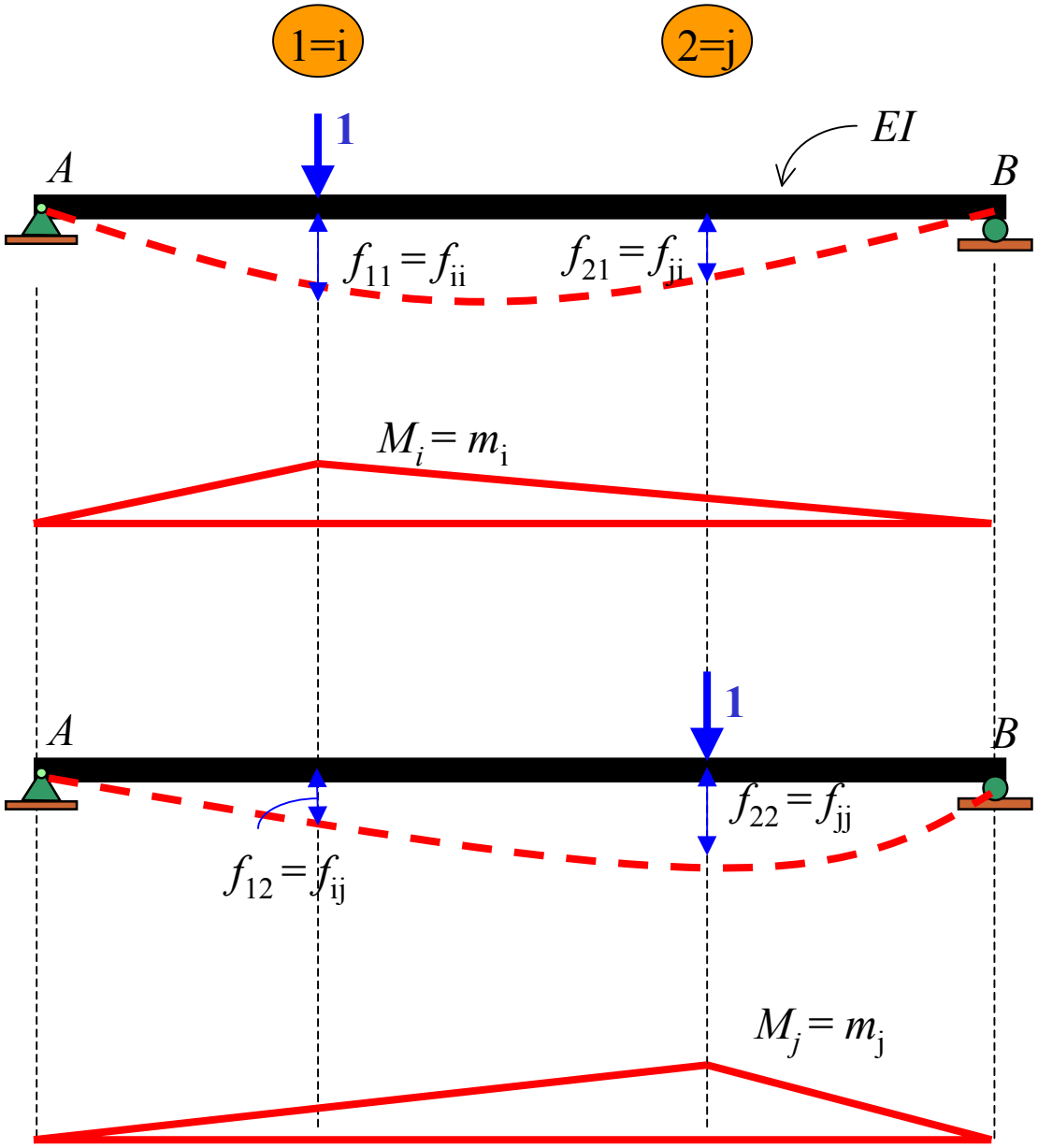
Stiffness matrix

Solve for displacement [D];

$$[D] = [K]^{-1}[Q] - [Q^f]$$

$$k_{ij} = \frac{1}{f_{ij}}$$

Maxwell's Theorem of reciprocal displacements



$$f_{ij}=\int m_i M_j \frac{dx}{EI}$$

$$=\int m_i m_j \frac{dx}{EI}$$

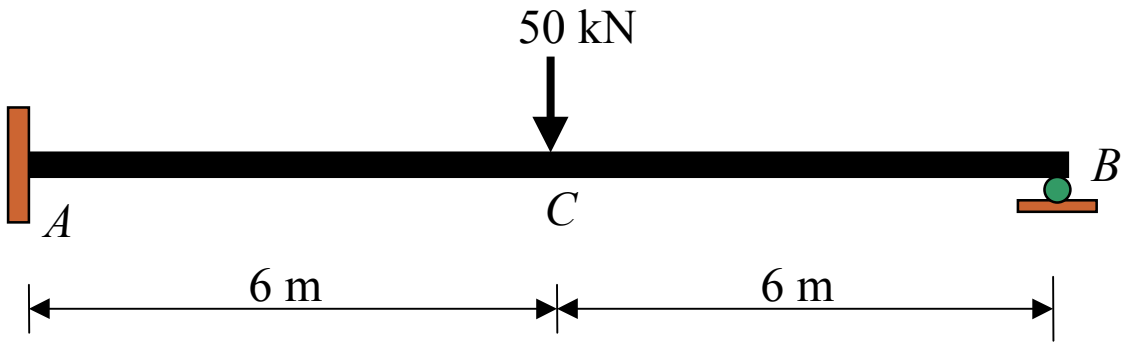
$$f_{ji}=\int m_j M_i \frac{dx}{EI}$$

$$=\int m_j m_i \frac{dx}{EI}$$

$$f_{ij}=f_{ji}$$

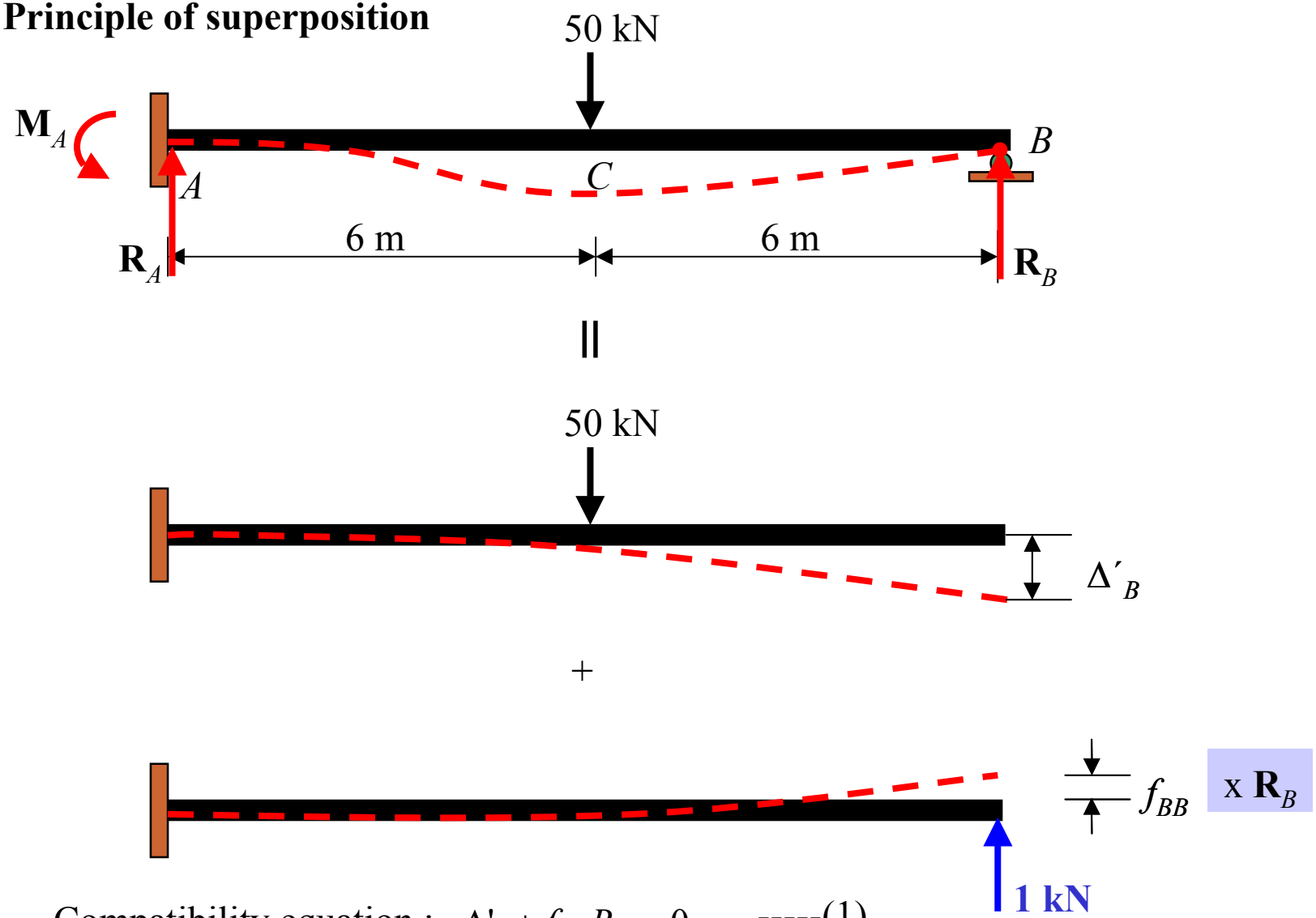
Example 1

Determine the **reaction at all supports** and **the displacement at *C***.



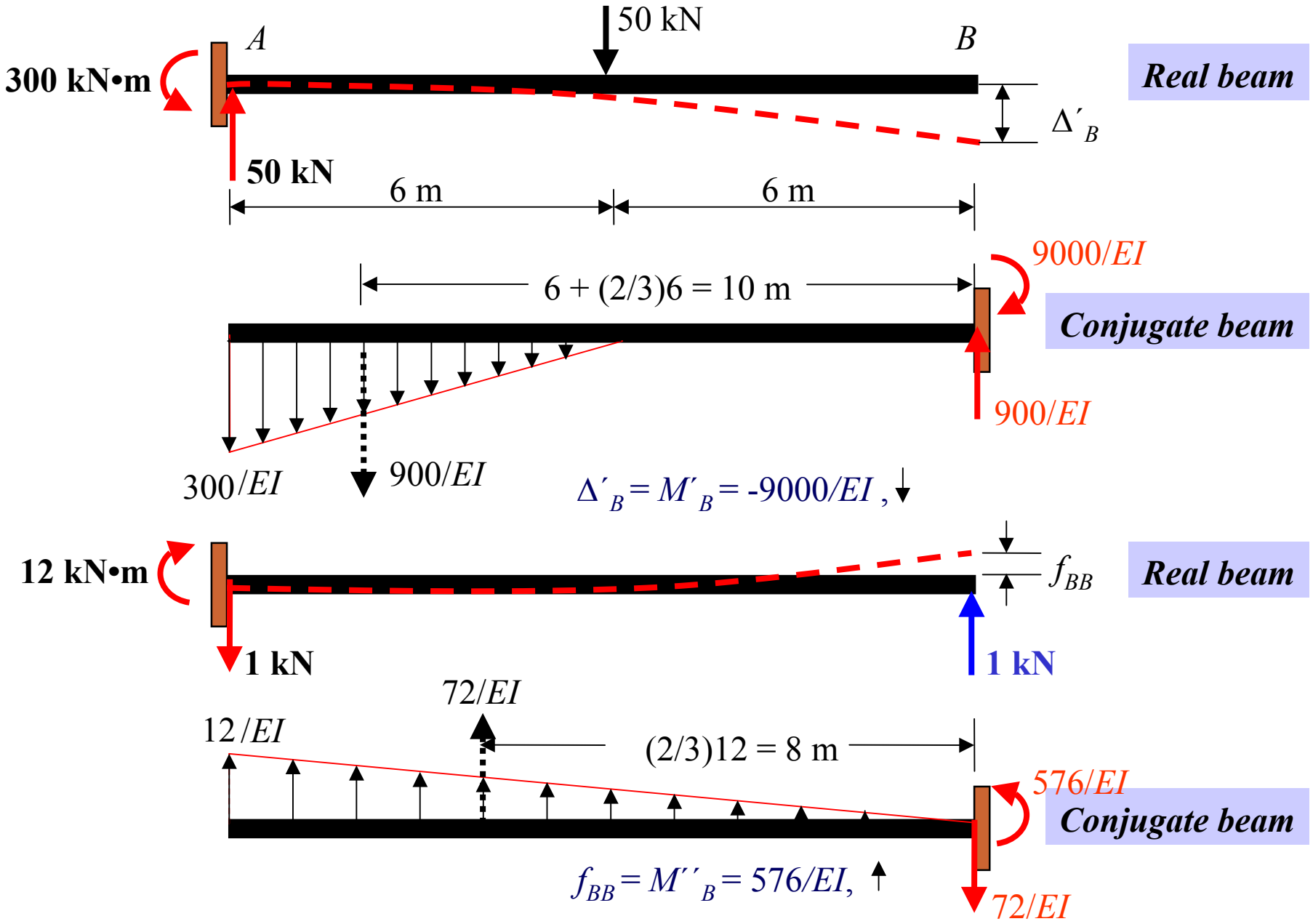
SOLUTION

- **Principle of superposition**



Compatibility equation : $\Delta'_B + f_{BB} R_B = 0$ -----(1)

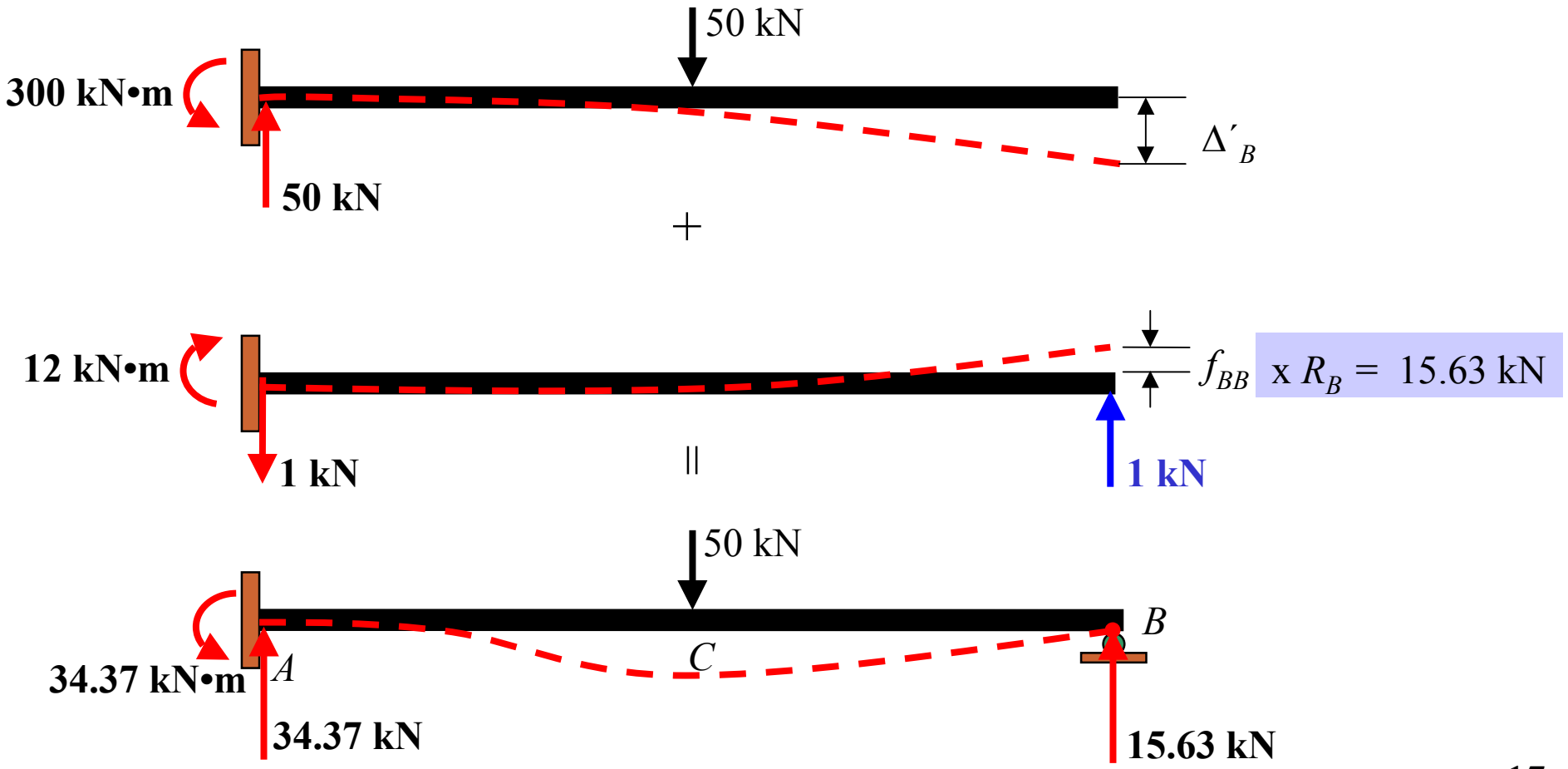
• **Use conjugate beam in obtaining Δ'_B and f_{BB}**



