# 3 Development of Block Diagrams for Control Systems

## 3.1 INTRODUCTION

The design of control systems can be divided into two distinct parts. One is concerned with the design of individual components, the other with the design of overall systems by utilizing existing components. The former belongs to the domain of instrumentation engineers; the latter, the domain of control engineers. This is a control text, so we are mainly concerned with utilization of existing components. Consequently, our discussion of control components stresses their functions rather than their structures.

Control components can be mechanical, electrical, hydraulic, pneumatic, or optical devices. Depending on whether signals are modulated or not, electrical devices again are divided into ac (alternating current) or dc (direct current) devices. Even a cursory introduction of these devices can easily take up a whole text, so this will not be attempted. Instead, we select a number of commonly used control components, discuss their functions, and develop their transfer functions. The loading problem will be considered in this development. We then show how these components are connected to form control systems. Block diagrams of these control systems are developed. Finally, we discuss manipulation of block diagrams. Mason's formula is introduced to compute overall transfer functions of block diagrams.

## 3.2 MOTORS

Motors are indispensable in many control systems. They are used to turn antennas, telescopes, and ship rudders; to close and open valves; to drive tapes in recorders, and rollers in steel mills; and to feed paper in printers. There are many types of motors: dc, ac, stepper, and hydraulic. The magnetic field of a dc motor can be excited by a circuit connected in series with the armature circuit; it can also be excited by a field circuit that is independent of the armature circuit or by a permanent magnet. We discuss in this text only separately excited dc motors. DC motors used in control systems, also called *servomotors*, are characterized by large torque to rotor-inertia ratios, small sizes, and better linear characteristics. Certainly, they are more expensive than ordinary dc motors.

### 3.2.1 Field-Controlled DC Motor

Most of the dc motors used in control systems can be modeled as shown in Figure 3.1. There are two circuits, called the field circuit and armature circuit. Let $i_f$ and $i_a$ be the field current and armature current, respectively. Then the torque $T$ generated by the motor is given by

$$T(t) = ki_a(t)i_f(t) \tag{3.1}$$

where $k$ is a constant. The generated torque is used to drive a load through a shaft. The shaft is assumed to be rigid. To simplify analysis, we consider only the viscous friction between the shaft and bearing. Let $J$ be the total moment of inertia of the load, the shaft, and the rotor of the motor; $\theta$, the angular displacement of the load; and $f$, the viscous friction coefficient of the bearing. Then we have, as developed in (2.5),

$$T(t) = J\frac{d^2\theta(t)}{dt^2} + f\frac{d\theta(t)}{dt} \tag{3.2}$$

This describes the relationship between the motor torque and load's angular displacement.

If the armature current $i_a$ is kept constant and the input voltage $u(t)$ is applied

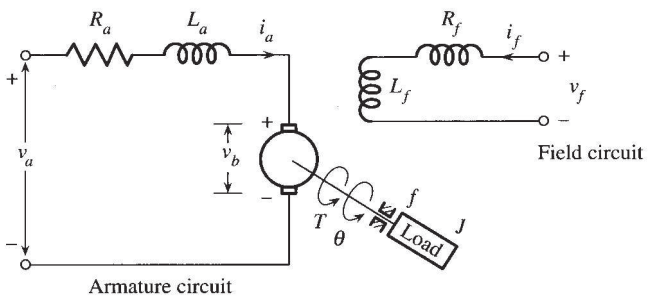



**Figure 3.1** DC motor.

to the field circuit, the motor is called a *field-controlled* dc motor. We now develop its block diagram.

