Example 2.8.2

Consider the state-variable equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -6 & -3.5 \\ 6 & 4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \tag{2.80a}$$

$$y = [-2 \quad -1]\mathbf{x} \tag{2.80b}$$

Equation (2.80a) is identical to (2.79a); (2.80b), however, is different from (2.79b). The characteristic polynomial of (2.80) is

$$\Delta(s) = s^2 + 2s - 3$$

Its eigenvalues are -3 and 1. The transfer function of (2.80) is, replacing [4 5] in (2.66) by $[-2 \quad -1]$,

$$\begin{aligned} G(s) &= [-2 \quad -1] \frac{1}{s^2 + 2s - 3} \begin{bmatrix} s - 4 & -3.5 \\ 6 & s + 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2 + 2s - 3} [-2 \quad -1] \begin{bmatrix} -s + 0.5 \\ s \end{bmatrix} = \frac{2s - 1 - s}{s^2 + 2s - 3} \\ &= \frac{s - 1}{(s - 1)(s + 3)} \end{aligned} \tag{2.81}$$

Thus the transfer function of (2.80) is $G(s) = 1/(s + 3)$. It has only one pole at -3, one less than the number of the eigenvalues of (2.80).

A state-variable equation is called *minimal* if the number of its eigenvalues or the dimension of the equation equals the number of the poles of its transfer function. The state-variable equation in (2.79) is such an equation. Every eigenvalue of a minimal state-variable equation will appear as a pole of its transfer function and there is no essential difference between using the state-variable equation or its transfer function to study the system. Thus, the state-variable equation is said to be *completely characterized* by its transfer function. As will be discussed in Chapter 11, every minimal state-variable equation has the properties of controllability and observability. If a state-variable equation is not minimal, such as the one in (2.80), then some eigenvalues will not appear as poles of its transfer function and the transfer function is said to have *missing poles*. In this case, the state-variable equation is not completely characterized by its transfer function and we cannot use the transfer function to carry out analysis and design. This situation is similar to the one in Section 2.5.

2.8 DISCRETIZATION OF STATE EQUATIONS

Consider the state-variable equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \tag{2.82a}$$

$$y(t) = \mathbf{c}\mathbf{x}(t) + du(t) \tag{2.82b}$$

This equation is defined at every instant of time and is called a *continuous-time equation.* If a digital computer is used to generate the input u, then $u(t)$ is (as will be discussed in Chapter 12) stepwise, as shown in Figure 2.23(a). Let

$$u(t) = u(k) \qquad \text{for } kT \leq t < (k+1)T,\ k = 1, 2, 3, \ldots \tag{2.83}$$

where T is called the *sampling period.* This type of signal is completely specified by the sequence of numbers $u(k)$, $k = 0, 1, 2, \ldots$, as shown in Figure 2.23(b). The signal in Figure 2.23(a) is called a *continuous-time or analog signal* because it is defined for all time; the signal in Figure 2.23(b) is called a *discrete-time signal* because it is defined only at discrete instants of time. Note that a continuous-time signal may not be a continuous function of time as shown in the figure.

If we apply a stepwise input to (2.82), the output $y(t)$ is generally not stepwise. However, if we are interested in only the output $y(t)$ at $t = kT$, $k = 0, 1, 2, \ldots$, or $y(k) := y(kT)$, then it is possible to develop an equation simpler than (2.82) to describe $y(k)$. We develop such an equation in the following.

The solution of (2.82) was developed in (2.72) as

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau \tag{2.84}$$

This equation holds for any t—in particular, for t at kT. Let $\mathbf{x}(k) := \mathbf{x}(kT)$ and $\mathbf{x}(k+1) := \mathbf{x}((k+1)T)$. Then we have

$$\mathbf{x}(k) = e^{\mathbf{A}kT}\mathbf{x}(0) + e^{\mathbf{A}kT}\int_0^{kT} e^{-A\tau}\mathbf{b}u(\tau)d\tau \tag{2.85}$$

$$\mathbf{x}(k+1) = e^{\mathbf{A}(k+1)T}\mathbf{x}(0) + e^{\mathbf{A}(k+1)T}\int_0^{(k+1)T} e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau \tag{2.86}$$

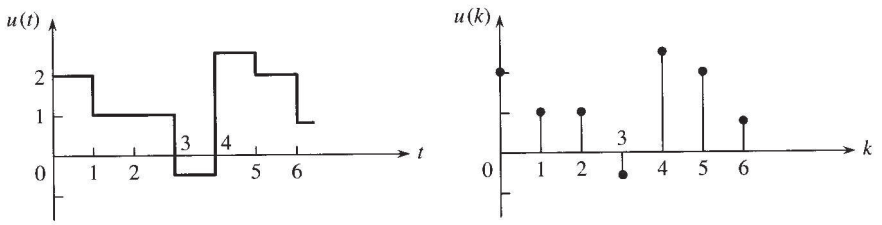


Figure 2.23 (a) Stepwise continuous-time signal. (b) Discrete-time signal.

