

Figure 3.4 Transfer functions of armature-controlled dc motor.

their angular velocities. Let w be the angular velocity of the motor shaft. Then we have $w(t) = d\theta/dt$ and

$$W(s) = s\Theta(s) = s \cdot \frac{k_m}{s(\tau_m s + 1)} \cdot U(s) = \frac{k_m}{\tau_m s + 1} \cdot U(s)$$

The transfer function of the motor from applied voltage u to angular velocity w is

$$\frac{W(s)}{U(s)} = \frac{k_m}{\tau_m s + 1} \tag{3.18}$$

as shown in Figure 3.4(b). Equation (3.18) differs from (3.15) by the absence of one pole at $s = 0$. Thus the transfer function of motors can be (3.15) or (3.18), depending on what is considered as the output. Therefore it is important to specify the input and output in using a transfer function. Different specifications lead to different transfer functions for the same system.

3.2.3 Measurement of Motor Transfer Functions

The transfer function of a motor driving a load is given by (3.15). To use the formula, we must use (3.16) and (3.17) to compute k_m and τ_m which, in turn, require k_t, k_b, R_a, L_a, J, and f. Although the first four constants may be supplied by the manufacturer of the motor, we still need J and f. The moment of inertia J of a load is not easily computable if it is not of regular shape. Neither can f be obtained analytically. Therefore, we may have to obtain J and f by measurements. If measurements are to be carried out, we may as well measure the transfer function directly. We discuss one way of measuring motor transfer functions in the following.

Let the transfer function of a motor driving a load be of the form shown in (3.15). If we apply a voltage and measure its angular velocity, then the transfer function reduces to (3.18). Let the applied voltage be a; the velocity can then be computed as

$$W(s) = \frac{k_m}{\tau_m s + 1} \cdot \frac{a}{s} = \frac{k_m a}{s} - \frac{k_m a \tau_m}{\tau_m s + 1}$$

Its inverse Laplace transform is

$$w(t) = k_m a - k_m a e^{-t/\tau_m} \tag{3.19}$$

for $t \geq 0$. Because τ_m is positive, the term e^{-t/τ_m} approaches zero as t approaches infinity and the response $w(t)$ is as shown in Figure 3.5(a). Thus we have

$$w(\infty) = k_m a \tag{3.20}$$

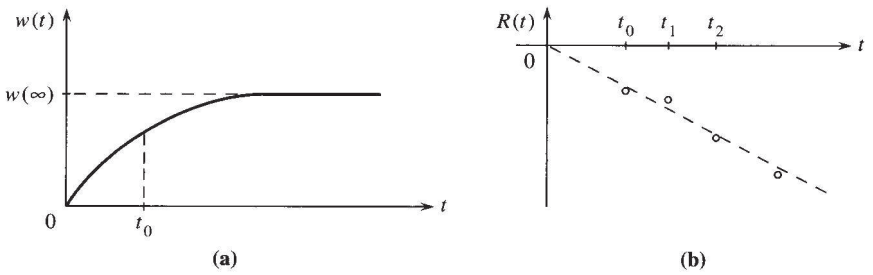


Figure 3.5 (a) Step response of motor. (b) Estimation of τ_m.

This is called the *final speed* or *steady-state speed.* If we apply a voltage of known magnitude a and measure the final speed $w(\infty)$, then the motor gain constant can be easily obtained as $k_m = w(\infty)/a$. To find the motor time constant, we rewrite (3.19) as

$$w(t) = k_m a(1 - e^{-t/\tau_m})$$

which implies, using (3.20),

$$e^{-t/\tau_m} = 1 - \frac{w(t)}{k_m a} = 1 - \frac{w(t)}{w(\infty)}$$

Thus we have

$$-\frac{t}{\tau_m} = \ln\left(1 - \frac{w(t)}{w(\infty)}\right) =: R(t) \tag{3.21}$$

where ln stands for the natural logarithm. Now if we measure the speed at any t, say $t = t_0$, then from (3.21) we have

$$\tau_m = \frac{-t_0}{\ln\left(1 - \dfrac{w(t_0)}{w(\infty)}\right)} \tag{3.22}$$

Thus the motor time constant can be obtained from the final speed and one additional measurement at any t.

Example 3.2.1

Consider an armature-controlled dc motor driving a load. We apply 5 V to the motor. The angular speed of the load at $t_0 = 2$ s is measured as 30 rad/s and the final speed is measured as 70 rad/s. What are the transfer functions from the applied voltage to the speed and from the applied voltage to the displacement?