In the field-controlled dc motor, the armature current $i_a(t)$ is constant. Therefore, (3.1) can be reduced to

$$T(t) = ki_a(t)i_f(t) =: k_f i_f(t) \tag{3.3}$$

where $k_f = ki_a$. From the field circuit, we have

$$L_f \frac{di_f(t)}{dt} + R_f i_f(t) = v_f(t) = u(t) \tag{3.4}$$

The application of the Laplace transform to (3.3) and (3.4) yields,[1] assuming zero initial conditions,

$$T(s) = k_f I_f(s)$$

$$L_f s I_f(s) + R_f I_f(s) = U(s)$$

which imply

$$I_f(s) = \frac{1}{L_f s + R_f} U(s)$$

and

$$T(s) = \frac{k_f}{L_f s + R_f} U(s)$$

Thus, if the generated torque is considered as the output of the motor, then the transfer function of the field-controlled dc motor is

$$G_m(s) := \frac{T(s)}{U(s)} = \frac{k_f}{L_f s + R_f} \tag{3.5}$$

This transfer function remains the same no matter what load the motor drives. Now we compute the transfer function of the load. The application of the Laplace transform to (3.2) yields, assuming zero initial conditions,

$$T(s) = Js^2\Theta(s) + fs\Theta(s)$$

which can be written as

$$G_l(s) := \frac{\Theta(s)}{T(s)} = \frac{1}{s(Js + f)}$$

This is the transfer function of the load if we consider the motor torque the input, and the load's angular displacement the output. Thus, the motor and load in

[1]Capital letters are used to denote the Laplace transforms of the corresponding lowercase letters. In the case of $T(t)$ and $T(s)$, we use the arguments to differentiate them.

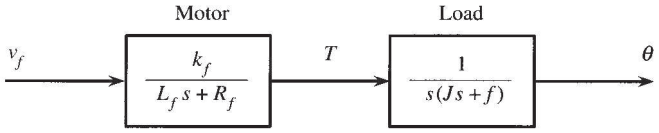



**Figure 3.2** Block diagram of field-controlled dc motor.

Figure 3.1 can be modeled as shown in Figure 3.2. It consists of two blocks: One represents the motor; the other, the load. Note that any change of the load will not affect the motor transfer function, so there is no loading problem in the connection.

The transfer function of the field-controlled dc motor driving a load is the product of $G_m(s)$ and $G_l(s)$ or

$$G(s) = G_m(s)G_l(s) = \frac{k_f}{s(L_f s + R_f)(Js + f)} = \frac{k_f/R_f f}{s(\tau_f s + 1)(\tau_m s + 1)} =: \frac{k_m}{(\tau_f s + 1)s(\tau_m s + 1)} \tag{3.6}$$

where $\tau_f := L_f/R_f$, $\tau_m := J/f$ and $k_m := k_f/R_f f$. The constant $\tau_f$ depends only on the electric circuit and is therefore called the motor *electrical time constant.* The constant $\tau_m$ depends only on the load and is called the motor *mechanical time constant.* The physical meaning of the time constant will be discussed in the next chapter. If the electrical time constant $\tau_f$ is much smaller than the mechanical time constant $\tau_m$, as is often the case in practice, then $G(s)$ is often approximated by

$$G(s) = \frac{k_m}{s(\tau_m s + 1)} \tag{3.7}$$

For a discussion of this type of approximation, see Reference [18]. Note that $\tau_m$ depends only on the load.

The field-controlled dc motor is used to drive a load, yet there is no loading problem in Figure 3.2. How could this be possible? Different loads will induce different $i_a$; even for the same load, $i_a$ will not be constant. Therefore, the loading problem is eliminated by keeping $i_a$ constant. In practice, it is difficult to keep $i_a$ constant, so the field-controlled dc motor is rarely used.

### 3.2.2 Armature-Controlled DC Motor

Consider the dc motor shown in Figure 3.1. If the field current $i_f(t)$ is kept constant or the field circuit is replaced by a permanent magnetic field, and if the input voltage is applied to the armature circuit, then the motor is called an *armature-controlled* dc motor. We now develop its transfer function. If $i_f(t)$ is constant, (3.1) can be written as

$$T(t) = k_t i_a(t) \tag{3.8}$$

where $k_t := ki_f(t)$ is a constant. When the motor is driving a load, a back electromotive force (back emf) voltage $v_b$ will develop in the armature circuit to resist the

applied voltage. The voltage $v_b(t)$ is linearly proportional to the angular velocity of the motor shaft:

$$v_b(t) = k_b \frac{d\theta(t)}{dt} \tag{3.9}$$

Thus the armature circuit in Figure 3.1 is described by

$$R_a i_a(t) + L_a \frac{di_a(t)}{dt} + v_b(t) = v_a(t) = u(t) \tag{3.10}$$

or

$$R_a i_a(t) + L_a \frac{di_a(t)}{dt} + k_b \frac{d\theta(t)}{dt} = u(t) \tag{3.11}$$