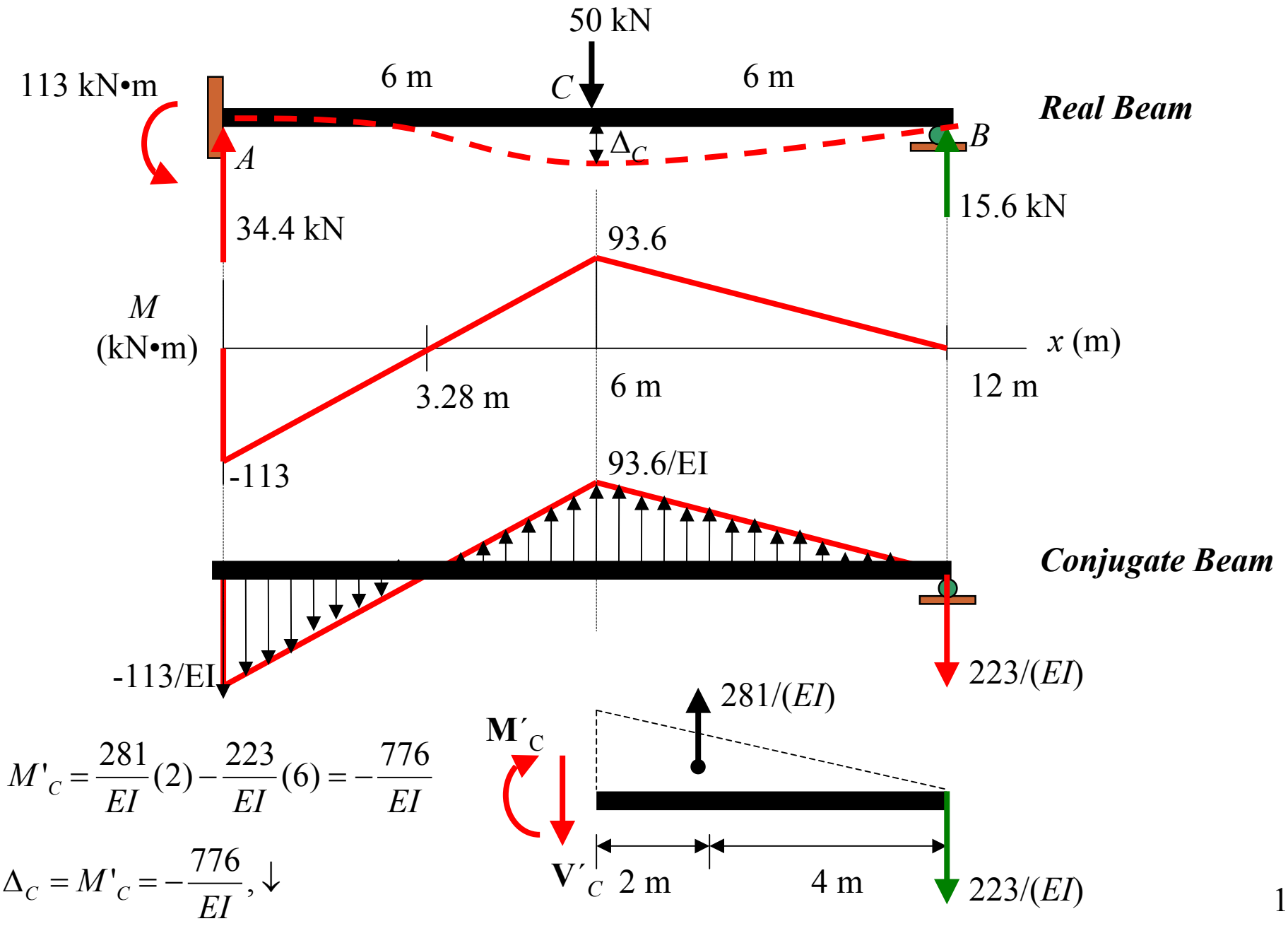
• **Substitute** Δ'_B and f_{BB} in Eq. (1)

$$+\uparrow: -\frac{9000}{\cancel{EI}} + \left(\frac{576}{\cancel{EI}}\right) R_B = 0$$

R_B = +15.63 kN, ↑ (same direction as 1 kN)



Use conjugate beam in obtaining the displacement

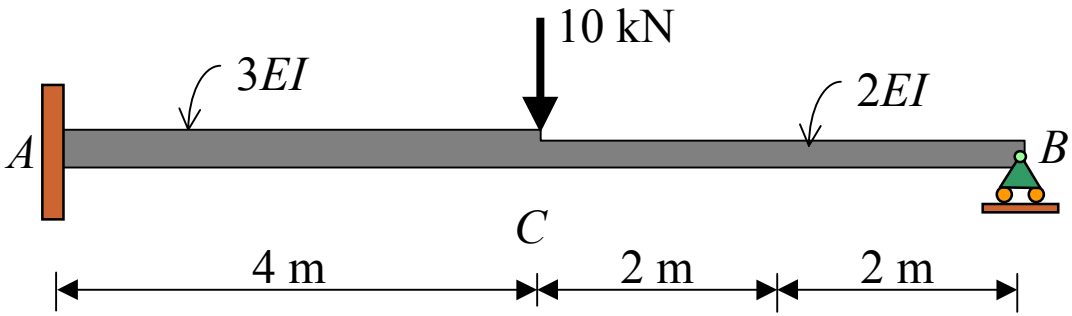


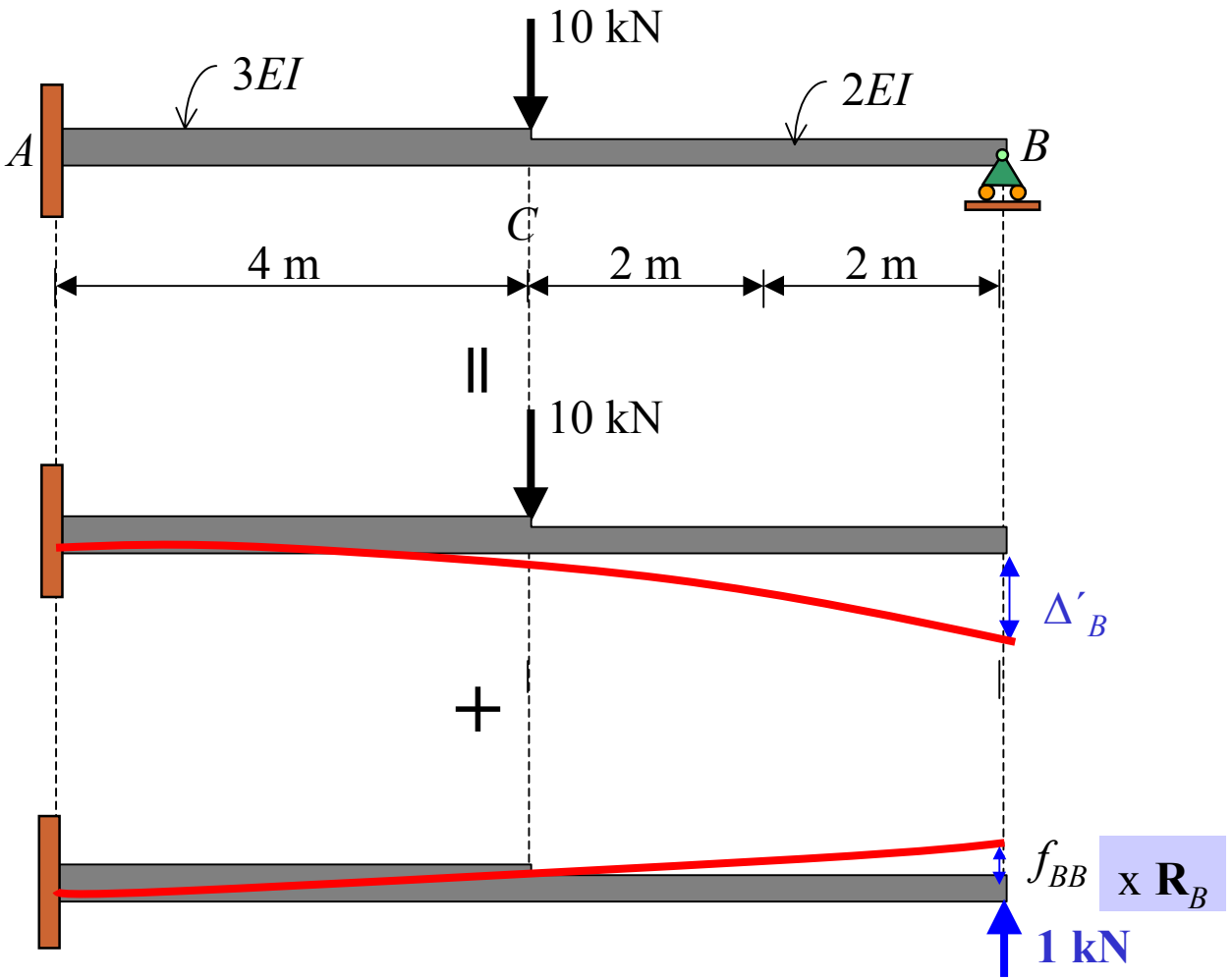
$$M'_C = \frac{281}{EI}(2) - \frac{223}{EI}(6) = -\frac{776}{EI}$$

$$\Delta_C = M'_C = -\frac{776}{EI}, \downarrow$$

Example 2

Determine the **reaction at all supports** and **the displacement at *C***. *Take* E = 200 GPa and I = $5(10^6)$ mm^4

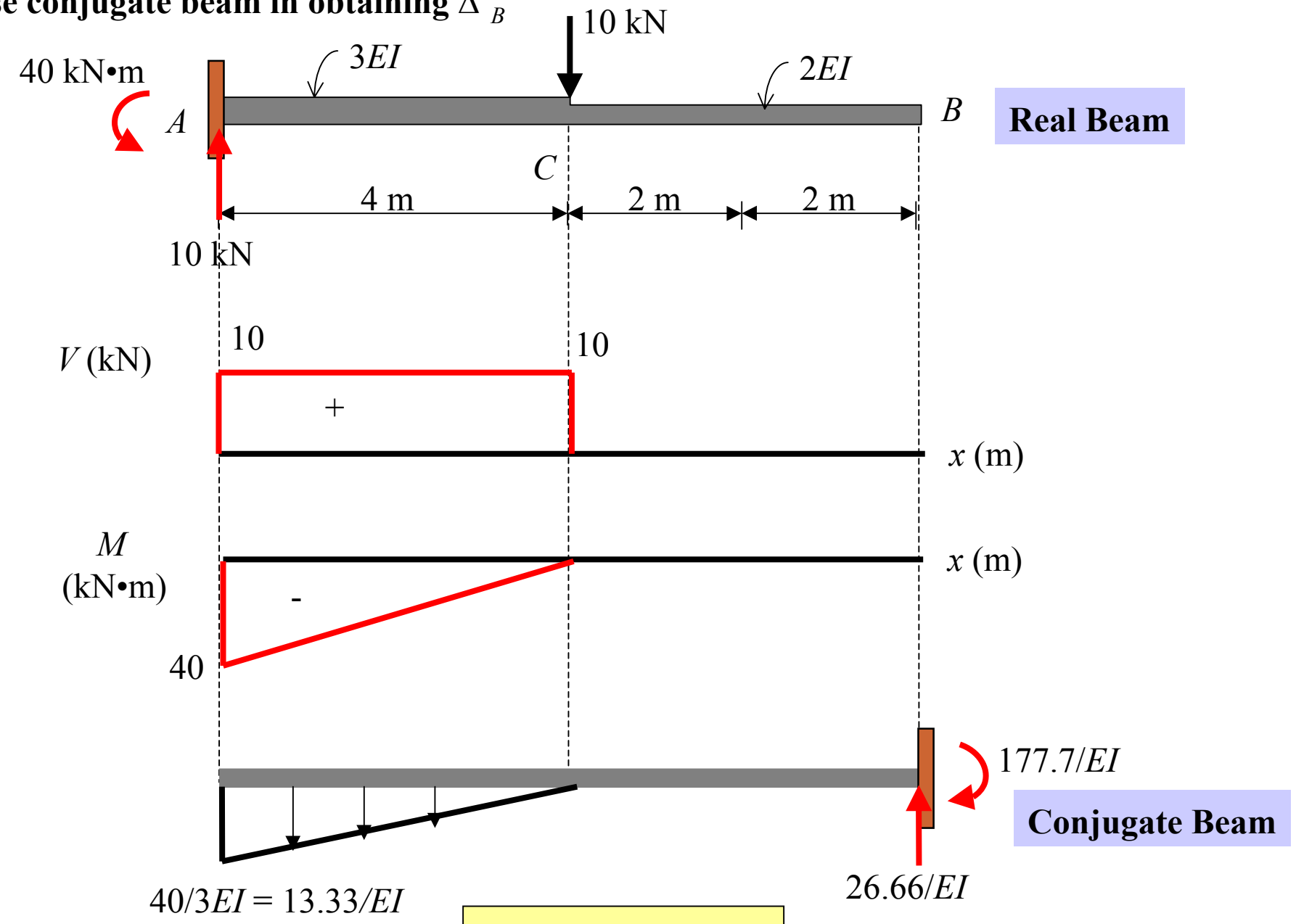




Compatibility equation:

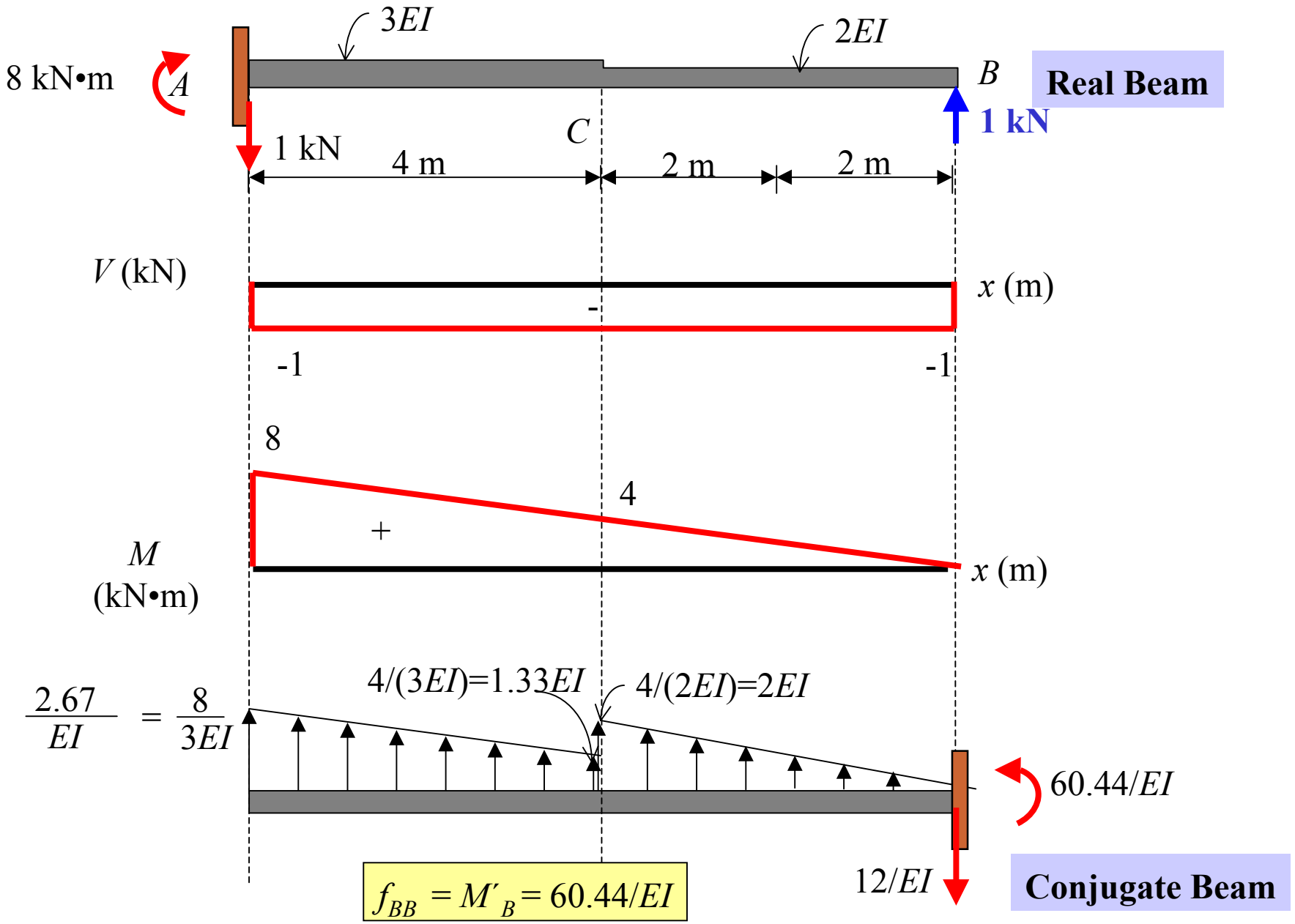
$$\Delta'_B + f_{BB} R_B = \Delta_B = 0$$