We rewrite (2.86) as

$$\mathbf{x}(k+1) = e^{\mathbf{A}T}\left[e^{\mathbf{A}kT}\mathbf{x}(0) + e^{\mathbf{A}kT}\int_0^{kT} e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau\right] + \int_{kT}^{(k+1)T} e^{\mathbf{A}((k+1)T-\tau)}\mathbf{b}u(\tau)d\tau \tag{2.87}$$

The term inside the brackets equals $\mathbf{x}(k)$. If the input $u(t)$ is stepwise as in (2.83), then $u(\tau)$ in the integrand of the last term of (2.87) equals $u(k)$ and can be moved outside the integration. Define $\alpha = (k+1)T - \tau$. Then $d\alpha = -d\tau$ and (2.87) becomes

$$\begin{aligned}\mathbf{x}(k+1) &= e^{\mathbf{A}T}\mathbf{x}(k) - \left(\int_T^0 e^{\mathbf{A}\alpha}d\alpha\right)\mathbf{b}u(k) \\ &= e^{\mathbf{A}T}\mathbf{x}(k) + \left(\int_0^T e^{A\alpha}d\alpha\right)\mathbf{b}u(k)\end{aligned} \tag{2.88}$$

This and (2.82b) can be written as

$$\mathbf{x}(k+1) = \tilde{\mathbf{A}}\mathbf{x}k + \tilde{\mathbf{b}}u(k) \tag{2.89a}$$

$$y(k) = \tilde{\mathbf{c}}\mathbf{x}(k) + \tilde{d}u(k) \tag{2.89b}$$

with

$$\tilde{\mathbf{A}} = e^{\mathbf{A}T} \quad \tilde{\mathbf{b}} = \left(\int_0^T e^{\mathbf{A}\tau}d\tau\right)\mathbf{b} \quad \tilde{\mathbf{c}} = \mathbf{c} \quad \tilde{d} = d \tag{2.89c}$$

This is called a *discrete-time state-variable equation.* It consists of a set of first-order *difference* equations. We use an example to illustrate how to obtain such an equation.

Example 2.9.1

Consider the continuous-time state-variable equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \tag{2.90a}$$

$$y = [2 \quad 1]\mathbf{x} \tag{2.90b}$$

Discretize the equation with sampling period $T = 0.1$.

First we compute

$$\mathcal{L}[e^{\mathbf{A}t}] = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+1 & -1 \\ 0 & s+1 \end{bmatrix}^{-1}$$

$$= \frac{1}{(s+1)^2}\begin{bmatrix} s+1 & 1 \\ 0 & s+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)^2} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$

Its inverse Laplace transform is, using Table A.1,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

Using an integration table, we can compute

$$\int_0^T e^{\mathbf{A}\tau}d\tau = \begin{bmatrix} 1 - e^{-T} & 1 - (1+T)e^{-T} \\ 0 & 1 - e^{-T} \end{bmatrix}$$

If $T = 0.1$, then

$$e^{0.1\mathbf{A}} = \begin{bmatrix} 0.9048 & 0.0905 \\ 0 & 0.9048 \end{bmatrix} \quad \text{and} \quad \left(\int_0^{0.1} e^{\mathbf{A}\tau}d\tau\right)\mathbf{b} = \begin{bmatrix} 0.0047 \\ 0.0952 \end{bmatrix} \tag{2.91}$$

Thus the discrete-time state-variable equation is

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.9048 & 0.0905 \\ 0 & 0.9048 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 0.0047 \\ 0.0952 \end{bmatrix}u(k) \tag{2.92a}$$

$$y(k) = [2 \quad 1]\mathbf{x}(k) \tag{2.92b}$$

This completes the discretization. This discretized equation is used on digital computers, as will be discussed in Chapter 5, to compute the response of the continuous-time equation in (2.90).

We have used the Laplace transform to compute $e^{\mathbf{A}T}$ in (2.91). If T is small, the infinite series in (2.67) may converge rapidly. For example, for $e^{\mathbf{A}T}$ in (2.91) with $T = 0.1$, we have

$$\mathbf{I} + T\mathbf{A} + \frac{1}{2}(T\mathbf{A})(T\mathbf{A}) = \begin{bmatrix} 0.9050 & 0.0900 \\ 0 & 0.9050 \end{bmatrix}$$

and

$$\mathbf{I} + T\mathbf{A} + \frac{1}{2}(T\mathbf{A})(T\mathbf{A}) + \frac{1}{6}(T\mathbf{A})(T\mathbf{A})(T\mathbf{A}) = \begin{bmatrix} 0.9048 & 0.0905 \\ 0 & 0.9048 \end{bmatrix}$$

The infinite series converges to the actual values in four terms. Therefore for small T, the infinite series in (2.67) is a convenient way of computing e^{AT}. In addition to the Laplace transform and infinite series, there are many ways (at least 17) of computing e^{AT}. See Reference [47]. In Chapter 5, we will introduce computer software to solve and to discretize state-variable equations.