Using (3.2), (3.8), and (3.11), we now develop a transfer function for the motor. The substitution of (3.8) into (3.2) and the application of the Laplace transform to (3.2) and (3.11) yield, assuming zero initial conditions,

$$k_t I_a(s) = Js^2\Theta(s) + fs\Theta(s) \tag{3.12}$$

$$R_a I_a(s) + L_a s I_a(s) + k_b s\Theta(s) = U(s) \tag{3.13}$$

The elimination of $I_a$ from these two equations yields

$$G(s) = \frac{\Theta(s)}{U(s)} = \frac{k_t}{s[(Js + f)(R_a + L_a s) + k_t k_b]} \tag{3.14}$$

This is the transfer function from $v_a = u$ to $\theta$ of the armature-controlled dc motor.

Using (3.8), (3.9), (3.10), and (3.12), we can draw a block diagram for the armature-controlled dc motor as shown in Figure 3.3. Although we can draw a block for the motor and a block for the load, the two blocks are not independent as in the case of field-controlled dc motors. A signal from inside the load is fed back to the input of the motor. This essentially takes care of the loading problem. Because of the loading, it is simpler to combine the motor and load into a single block with the transfer function given in (3.14).

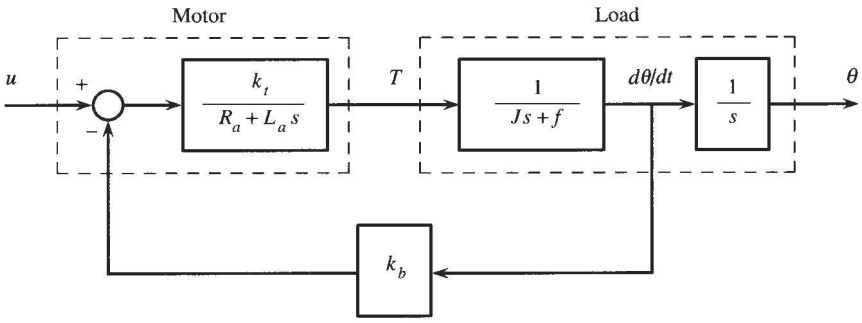



**Figure 3.3** Block diagram of armature-controlled dc motor.

In application, the armature inductance $L_a$ is often set to zero.[2] In this case, (3.14) reduces to

$$G(s) = \frac{\Theta(s)}{U(s)} = \frac{k_t}{s(JR_a s + k_t k_b + fR_a)} = \frac{k_m}{s(\tau_m s + 1)} \tag{3.15}$$

where

$$k_m := \frac{k_t}{k_t k_b + fR_a} =: \text{motor gain constant} \tag{3.16}$$

and

$$\tau_m := \frac{JR_a}{k_t k_b + fR_a} =: \text{motor time constant} \tag{3.17}$$

The time constant in (3.17) depends on the load as well as the armature circuit. This is in contrast to (3.7) where $\tau_m$ depends only on the load. The electrical and mechanical time constants are well defined in (3.6) and (3.7), but this is not possible in (3.14) and (3.15). We often call (3.14) and (3.15) *motor transfer functions*, although they are actually the transfer functions of the motor *and* load. The transfer function of the same motor will be different if it drives a different load.

**Exercise 3.2.1**

Consider an armature-controlled dc motor with

$$R_a = 20\ \Omega \qquad k_t = 1\ \text{N·m/A} \qquad k_b = 3\ \text{V·s/rad}$$

If the motor is used to drive a load with moment of inertia 2 N·m·s$^2$/rad and with viscous friction coefficient 0.1 N·m·s/rad, what is its transfer function? If the moment of inertia of the load and the viscous friction coefficient are doubled, what is its transfer function? Do the time and gain constants equal half of those of the first transfer function?