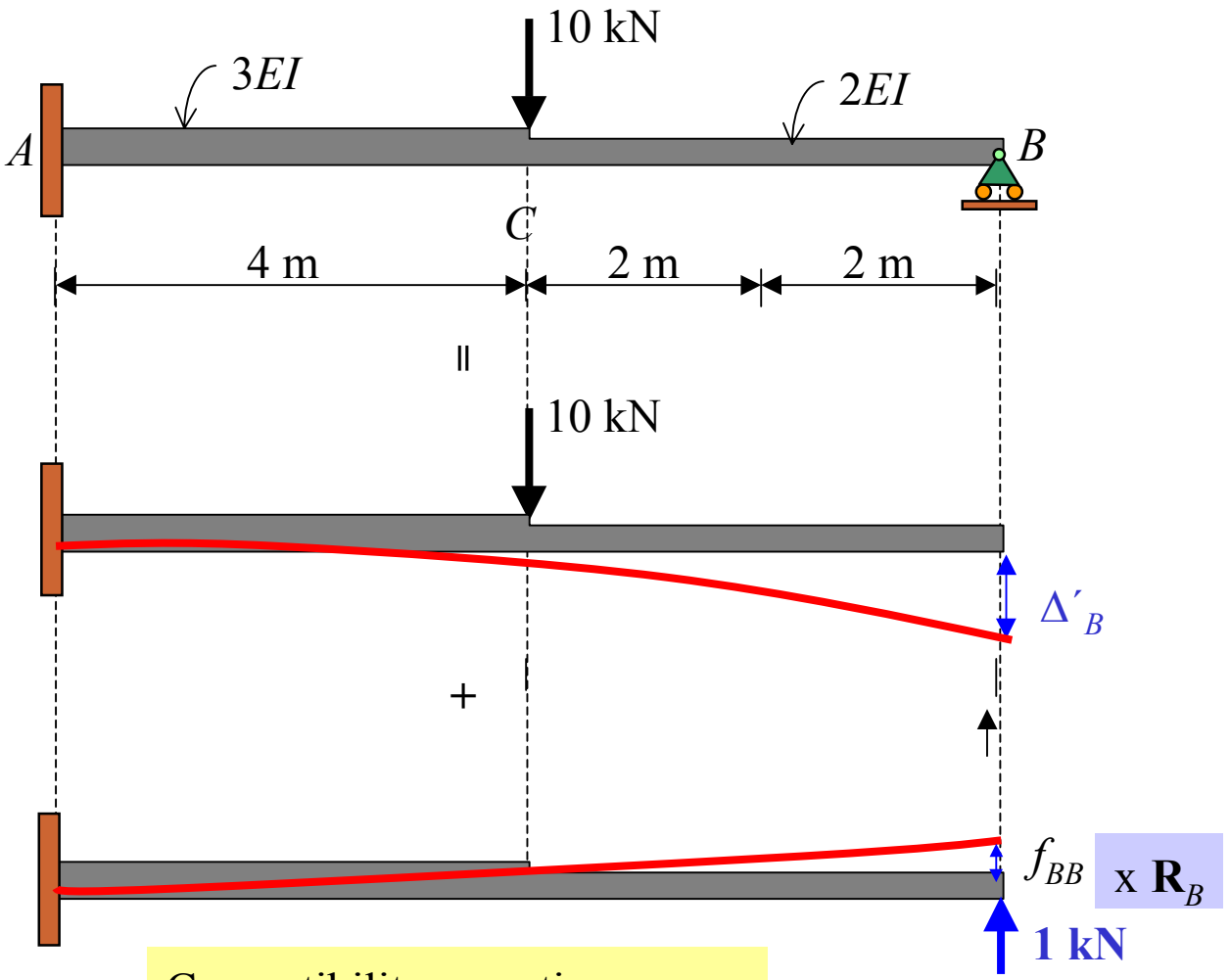
• **Use conjugate beam in obtaining Δ'_B**



$\Delta'_B = M'_B = 177.7/EI$

• **Use conjugate beam for f_{BB}**





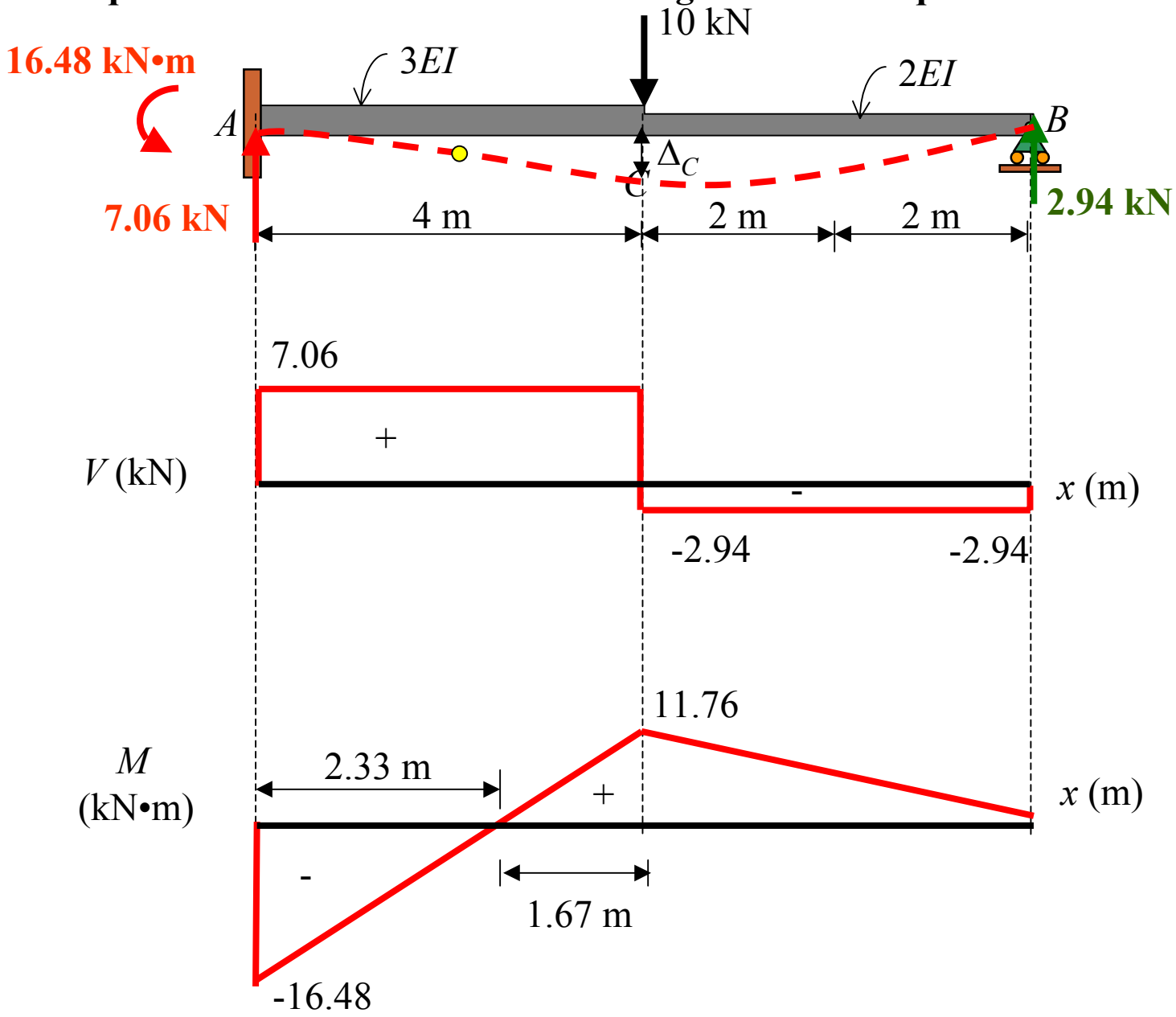
Compatibility. equation:

$$\Delta'_B + f_{BB} R_B = \Delta_B = 0$$

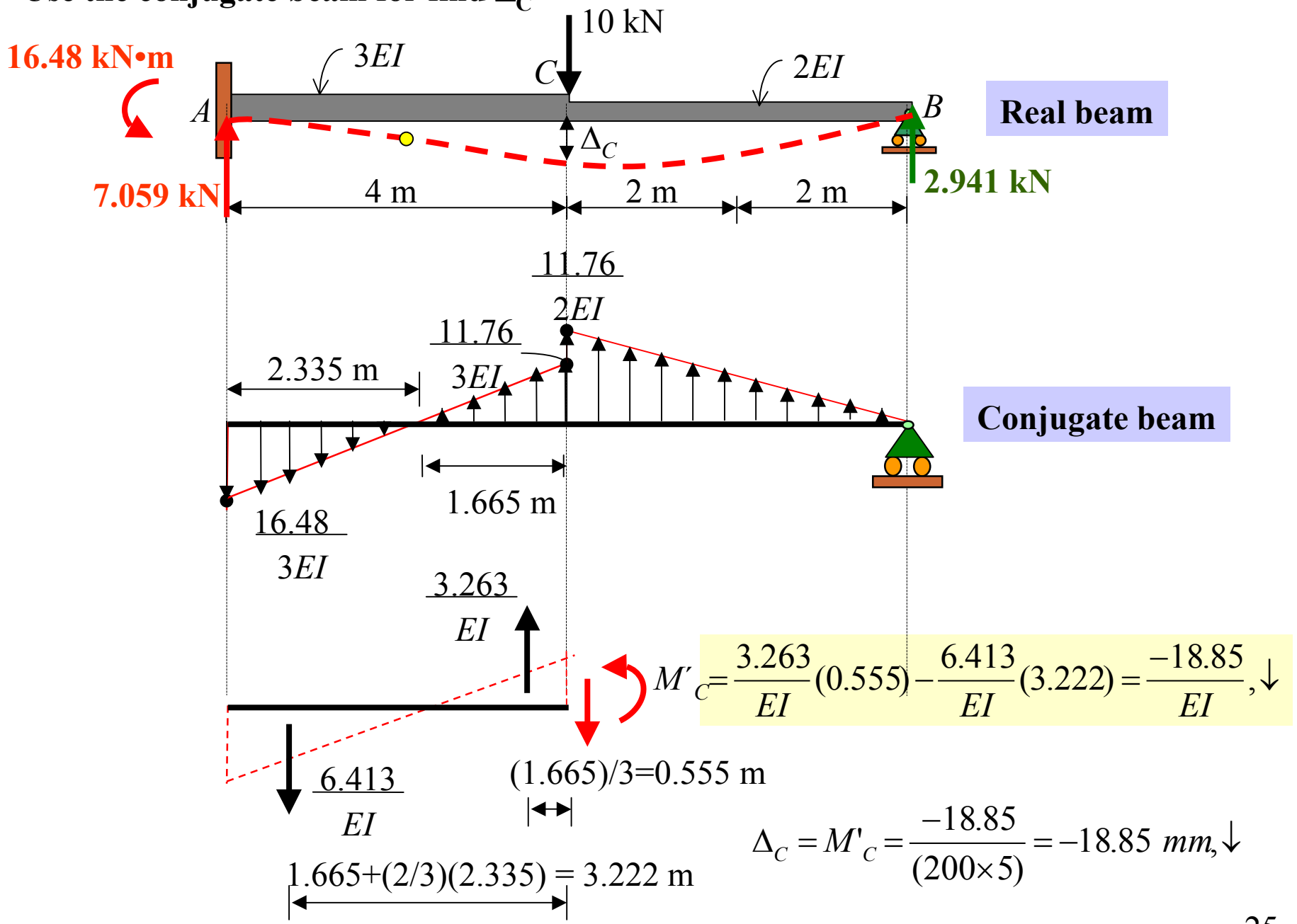
$$+\uparrow: -\frac{-177.7}{EI} + \left(\frac{60.44}{EI}\right) R_B = 0$$

$R_B =$ +2.941 kN, (same direction as 1 kN)

• **The quantitative shear and moment diagram and the qualitative deflected curve**

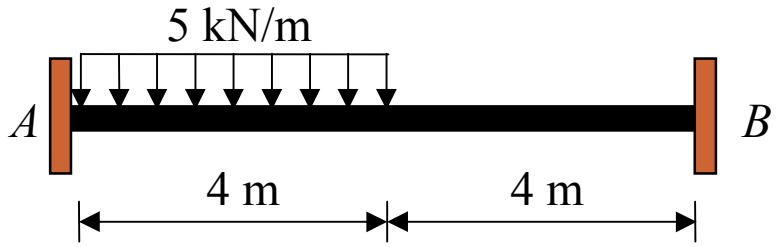


• **Use the conjugate beam for find Δ_C**



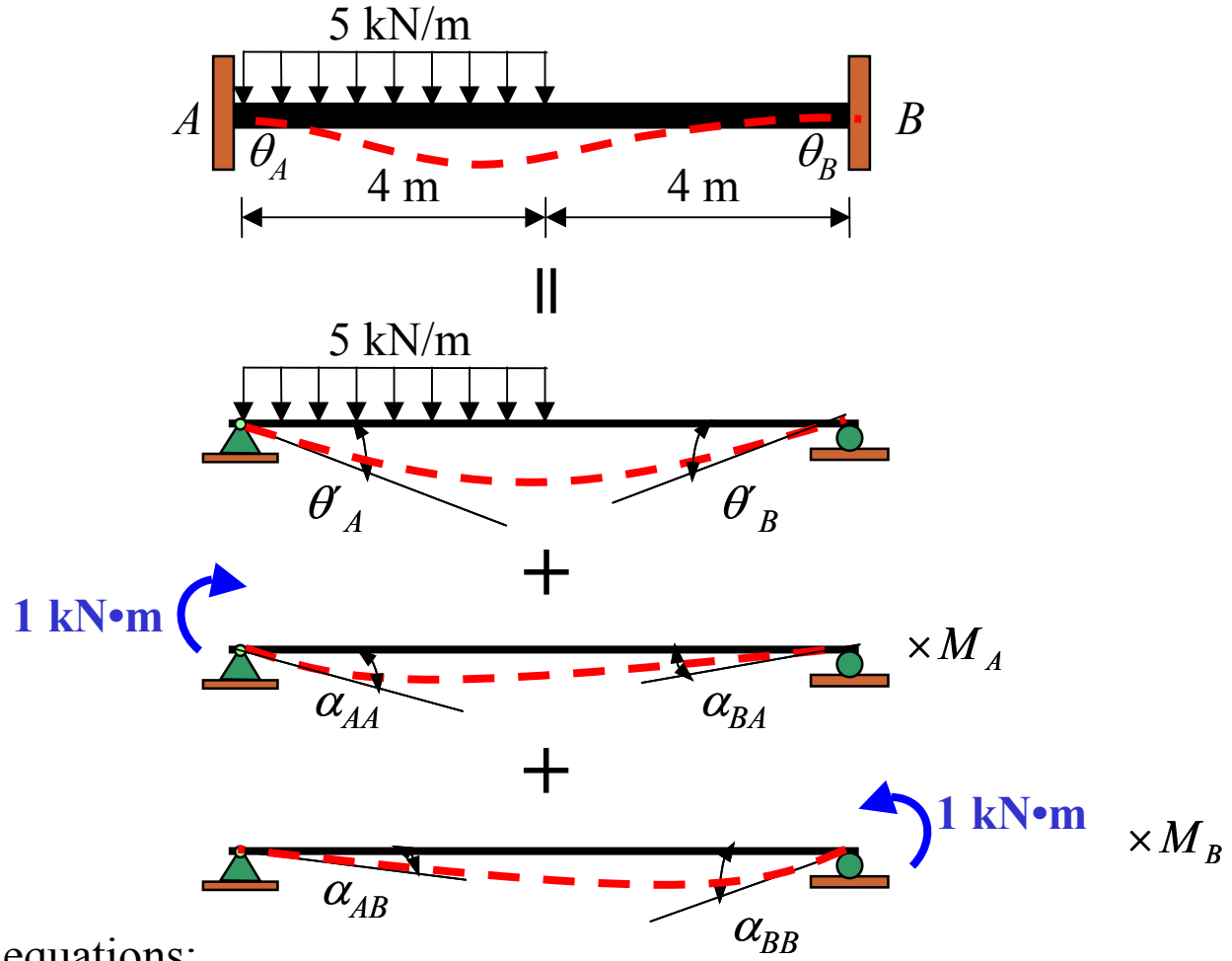
Example 3

Draw the quantitative Shear and moment diagram and the qualitative deflected curve for the beam shown below.*EI* is constant. Neglect the effects of axial load.