From (3.20) and (3.22), we have

$$k_m = \frac{w(\infty)}{a} = \frac{70}{5} = 14$$

and

$$\tau_m = \frac{-2}{\ln\left(1 - \frac{30}{70}\right)} = \frac{-2}{\ln 0.57} = \frac{-2}{-0.5596} = 3.57$$

Thus the transfer functions are

$$\frac{W(s)}{U(s)} = \frac{14}{3.57s + 1} \qquad \frac{\Theta(s)}{U(s)} = \frac{14}{s(3.57s + 1)}$$

Exercise 3.2.2

Consider an armature-controlled dc motor driving a load. We apply 10 volts to the motor. The velocity of the load is measured as 20 cycles per second at $t_0 = 5$ s, and its steady-state velocity is measured as 30 cycles per second. What is its transfer function from the applied voltage u to the angular displacement $\theta(t)$? What is its unit-step response? What is $\theta(\infty)$? What does it mean physically?

[**Answers:** $G(s) = 18.84/s(4.55s + 1) = 4.14/s(s + 0.22)$, $\theta(t) = 85.54e^{-0.22t} - 85.54 + 18.82t$, $\theta(\infty) = \infty$, it means that the load will continue to turn with steady-state speed 18.82 rad/s and the displacement from the original position becomes larger and larger.]

In (3.22), in addition to the final speed, we used only one datum to compute τ_m. A more accurate τ_m can be obtained by using more data. We plot $R(t)$ in (3.21) for a number of t as shown in Figure 3.5(b), and draw from the origin a straight line passing through these points. Then the slope of the straight line equals $1/\tau_m$. This is a more accurate way of obtaining the motor time constant. This method can also check whether or not the simplified transfer functions in (3.18) and (3.15) can be used. If all points in the plot are very close to the straight line, then (3.18) and (3.15) are adequate. If not, more accurate transfer functions such as the one in (3.14) must be used to describe the motor.

The problem of determining τ_m and k_m in (3.18) from measurements is called *parameter estimation.* In this problem, the form of the transfer function is assumed to be known, but its parameters are not known. We then determine the parameters from measurements. This is a special case of the identification problem where neither the form nor the parameters are assumed to be known. If no noise exists in measurements, then the identification problem is not difficult. See Reference [15]. However, in practice, noise often arises in measurements. This makes the problem difficult. For a different method of identification, see Section 8.3.2.

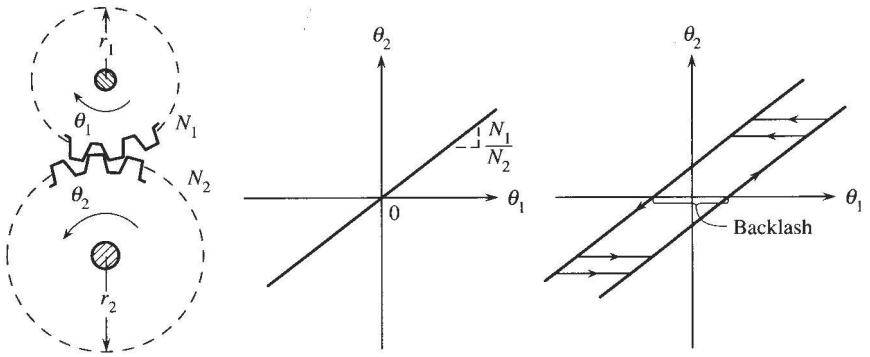


Figure 3.6 Gear train and its characteristics.

3.3 GEARS

Gears are used to convert high speed and small torque into low speed and high torque or the converse. The most popular type of gear is the spur gear shown in Figure 3.6(a). Let r_i be the radius; N_i, the number of teeth; θ_i, the angular displacement, and T_i, the torque for gear i, $i = 1, 2$. Clearly, the number of teeth on a gear is linearly proportional to its radius; hence $N_1/r_1 = N_2/r_2$. The linear distance traveled along the surfaces of both gears are the same, thus $\theta_1 r_1 = \theta_2 r_2$. The linear forces developed at the contact point of both gears are equal, thus $T_1/r_1 = T_2/r_2$. These equalities can be combined to yield