[**Answers:** $0.2/s(8s + 1)$, $0.14/s(11.4s + 1)$, no.]

The input and output of the transfer function in (3.15) are the applied electrical voltage and the angular position of the motor shaft as shown in Figure 3.4(a). If motors are used in velocity control systems to drive tapes or conveyers, we are no longer interested in their angular positions $\theta$. Instead we are mainly concerned with

[2]In engineering, it is often said that the electrical time constant is much smaller than the mechanical time constant, thus we can set $L_a = 0$. If we write (3.14) as $G(s) = k_m/s(as + 1)(bs + 1)$, then $a$ and $b$ are called time constants. Unlike (3.6), $a$ and $b$ depend on both electrical and mechanical parts of the motor. Therefore, electrical and mechanical time constants in armature-controlled dc motors are no longer as well defined as in field-controlled dc motors.

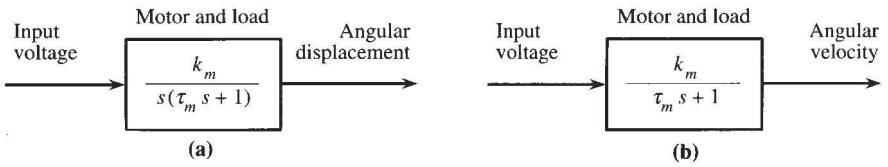



**Figure 3.4** Transfer functions of armature-controlled dc motor.

their angular velocities. Let $w$ be the angular velocity of the motor shaft. Then we have $w(t) = d\theta/dt$ and

$$W(s) = s\Theta(s) = s \cdot \frac{k_m}{s(\tau_m s + 1)} \cdot U(s) = \frac{k_m}{\tau_m s + 1} \cdot U(s)$$

The transfer function of the motor from applied voltage $u$ to angular velocity $w$ is

$$\frac{W(s)}{U(s)} = \frac{k_m}{\tau_m s + 1} \tag{3.18}$$

as shown in Figure 3.4(b). Equation (3.18) differs from (3.15) by the absence of one pole at $s = 0$. Thus the transfer function of motors can be (3.15) or (3.18), depending on what is considered as the output. Therefore it is important to specify the input and output in using a transfer function. Different specifications lead to different transfer functions for the same system.

### 3.2.3 Measurement of Motor Transfer Functions

The transfer function of a motor driving a load is given by (3.15). To use the formula, we must use (3.16) and (3.17) to compute $k_m$ and $\tau_m$ which, in turn, require $k_t$, $k_b$, $R_a$, $L_a$, $J$, and $f$. Although the first four constants may be supplied by the manufacturer of the motor, we still need $J$ and $f$. The moment of inertia $J$ of a load is not easily computable if it is not of regular shape. Neither can $f$ be obtained analytically. Therefore, we may have to obtain $J$ and $f$ by measurements. If measurements are to be carried out, we may as well measure the transfer function directly. We discuss one way of measuring motor transfer functions in the following.

Let the transfer function of a motor driving a load be of the form shown in (3.15). If we apply a voltage and measure its angular velocity, then the transfer function reduces to (3.18). Let the applied voltage be $a$; the velocity can then be computed as

$$W(s) = \frac{k_m}{\tau_m s + 1} \cdot \frac{a}{s} = \frac{k_m a}{s} - \frac{k_m a \tau_m}{\tau_m s + 1}$$

Its inverse Laplace transform is

$$w(t) = k_m a - k_m a e^{-t/\tau_m} \tag{3.19}$$

for $t \geq 0$. Because $\tau_m$ is positive, the term $e^{-t/\tau_m}$ approaches zero as $t$ approaches infinity and the response $w(t)$ is as shown in Figure 3.5(a). Thus we have

$$w(\infty) = k_m a \tag{3.20}$$