SOLUTION

- **Principle of superposition**

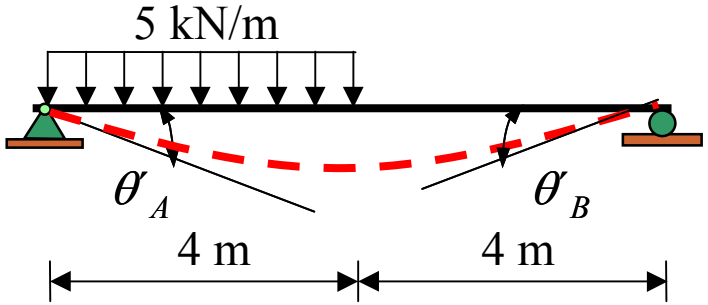


Compatibility equations:

$$\theta_A = 0 = \theta'_A + f_{AA}M_A + f_{AB}M_B \quad ---(1)$$

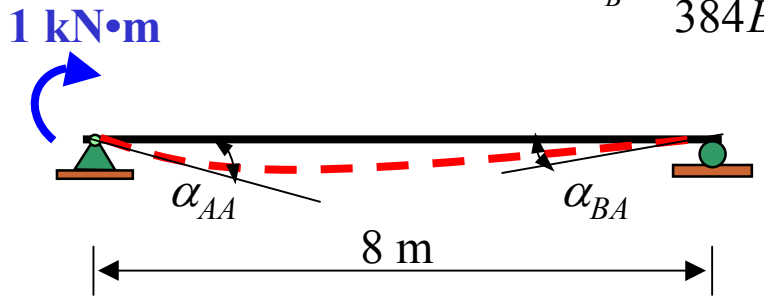
$$\theta_B = 0 = \theta'_B + f_{BA}M_A + f_{BB}M_B \quad ---(2)$$

- **Use formula provided in obtaining** θ'_A, θ'_B, α_{AA}, α_{BA}, α_{BB}, α_{AB}



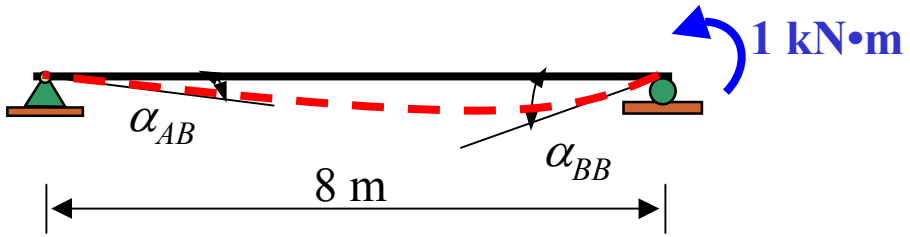
$$\theta'_A = \frac{3wL^3}{128EI} = \frac{3(5)(8)^3}{128EI} = \frac{60}{EI}$$

$$\theta'_B = \frac{7wL^3}{384EI} = \frac{7(5)(8)^3}{384EI} = \frac{46.67}{EI}$$



$$\alpha_{AA} = \frac{M_o L}{3EI} = \frac{1(8)}{3EI} = \frac{2.667}{EI}$$

$$\alpha_{BA} = \frac{M_o L}{6EI} = \frac{1(8)}{6EI} = \frac{1.333}{EI}$$

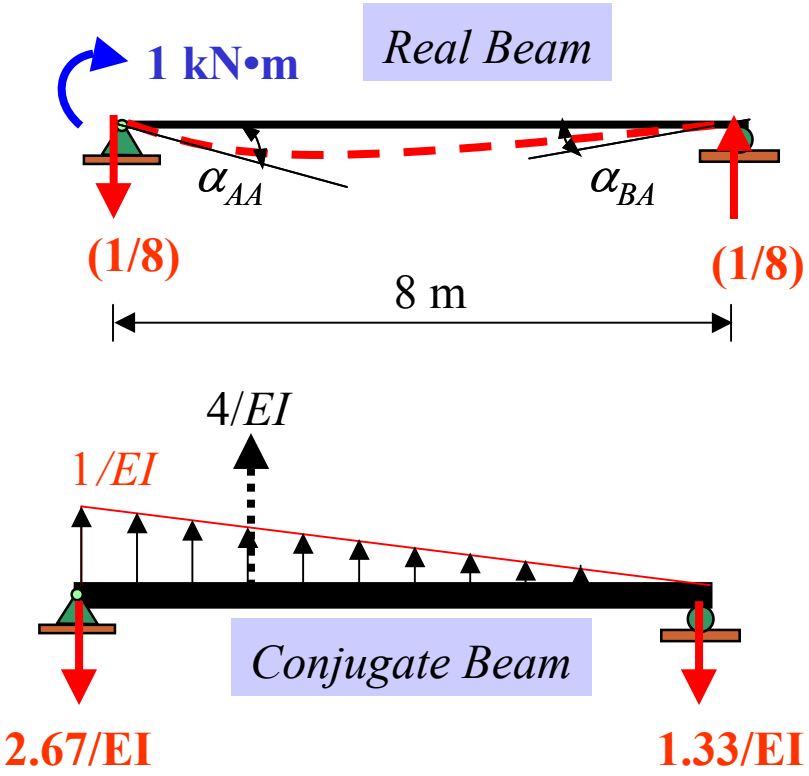


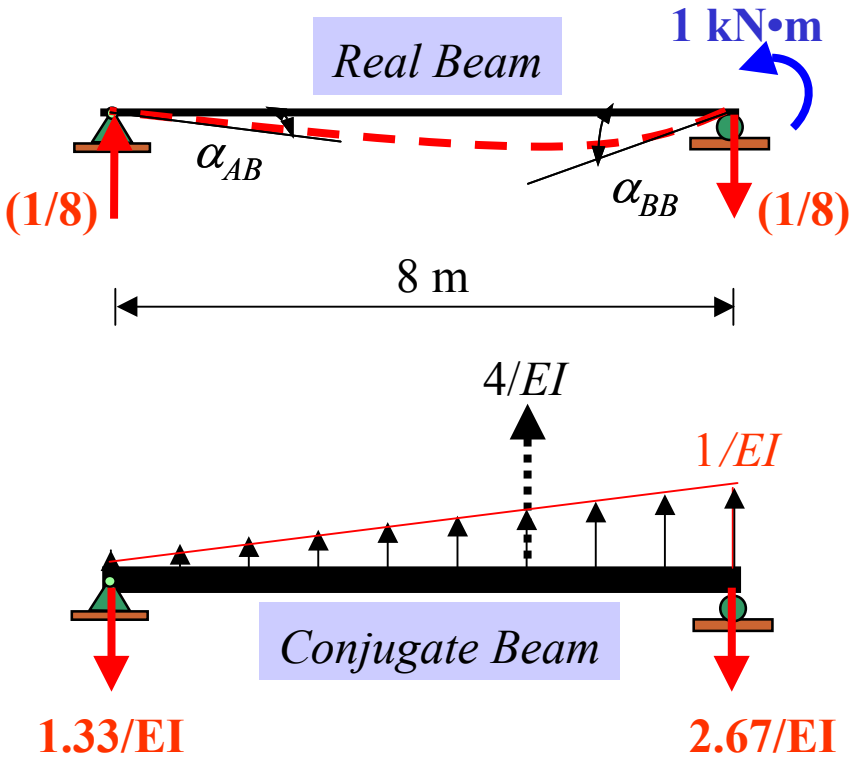
$$\alpha_{BB} = \frac{M_o L}{3EI} = \frac{1(8)}{3EI} = \frac{2.667}{EI}$$

$$\alpha_{AB} = \frac{M_o L}{6EI} = \frac{1(8)}{6EI} = \frac{1.333}{EI}$$

Note: *Maxwell's theorem of reciprocal displacement,* $\alpha_{AB} = \alpha_{BA}$

• **Use conjugate beam for** α_{AA}, α_{BA}, α_{BB}, α_{AB}



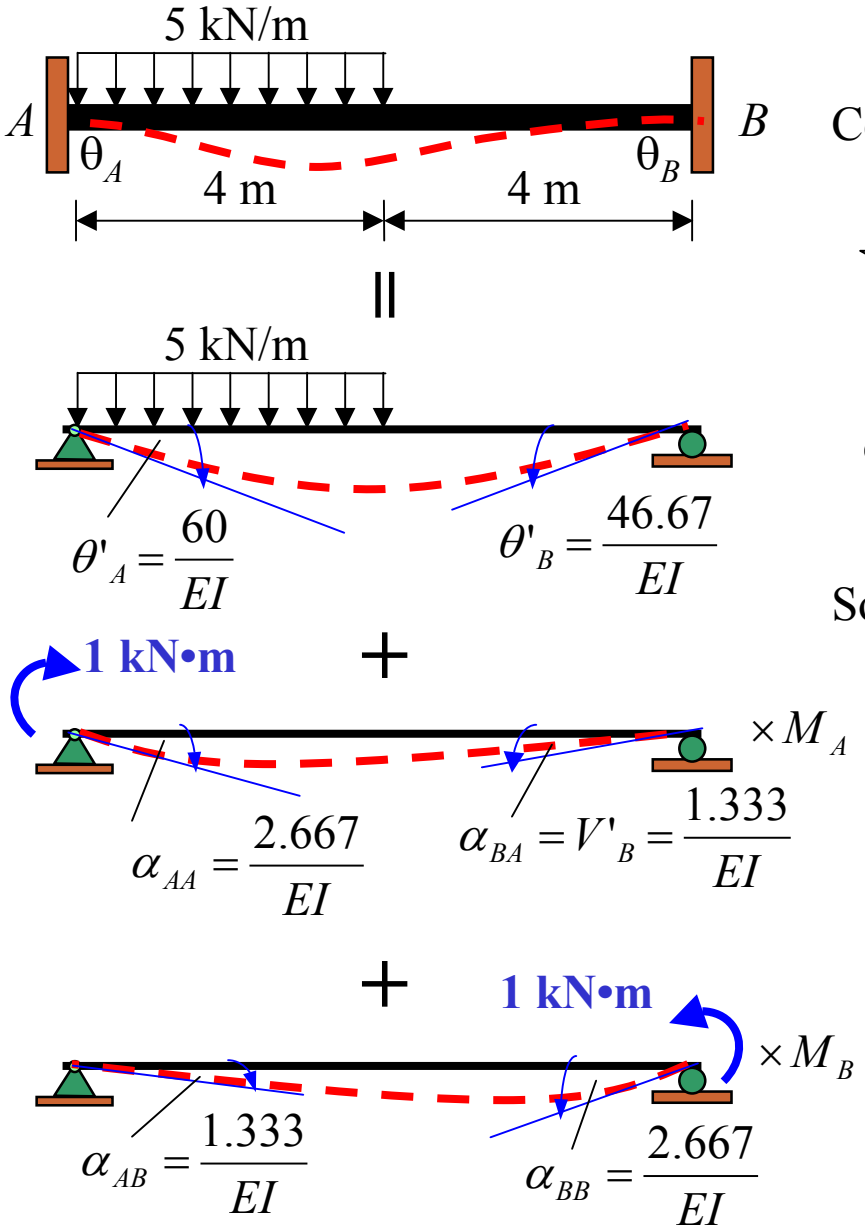


$$\alpha_{AA} = V'_A = \frac{-2.667}{EI}$$

$$\alpha_{BA} = V'_B = \frac{1.333}{EI}$$

$$\alpha_{AB} = V'_A = \frac{-1.333}{EI}$$

$$\alpha_{BB} = V'_B = \frac{2.667}{EI}$$



Compatibility equation

$$\frac{60}{EI}+\left(\frac{2.667}{EI}\right)M_A+\left(\frac{1.333}{EI}\right)M_B=0$$

$$\frac{46.67}{EI}+\left(\frac{1.333}{EI}\right)M_A+\left(\frac{2.667}{EI}\right)M_B=0$$

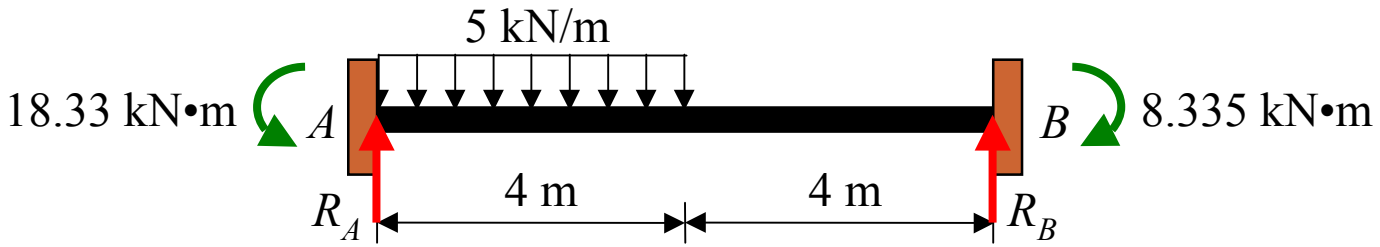
Solve simultaneous equations,

M_A = -18.33 kN•m, +

M_B = -8.335 kN•m, +

M_A = -18.33 kN•m,

M_B = -8.335 kN•m,

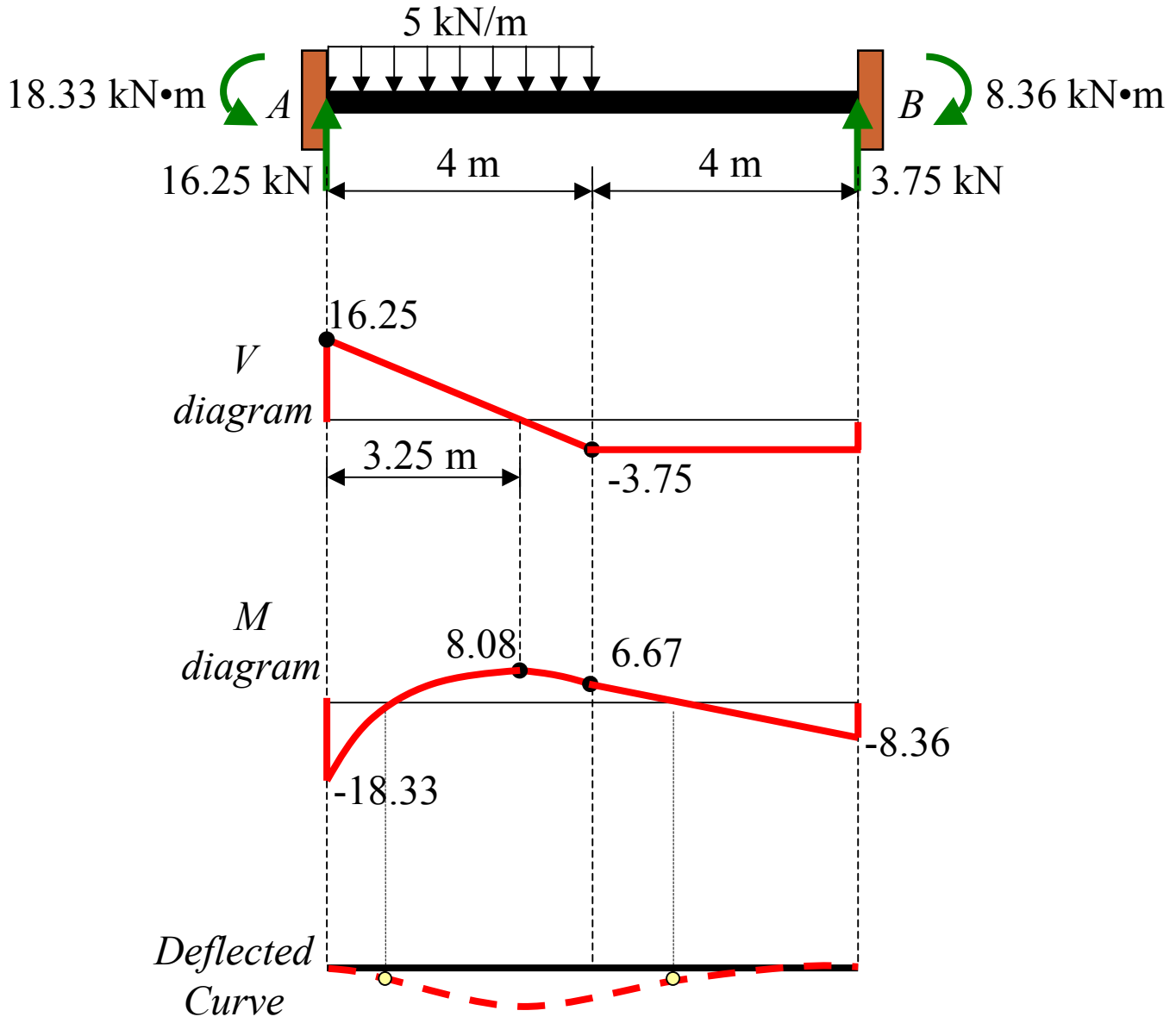


$+\circlearrowleft \Sigma M_A = 0$: $18.33 - 20(2) + R_B(8) - 8.355 = 0$ $\quad R_B = 3.753$ kN, ↑

3.753

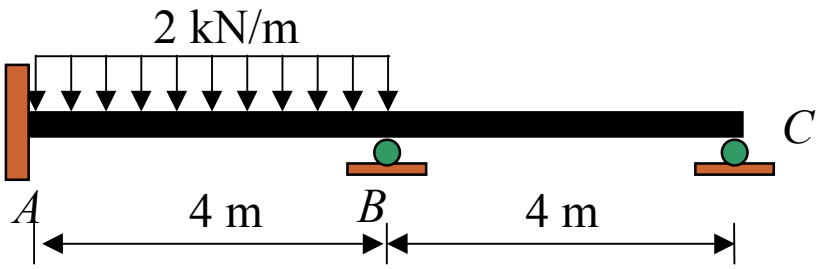
$+\uparrow \Sigma F_y = 0$: $R_A + R_B - 20 = 0$ $\quad R_a = 16.25$ kN, ↑