$$\frac{T_1}{T_2} = \frac{N_1}{N_2} = \frac{\theta_2}{\theta_1} \tag{3.23}$$

This is a linear equation as shown in Figure 3.6(b) and is obtained under idealized conditions. In reality, backlash exists between coupled gears. So long as the gears are rotating in one direction, the teeth will remain in contact. However, reversal of the direction of the driving gear will disengage the teeth and the driven gear will remain stationary until reengagement.[3] Therefore, the relationship between θ_1 and θ_2 should be as shown in Figure 3.6(c), rather than in Figure 3.6(b). Keeping the backlash small will increase the friction between the teeth and wear out the teeth faster. On the other hand, an excessive amount of backlash will cause what is called the *chattering, hunting*, or *limited-cycle* problem in control systems. To simplify analysis and design, we use the linear equation in (3.23).

Consider an armature-controlled dc motor driving a load through a gear train as shown in Figure 3.7(a). The numbers of teeth are N_1 and N_2. Let the total moment

[3]Here we assume that the mass of the gear is zero, or that the gear has no inertia.

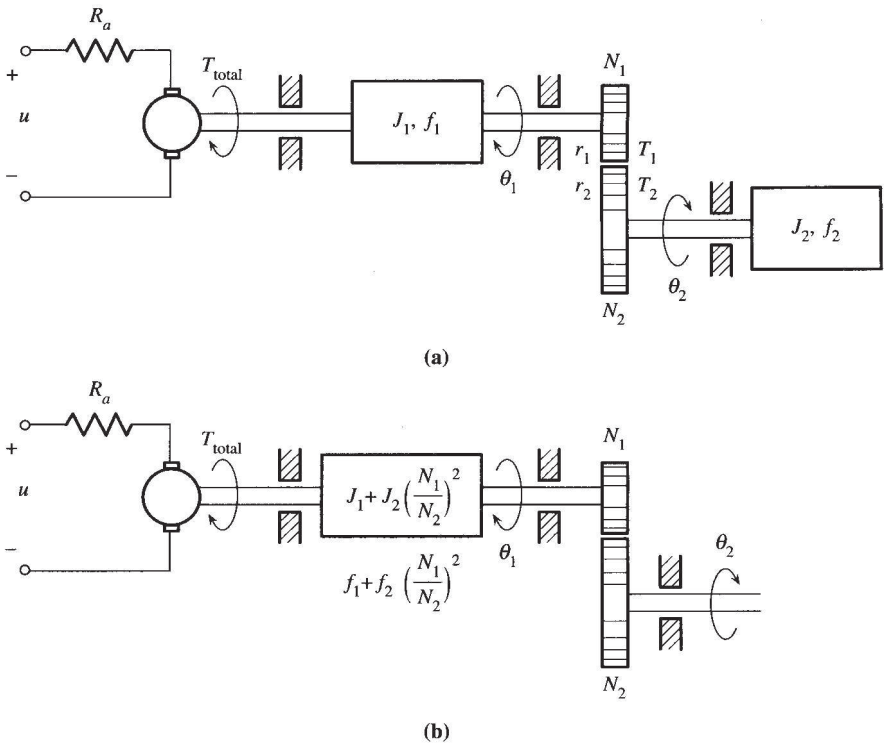


Figure 3.7 Gear train and its equivalence.

of inertia (including rotor, motor shaft, and gear 1) and viscous friction coefficient on the motor shaft be, respectively, J_1 and f_1, and those on the load shaft be J_2 and f_2. The torque generated by the motor must drive J_1, overcome f_1, and generate a torque T_1 at gear 1 to drive the second gear. Thus we have

$$T_{\text{motor}} = J_1 \frac{d^2\theta_1(t)}{dt^2} + f_1 \frac{d\theta_1(t)}{dt} + T_1 \tag{3.24}$$

Torque T_1 at gear 1 generates a torque T_2 at gear 2, which in turn drives J_2 and overcomes f_2. Thus we have

$$T_2 = J_2 \frac{d^2\theta_2(t)}{dt^2} + f_2 \frac{d\theta_2(t)}{dt}$$

which, using $\theta_2 = (N_1/N_2)\theta_1$, becomes

$$T_2 = J_2 \frac{N_1}{N_2} \frac{d^2\theta_1(t)}{dt^2} + f_2 \frac{N_1}{N_2} \frac{d\theta_1(t)}{dt} \tag{3.25}$$