PROBLEMS

2.1. Consider the pendulum system shown in Figure P2.1, where the applying force u is the input and the angular displacement θ is the output. It is assumed that there is no friction in the hinge and no air resistance of the mass. Is the system linear? Is it time-invariant? Find a differential equation to describe it. Is the differential equation LTIL?

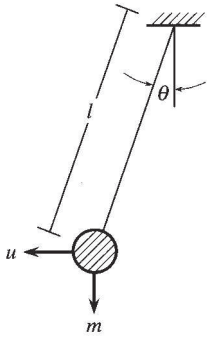


Figure P2.1

2.2. Consider the pendulum system in Figure P2.1. It is assumed that $\sin\theta$ and $\cos\theta$ can be approximated by θ and 1, respectively, if $|\theta| < \pi/4$ radians. Find an LTIL differential equation to describe the system.

2.3. **a.** Consider the system shown in Figure P2.3(a). The two blocks with mass m_1 and m_2 are connected by a rigid shaft. It is assumed that there is no friction between the floor and wheels. Find a differential equation to describe the system.

b. If the shaft is long and flexible, then it must be modeled by a spring with spring constant k_2 and a dashpot. A dashpot is a device that consists of a

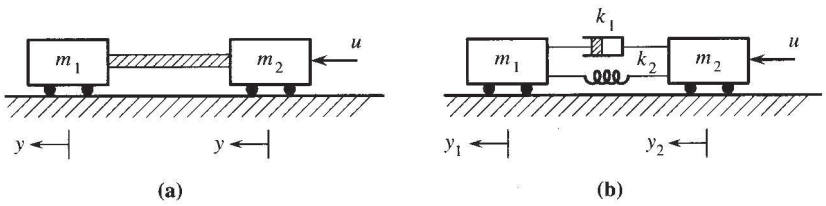


Figure P2.3

piston and oil-filled cylinder and is used to provide viscous friction. It is assumed that the force generated by the dashpot equals $k_1(dy_2/dt - dy_1/dt)$. Verify that the system is described by

$$m_2 \frac{d^2y_2(t)}{dt^2} = u(t) - k_1\left(\frac{dy_2(t)}{dt} - \frac{dy_1(t)}{dt}\right) - k_2(y_2(t) - y_1(t))$$

and

$$m_1 \frac{d^2y_1(t)}{dt^2} = -k_1\left(\frac{dy_1(t)}{dt} - \frac{dy_2(t)}{dt}\right) - k_2(y_1(t) - y_2(t))$$

2.4. If a robot arm is long, such as the one on a space shuttle, then its employment must be modeled as a flexible system, as shown in Figure P2.4. Define

T: Applied torque

k_1: Viscous-friction coefficient

k_2: Torsional spring constant

J_i: Moment of inertia

θ_i: Angular displacement

Find an LTIL differential equation to describe the system.

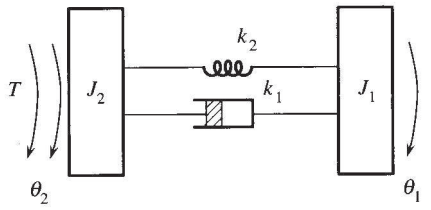
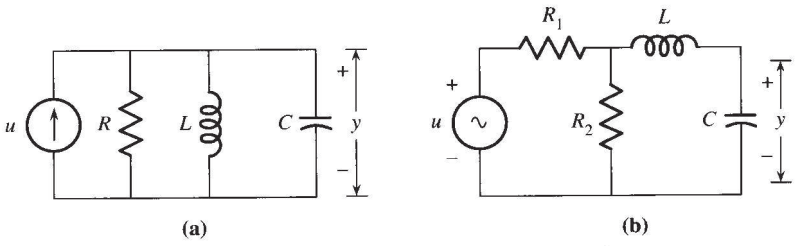


Figure P2.4

2.5. Find LTIL differential equations to describe the networks in Figure P2.5.



(a) (b)

Figure P2.5

2.6. Consider the network shown in Figure P2.6(a), in which T is a tunnel diode with characteristics as shown. Show that if v lies between a and b, then the circuit can be modeled as shown in Figure P2.6(b). Show also that if v lies between c and d, then the circuit can be modeled as shown in Figure P2.6(c),