• **Quantitative shear and bending diagram and qualitative deflected curve**

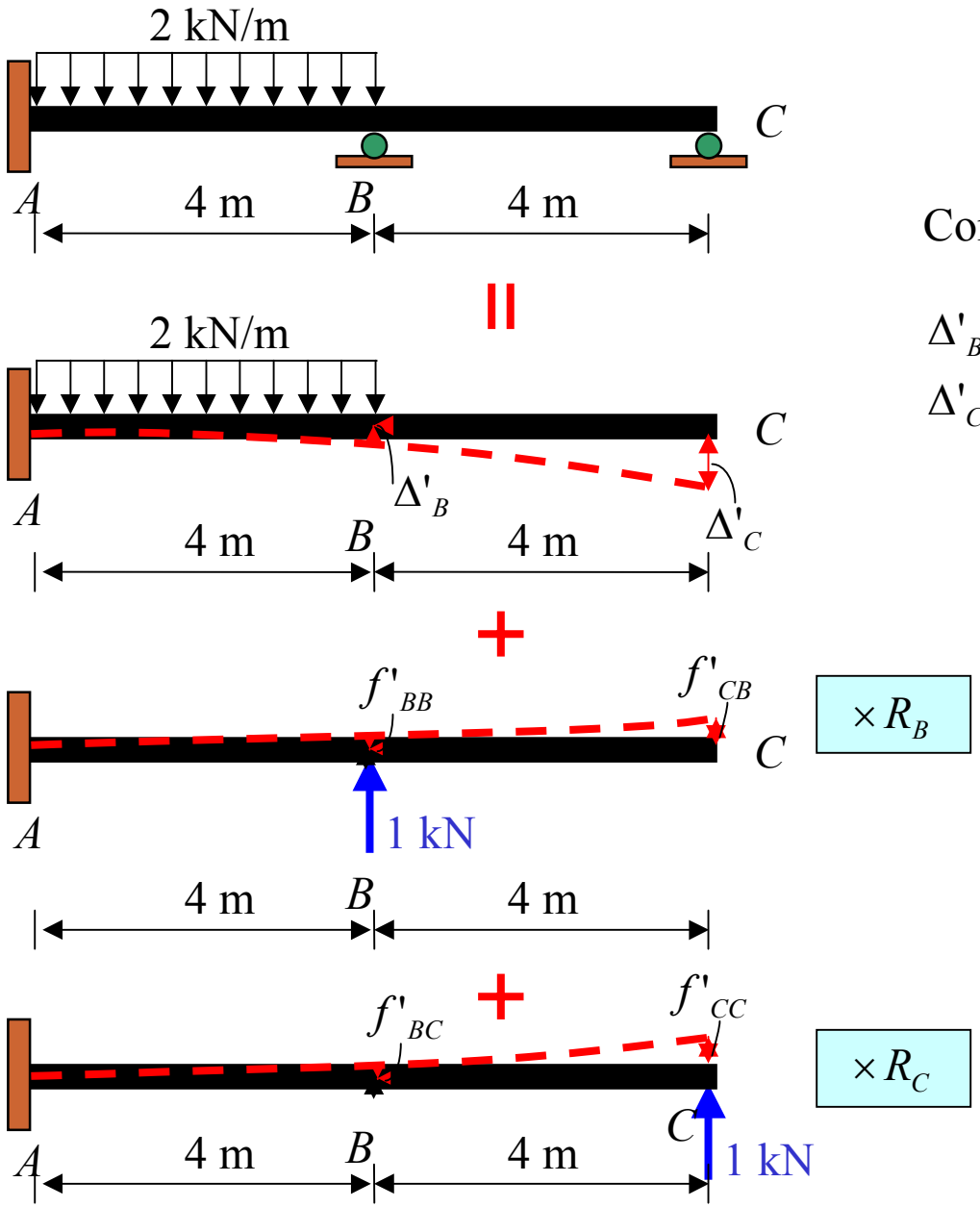


Example 4

Determine the reactions at the supports for the beam shown and draw the quantitative shear and moment diagram and the qualitative deflected curve. *EI* is constant.



• **Principle of superposition**

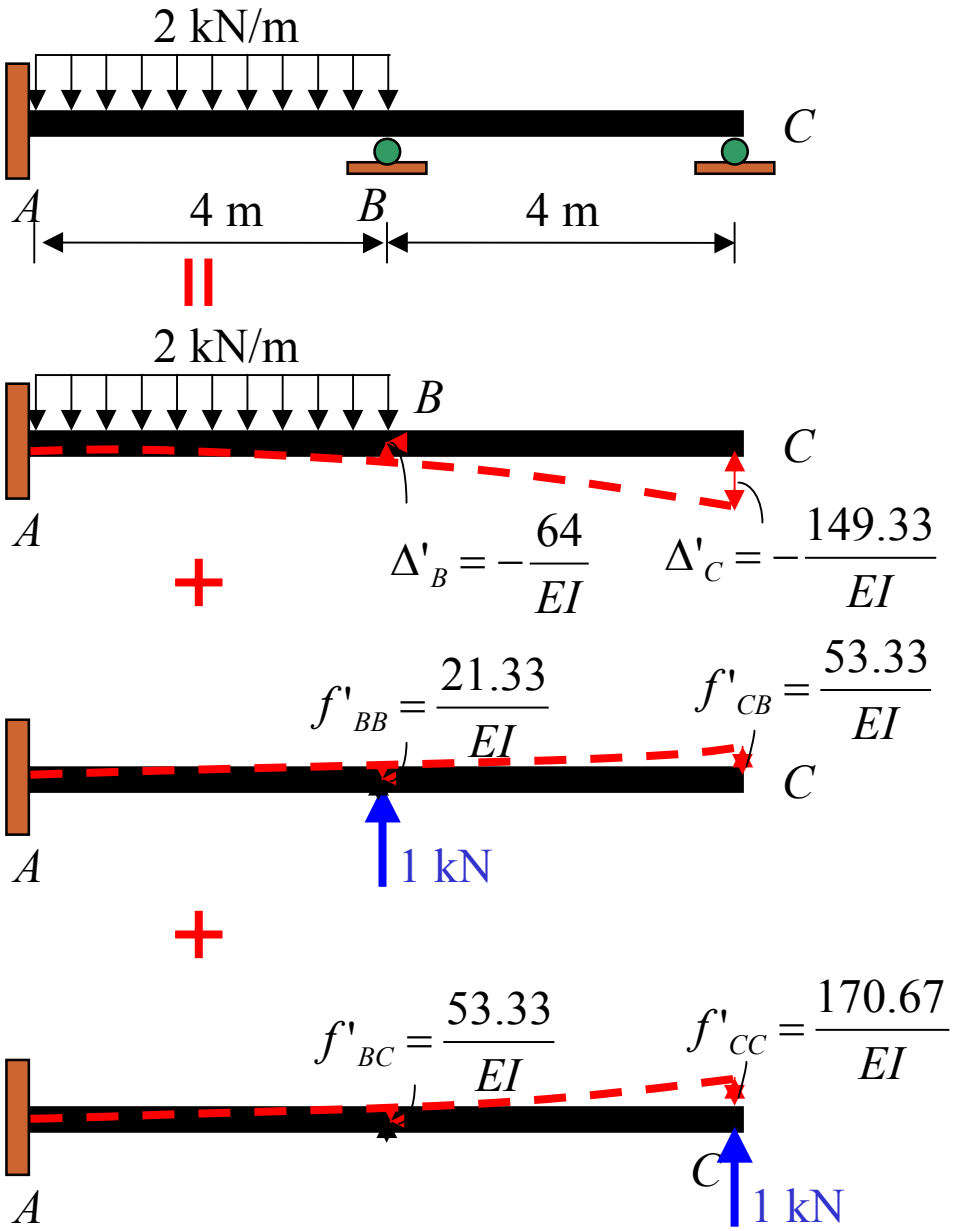


Compatibility equations:

$$\Delta'_B + f'_{BB} R_B + f'_{CB} R_C = 0 \quad ---(1)$$

$$\Delta'_C + f'_{BC} R_B + f'_{CC} R_C = 0 \quad ---(2)$$

• **Solve equation**



$\times R_B$

$\times R_C$

Compatibility equations:

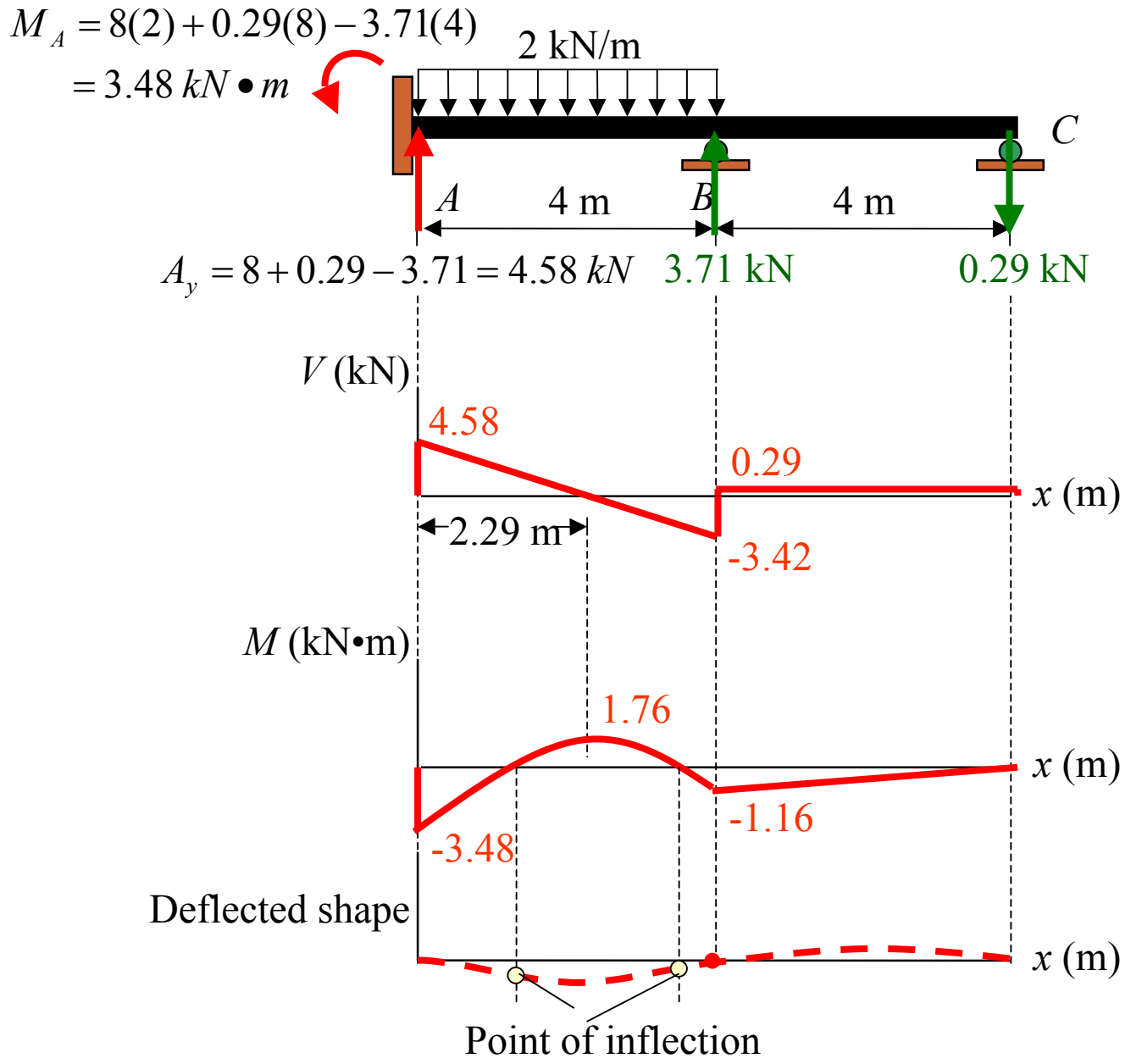
$$-\frac{64}{EI}+\frac{21.33}{EI}R_B+\frac{53.33}{EI}R_C=0 \quad --(1)$$

$$-\frac{149.33}{EI}+\frac{53.33}{EI}R_B+\frac{170.67}{EI}R_C=0 \quad --(2)$$

$$R_B = 3.71\,kN,\uparrow$$

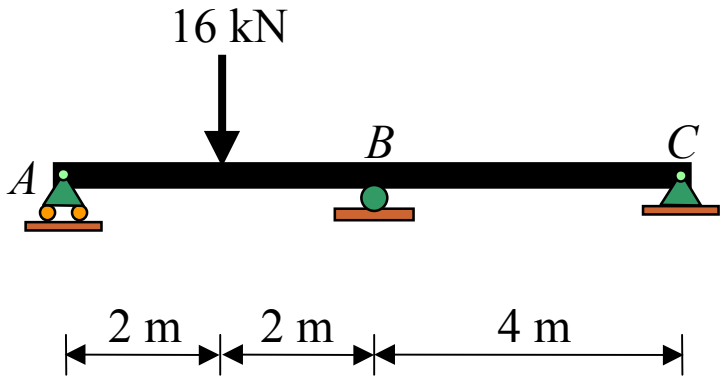
$$R_C = -0.29\,kN,\downarrow$$

• **Diagram**


$M_A = 8(2) + 0.29(8) - 3.71(4)$
$= 3.48\ kN \bullet m$
2 kN/m
C
A
4 m
B
4 m
$A_y = 8 + 0.29 - 3.71 = 4.58\ kN$
3.71 kN
0.29 kN
V (kN)
4.58
0.29
x (m)
2.29 m
-3.42
M (kN•m)
1.76
x (m)
-1.16
-3.48
Deflected shape
x (m)
Point of inflection

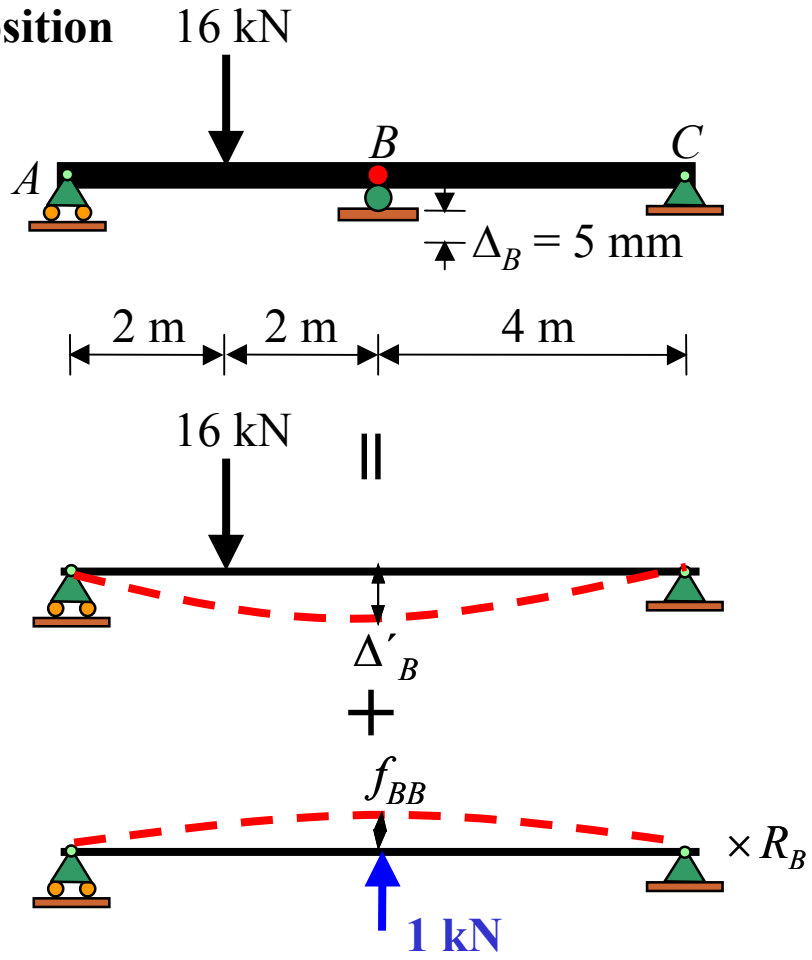
Example 5

Draw the quantitative Shear and moment diagram and the qualitative deflected curve for the beam shown below.
(a) The support at *B* does not settle
(b) The support at *B* settles 5 mm.
Take E = 200 GPa, $I = 60(10^6)$ mm^4.