The substitution of this T_2 into $T_1 = (N_1/N_2)T_2$ and then into (3.24) yields

$$\begin{aligned} T_{\text{motor}} &= J_1 \frac{d^2\theta_1(t)}{dt^2} + f_1 \frac{d\theta_1(t)}{dt} + J_2 \left(\frac{N_1}{N_2}\right)^2 \frac{d^2\theta_1(t)}{dt^2} \\ &\quad + f_2 \left(\frac{N_1}{N_2}\right)^2 \frac{d\theta_1(t)}{dt} \\ &= J_{1\text{eq}} \frac{d^2\theta_1(t)}{dt^2} + f_{1\text{eq}} \frac{d\theta_1(t)}{dt} \end{aligned} \tag{3.26}$$

where

$$J_{1\text{eq}} := J_1 + J_2 \left(\frac{N_1}{N_2}\right)^2 \qquad f_{1\text{eq}} := f_1 + f_2 \left(\frac{N_1}{N_2}\right)^2 \tag{3.27}$$

This process transfers the load and friction on the load shaft into the motor shaft as shown in Figure 3.7(b). Using this equivalent diagram, we can now compute the transfer function and develop a block diagram for the motor and load.

Once the load and friction on the load shaft are transferred to the motor shaft, we can simply disregard the gear train and the load shaft. If the armature inductance L_a is assumed to be zero, then the transfer function of the motor and load from u to θ_1 is, as in (3.15),

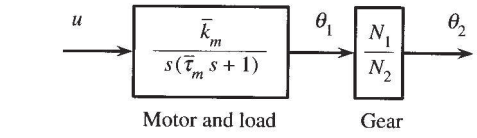
$$G(s) = \frac{\Theta_1(s)}{U(s)} = \frac{\bar{k}_m}{s(\bar{\tau}_m s + 1)} \tag{3.28}$$

with (3.16) and (3.17) modified as

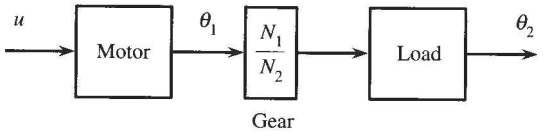
$$\bar{k}_m = \frac{k_t}{k_t k_b + f_{1\text{eq}} R_a} \tag{3.29a}$$

and

$$\bar{\tau}_m = \frac{J_{1\text{eq}} R_a}{k_t k_b + f_{1\text{eq}} R_a} \tag{3.29b}$$



(a)



(b)

Figure 3.8 (a) Block diagram. (b) Incorrect block diagram.

The transfer function from θ_1 to θ_2 is N_1/N_2. Thus the block diagram of the motor driving a load through a gear train is as shown in Figure 3.8(a). We remark that because of the loading, it is not possible to draw a block diagram as shown in Figure 3.8(b).

3.4 TRANSDUCERS

Transducers are devices that convert signals from one form to another—for example, from a mechanical shaft position, temperature, or pressure to an electrical voltage. They are also called *sensing devices*. There are all types of transducers such as thermocouples, strain gauges, pressure gauges, and others. We discuss in the following only potentiometers and tachometers.

Potentiometers

The potentiometer is a device that can be used to convert a linear or angular displacement into a voltage. Figure 2.14 shows a wire-wound potentiometer with its characteristic. The potentiometer converts the angular displacement $\theta(t)$ into an electric voltage $v(t)$ described by

$$v(t) = k\theta(t) \tag{3.30}$$

where k is a constant and depends on the applied voltage and the type of potentiometer used. The application of the Laplace transform to (3.30) yields

$$V(s) = k\Theta(s) \tag{3.31}$$

Thus the transfer function of the potentiometer is a constant. Figure 3.9 shows three commercially available potentiometers.

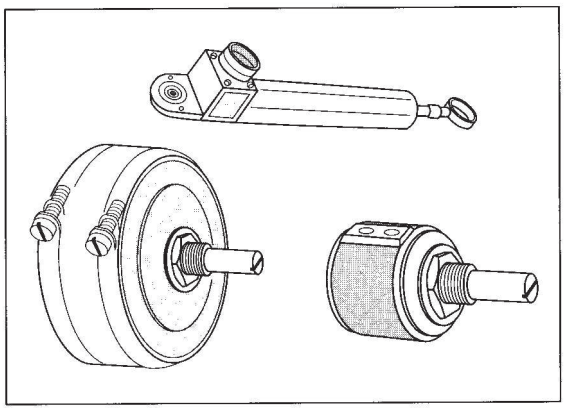


Figure 3.9 Potentiometers.