SOLUTION

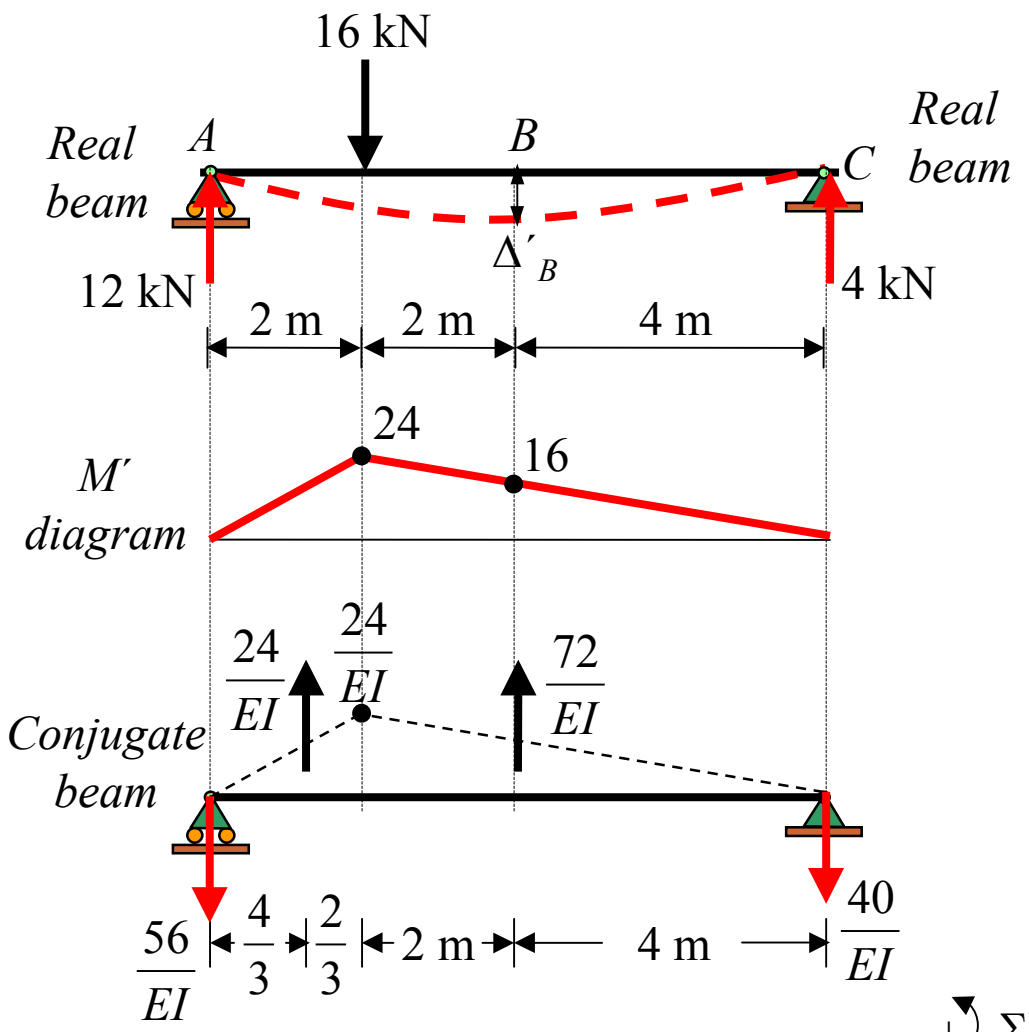
- **Principle of superposition**

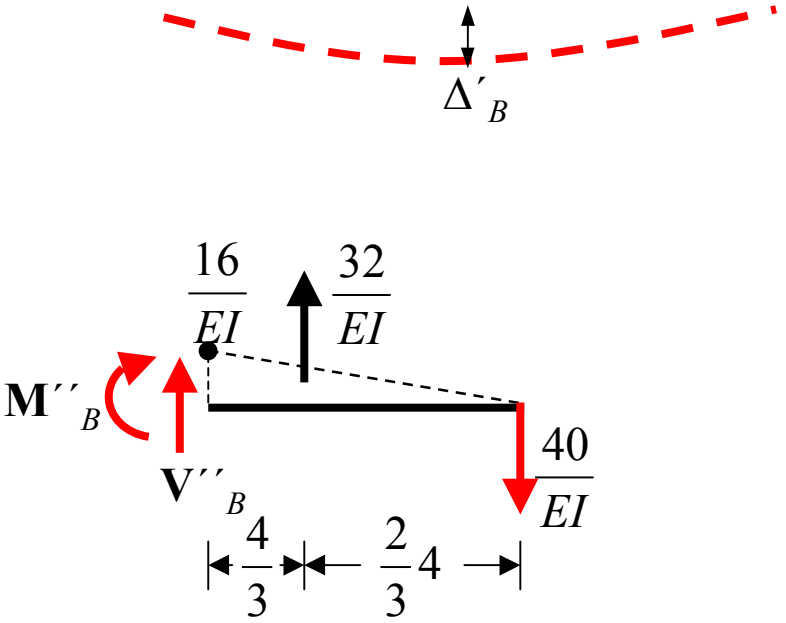


Compatibility equation : $+\uparrow \quad \Delta_B = 0 = \Delta'_B + f_{BB} R_B$ -----(1) : no settlement

$+\uparrow \quad \Delta_B = -0.005m = \Delta'_B + f_{BB} R_B$ -----(2) : with settlement

• **Use conjugate beam method in obtaining** Δ'_B

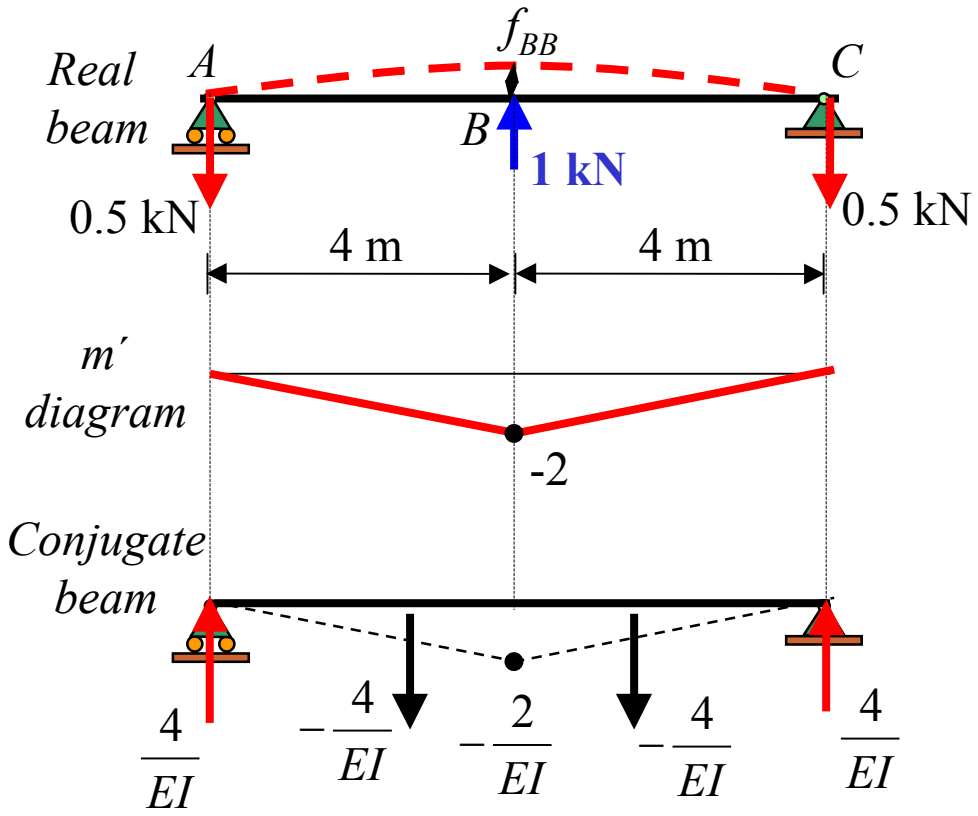


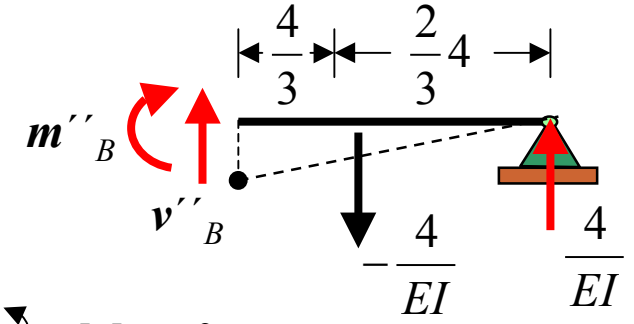


$$+\circlearrowleft \Sigma M_B = 0: \quad -M''_B + \frac{32}{EI}\left(\frac{4}{3}\right) - \frac{40}{EI}(4) = 0$$

$$\Delta'_B = M''_B = -\frac{117.33}{EI}, \downarrow$$

• **Use conjugate beam method in obtaining f_{BB}**

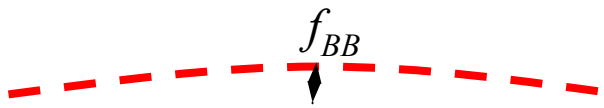




$+\circlearrowleft \Sigma M_B = 0$:

$$-m''_B - \frac{4}{EI}\left(\frac{4}{3}\right) + \frac{4}{EI}(4) = 0$$

$$f_{BB} = m''_B = \frac{10.67}{EI}, \uparrow$$

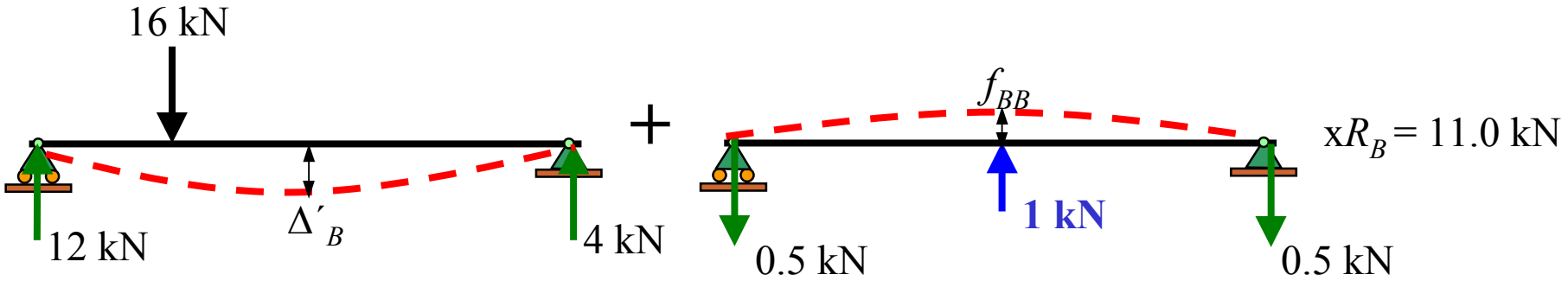


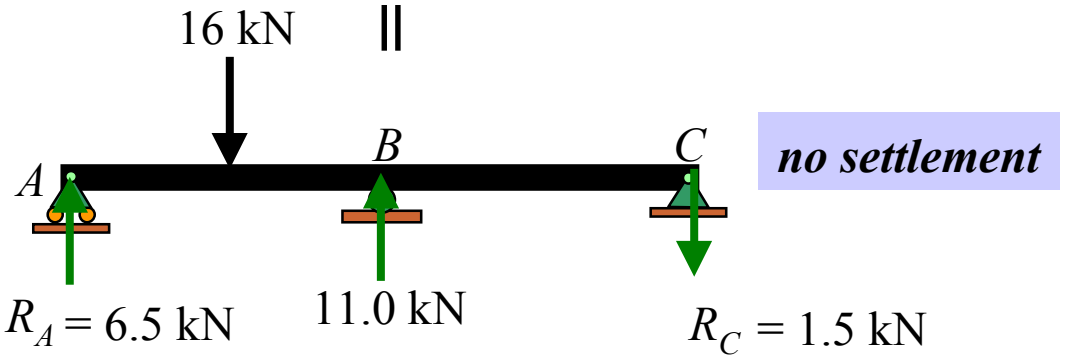
• **Substitute** Δ'_B and f_{BB} in Eq. (1)

$$\Delta'_B = -\frac{117.33}{EI}, \downarrow$$

$$f_{BB} = \frac{10.67}{EI}, \uparrow$$

$$+\uparrow; \quad 0 = -\frac{117.33}{EI} + \frac{10.67}{EI} R_B, \qquad R_B = 11.0 \text{ kN}, \uparrow$$





no settlement

• **Substitute** Δ'_B and f_{BB} in Eq. (2)

$$\Delta'_B = -\frac{117.33}{EI}, \downarrow$$

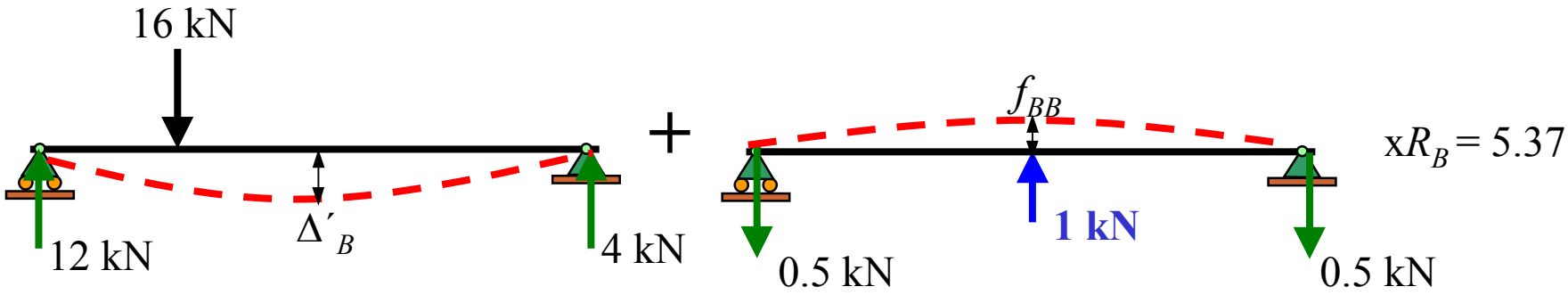
$$f_{BB} = \frac{10.67}{EI}, \uparrow$$

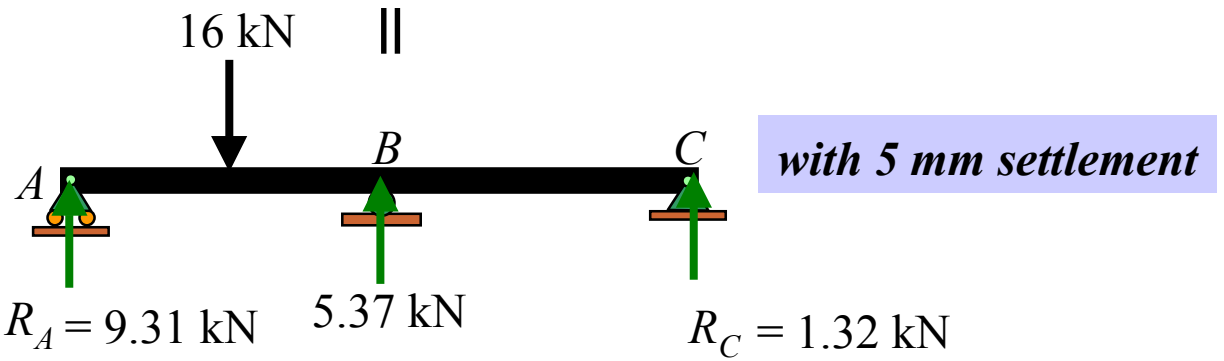
$$+\uparrow; \quad -0.005m = -\frac{117.33}{EI} + \frac{10.67}{EI} R_B$$

$$(-0.005m)EI = -117.33 + 10.67 R_B$$

$$(-0.005)(200 \times 60) = -117.33 + 10.67 R_B$$

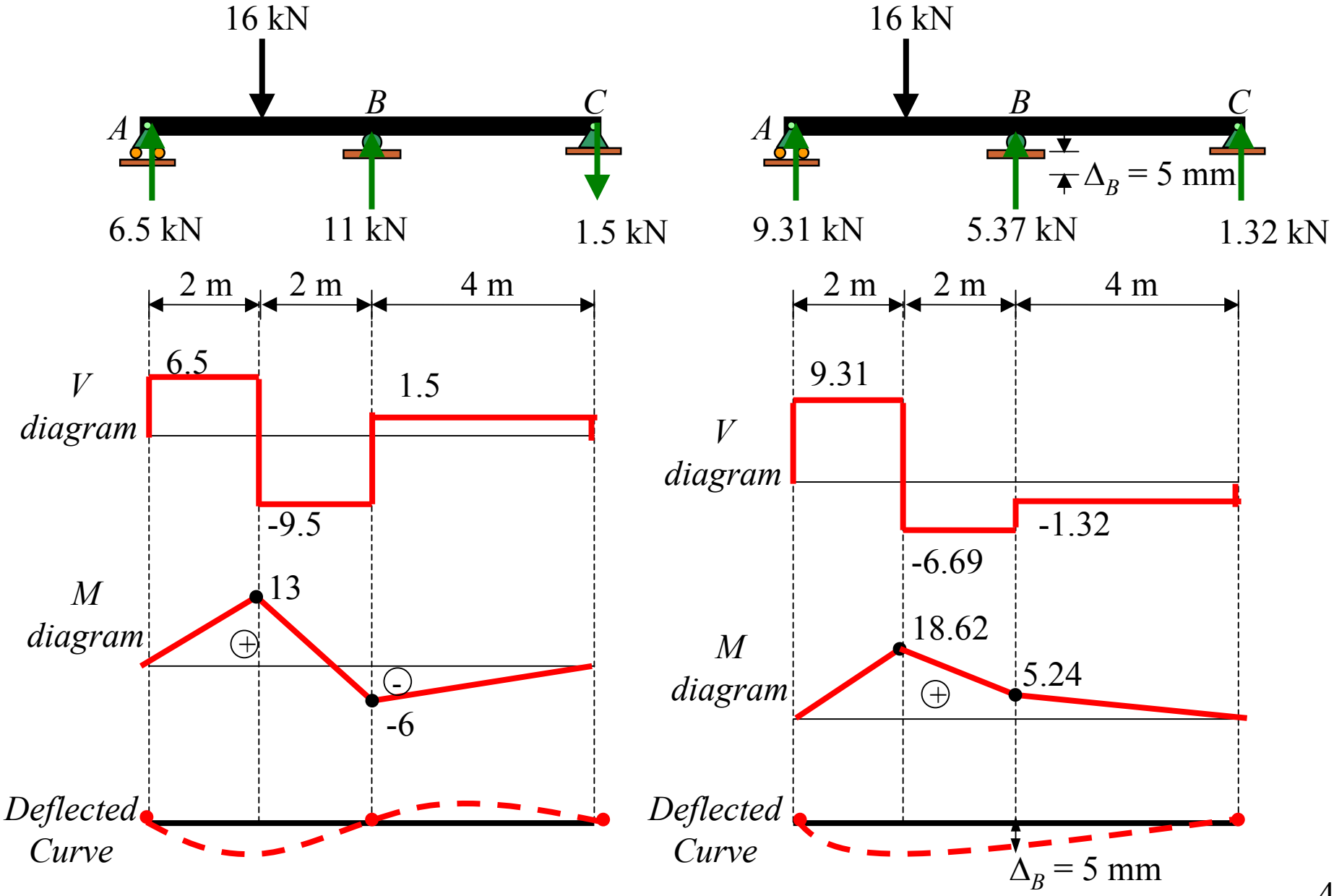
$$R_B = 5.37 \text{ kN}, \uparrow$$





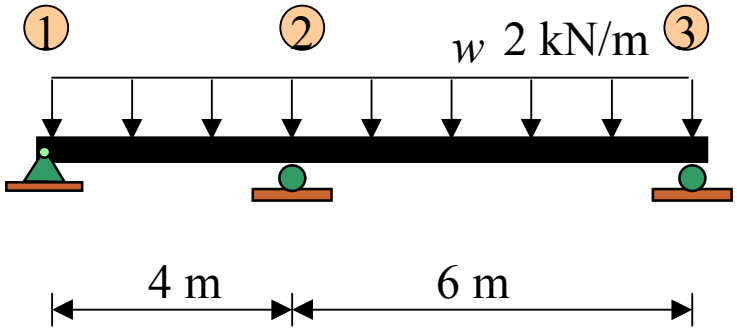
with 5 mm settlement

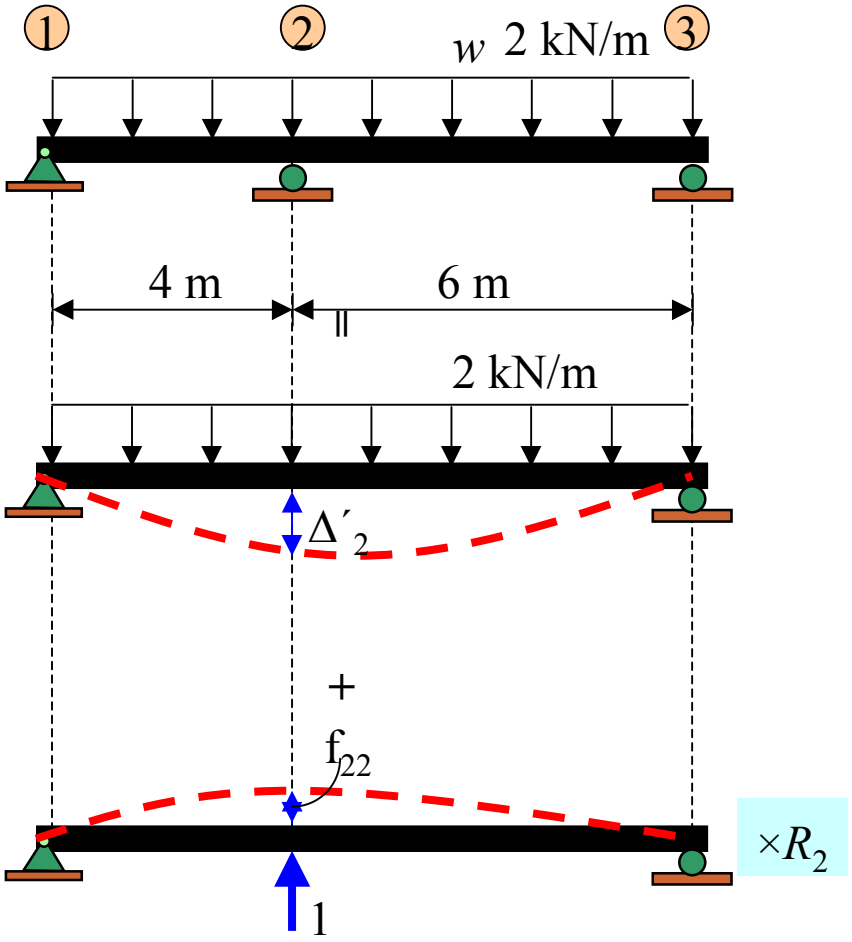
• **Quantitative shear and bending diagram and qualitative deflected curve**



Example 6

Calculate supports reactions and draw the bending moment diagrams for (a) $D_2 = 0$ and (b) $D_2 = 2$ mm.





Compatibility equation:

$$\Delta'_2 + f_{22}R_2 = \Delta_2$$

Use formulations provided :

$$\Delta'_2 = -\frac{wx}{24EI}(x^3 - 2Lx^2 + L^3)$$

$$= -\frac{(2)(4)}{24EI}(4^3 - 2\times 10\times 4^2 + 10^3)$$

$$= -\frac{248}{(200)(200)} = -6.2mm, \quad \downarrow$$

$$f_{22} = \frac{Pbx}{6LEI}(L^2 - b^2 - x^2)$$

$$= -\frac{1\times 6\times 4}{6\times 10EI}(10^2 - 6^2 - 4^2)$$

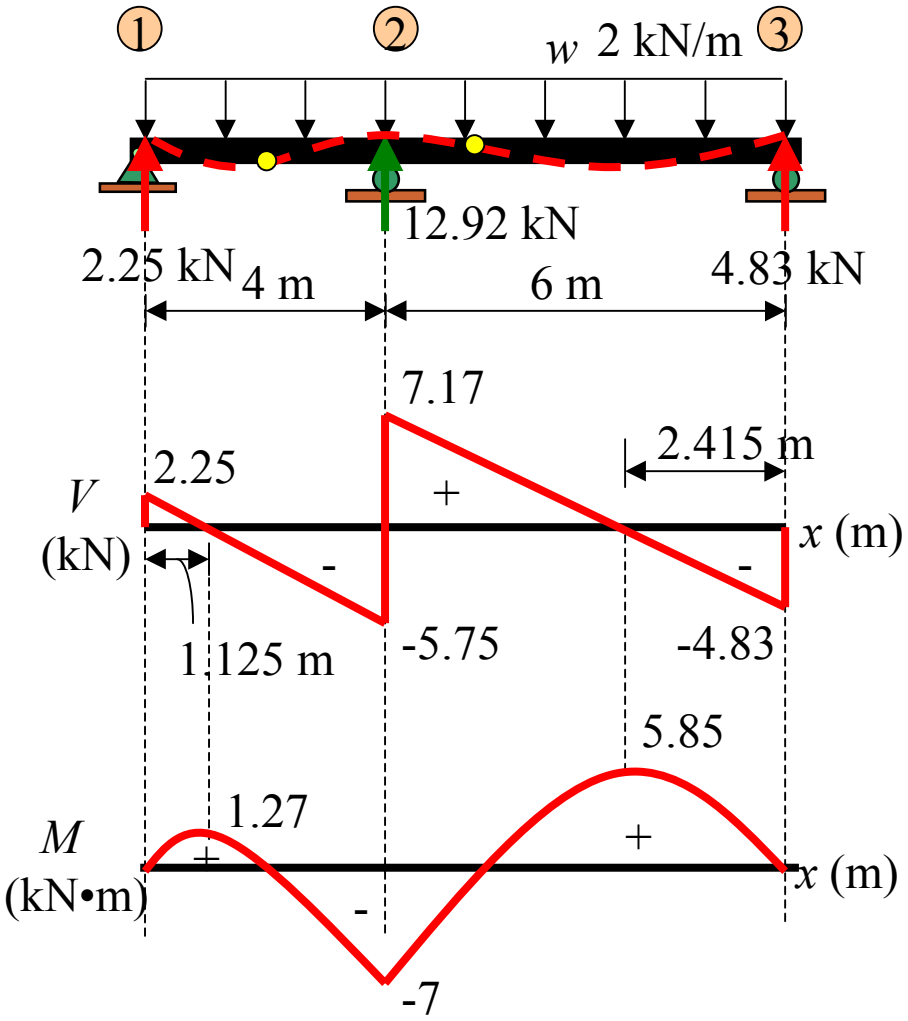
$$= \frac{19.2}{(200)(200)} = +0.48\,mm, \quad \uparrow$$

For $\Delta_2 = 0$

Compatibility equation:

$$-6.2 + 0.48R_2 = 0$$

$$R_2 = 12.92 \text{ kN}$$

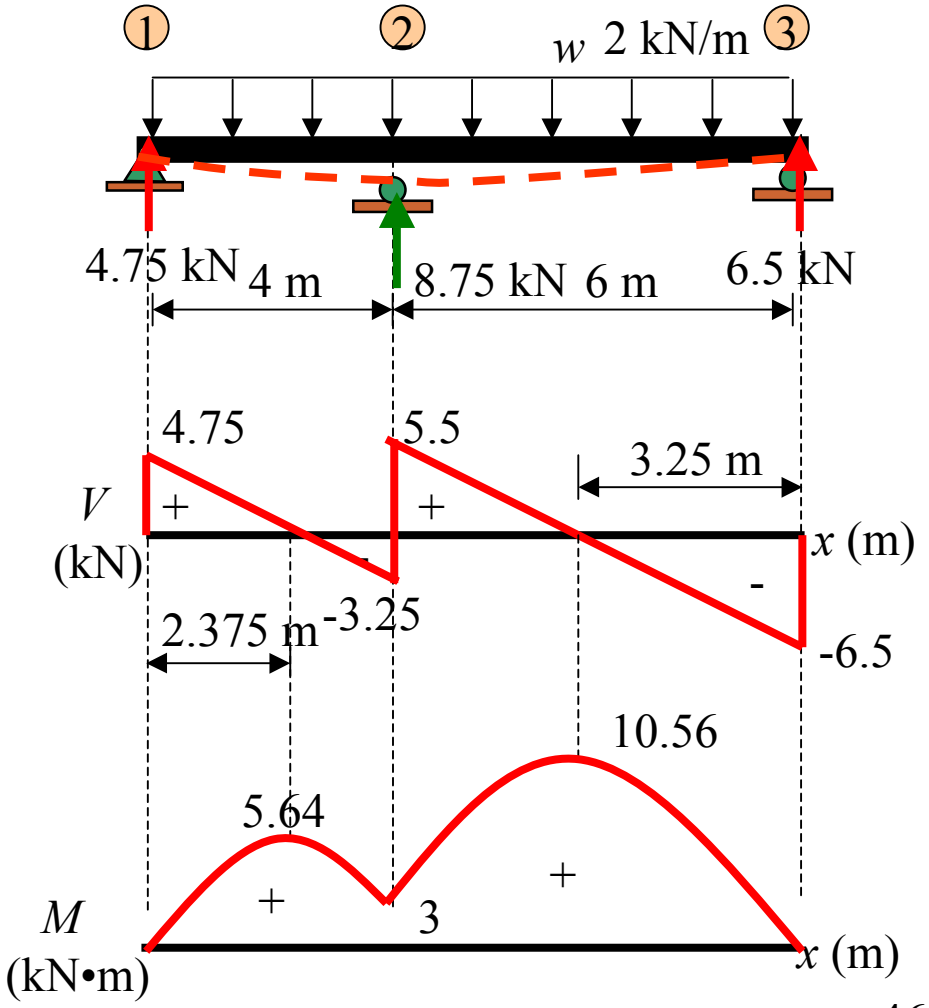


For $\Delta_2 = 2$ mm

Compatibility.equation:

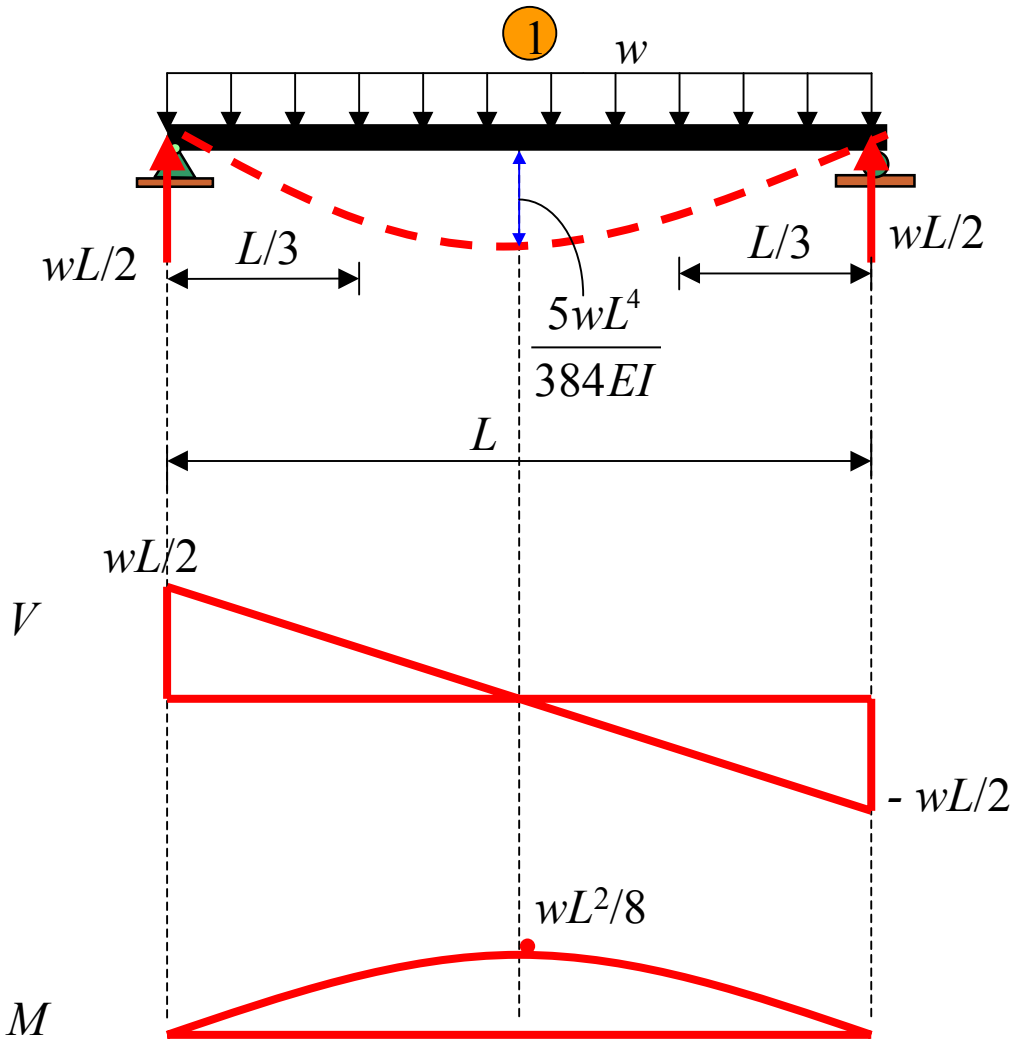
$$-6.2 + 0.48R_2 = -2$$

$$R_2 = 8.75 \text{ kN}$$

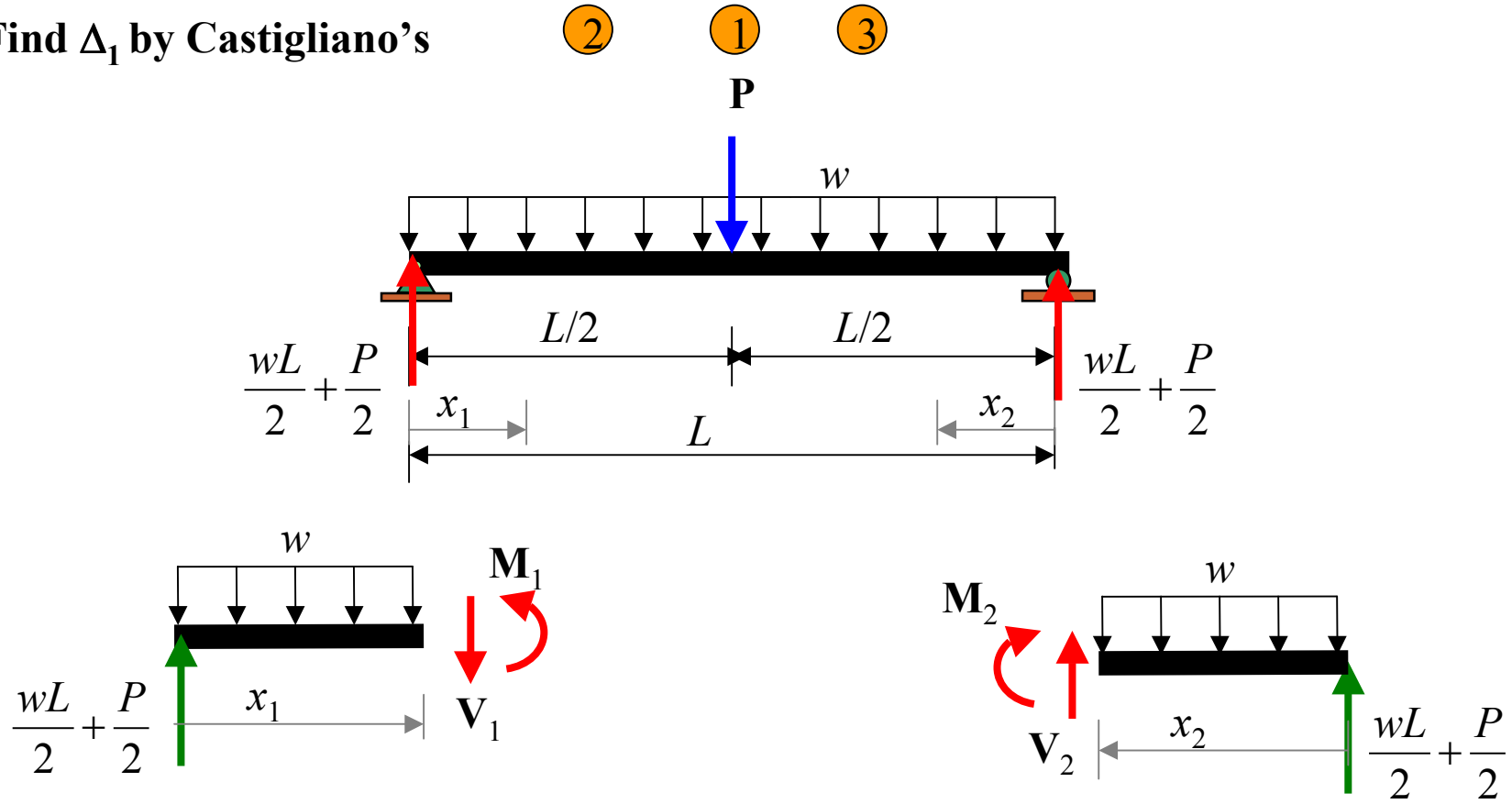


APPENDIX

Basic Beams: Single span



• **Find Δ_1 by Castigliano's**



$+\circlearrowleft \Sigma M = 0;$

$$M_1 + \frac{wx_1^{\,2}}{2} - (\frac{wL}{2} + \frac{P}{2})x_1 = 0$$

$$M_1 = -\frac{wx_1^{\,2}}{2} + (\frac{wL}{2} + \frac{P}{2})x_1 = M_2$$

$$M_1 = -\frac{wx_1^{\,2}}{2} + (\frac{wL}{2} + \frac{P}{2})x_1 = M_2$$

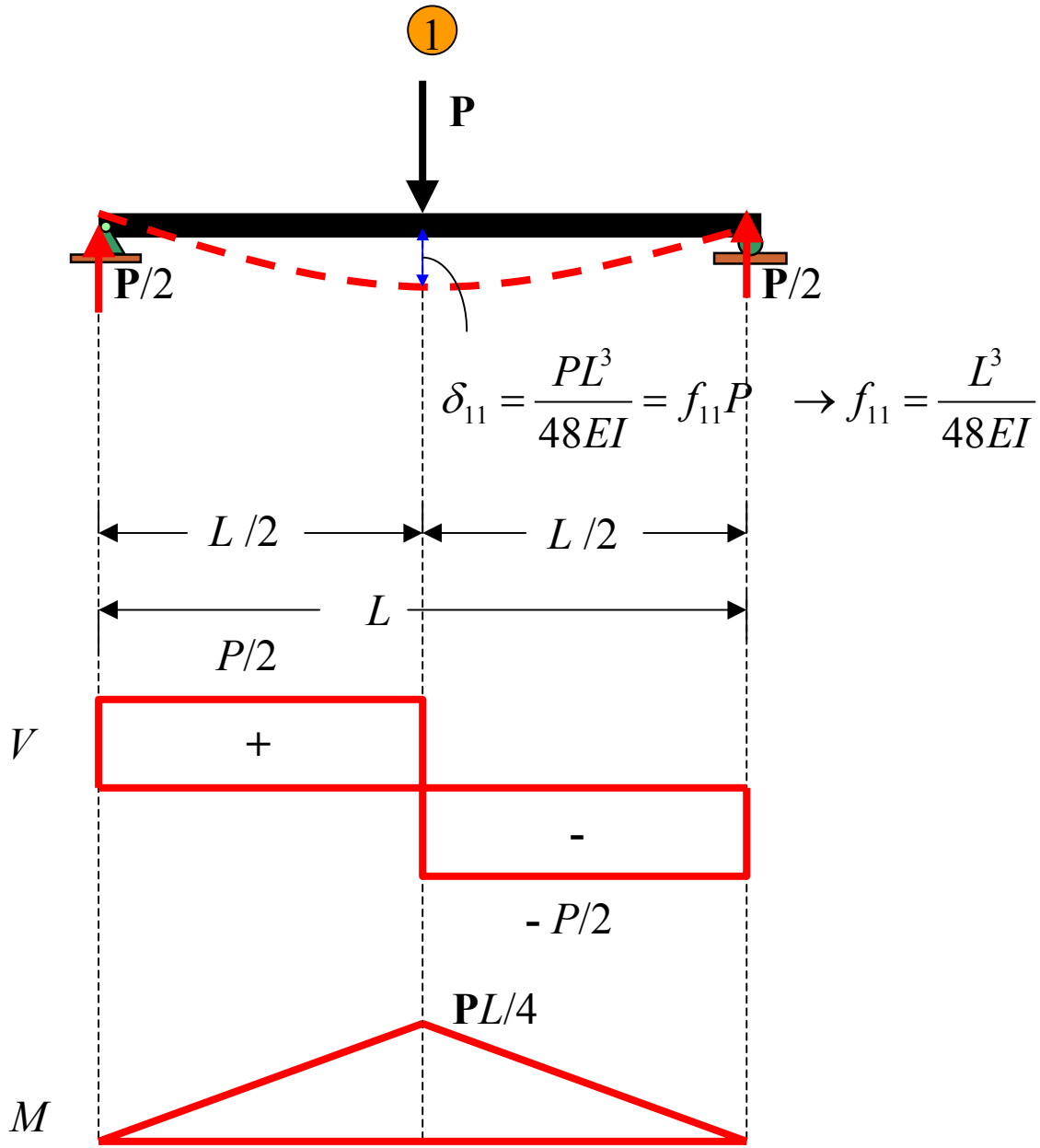
$$\Delta_1 = \Delta_{\max} = \int_0^{L/2} (\frac{\partial M_1}{\partial P}) M_1 \frac{dx_1}{EI} + \int_0^{L/2} (\frac{\partial M_2}{\partial P}) M_2 \frac{dx_2}{EI}$$

$$= 2\int_0^{L/2} (\frac{\partial M_1}{\partial P}) M_1 \frac{dx_1}{EI}$$

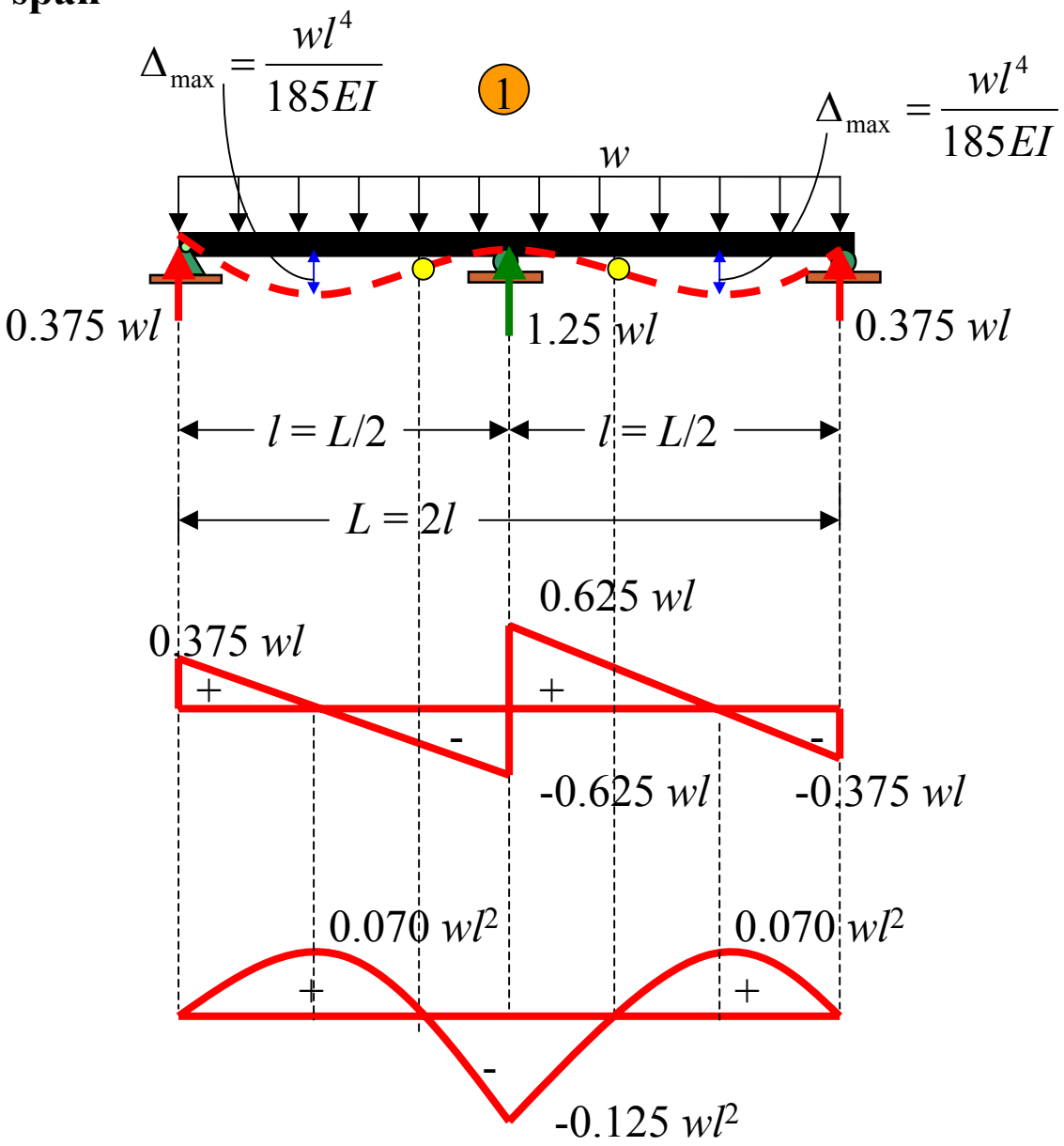
$$= \frac{2}{EI} \int_0^{L/2} (\frac{x_1}{2})(\frac{-wx_1^{\,2}}{2} + (\frac{wL}{2} + \frac{\overset{0}{\nearrow}P}{2})x_1)dx_1$$

$$= \frac{5wL^4}{384EI}$$

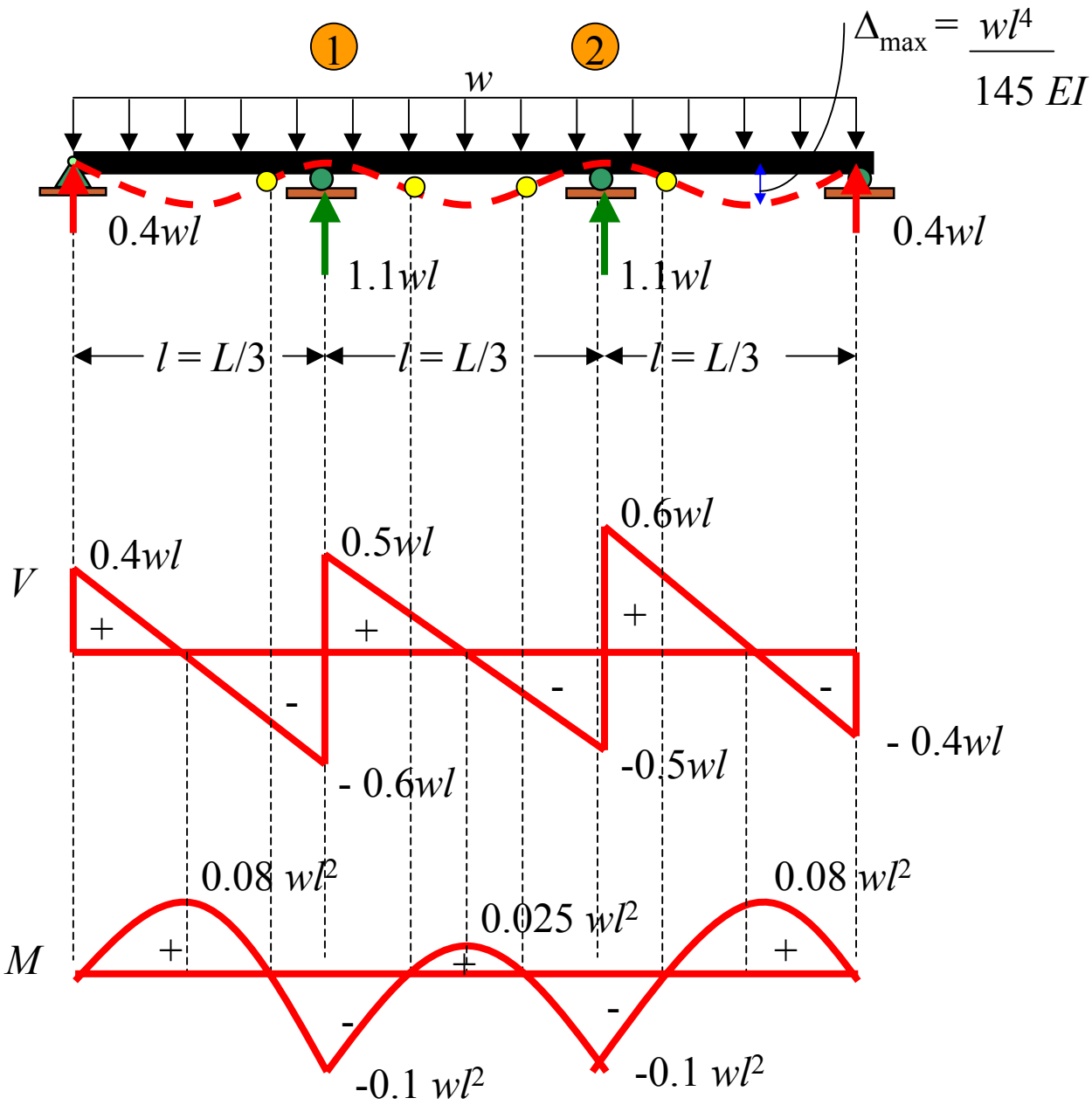
Single span


1
P
P/2
P/2
$\delta_{11} = \frac{PL^3}{48EI} = f_{11}P \quad \rightarrow f_{11} = \frac{L^3}{48EI}$
L /2
L /2
L
P/2
V
+
-
- P/2
PL/4
M

Double span


$\Delta_{max} = \frac{wl^4}{185EI}$
1
$\Delta_{max} = \frac{wl^4}{185EI}$
w
0.375 wl
1.25 wl
0.375 wl
l = L/2
l = L/2
L = 2l
0.625 wl
0.375 wl
+
+
-
-
-0.625 wl
-0.375 wl
0.070 wl²
0.070 wl²
+
+
-
-0.125 wl²

Triple span


1
2
w
$\Delta_{max} = \frac{wl^4}{145\ EI}$
0.4wl
1.1wl
1.1wl
0.4wl
l = L/3
l = L/3
l = L/3
V
0.4wl
0.5wl
0.6wl
+
+
+
-
-
-
- 0.6wl
-0.5wl
- 0.4wl
M
0.08 wl²
0.025 wl²
0.08 wl²
+
+
+
-
-
-0.1 wl²
-0.1 wl²