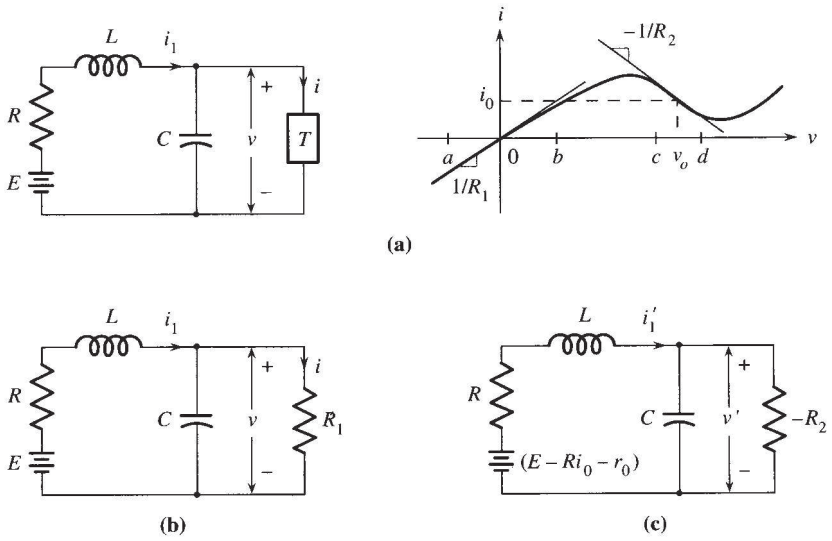


Figure P2.6

where $i_1' = i_1 - i_o$, $v' = v - v_o$. These linearized models are often called *linear incremental models*, or *small-signal models*.

2.7. Consider the LTIL differential equation

$$\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 3y(t) = 2\frac{du(t)}{dt} + u(t)$$

a. Find its zero-input response due to $y(0^-) = 1$ and $\dot{y}(0^-) = 2$.

b. What are its characteristic polynomial and modes?

c. Find the set of all initial conditions such that the mode 3 will not be excited.

d. Find the set of all initial conditions such that the mode -1 will not be excited.

e. Plot (c) and (d) on a two-dimensional plane with y and $\dot{y}$ as coordinates. If a set of initial conditions is picked randomly, what is the probability that the two modes will be excited?

2.8. Consider the differential equation in Problem 2.7

a. Find its unit-step response, that is, the zero-state response due to a unit-step input $u(t)$.

b. What are its transfer function, its poles and zeros?

c. Find a bounded input that will not excite pole 3.

d. Find a bounded input that will not excite pole -1.

2.9. Consider the differential equation in Problem 2.7. Find the response due to $y(0^-) = 1$, $\dot{y}(0^-) = 2$, and $u(t) = 2$, for $t \geq 0$.

2.10. Find the transfer function from u to θ of the system in Problem 2.2.

2.11. Find the transfer functions for the networks in Figure P2.5. Compute them from differential equations. Also compute them using the concept of Laplacian impedances.

2.12. Find the transfer functions from u to y, u to y_1, and u to y_2 of the systems in Figure P2.3.

2.13. Consider a system. If its zero-state response due to a unit-step input is measured as $y(t) = 1 - e^{-2t} + \sin t$, where is the transfer function of the system?

2.14. Consider a system. If its zero-state response due to $u(t) = \sin 2t$ is measured as $y(t) = e^{-t} - 2e^{-3t} + \sin 2t + \cos 2t$, what is the transfer function of the system?

2.15. Consider the LTIL system described by

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} - 3y(t) = \frac{du(t)}{dt} - u(t)$$

a. Find the zero-input response due to $y(0^-) = 1$ and $\dot{y}(0^-) = 2$. What are the modes of the system?

b. Find the zero-state response due to $u(t) = 1$, for $t \geq 0$.

c. Can you detect all modes of the system from the zero-state response?

d. Is the system completely characterized by its transfer function? Will a serious problem arise if we use the transfer function to study the system?

2.16. Repeat Problem 2.15 for the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} - 3y(t) = \frac{du(t)}{dt} + 3u(t)$$

2.17. Consider a transfer function $G(s)$ with poles -1 and $-2 \pm j1$, and zero 3. Can you determine $G(s)$ uniquely? If it is also known that $G(2) = -0.1$, can you now determine $G(s)$ uniquely?

2.18. Find the unit-step response of the following systems and plot roughly the responses:

a. $G_1(s) = \dfrac{2}{(s+1)(s+2)}$

b. $G_2(s) = \dfrac{20(s+0.1)}{(s+1)(s+2)}$

c. $G_3(s) = \dfrac{0.2(s+10)}{(s+1)(s+2)}$

d. $G_4(s) = \dfrac{-20(s-0.1)}{(s+1)(s+2)}$

e. $G_5(s) = \dfrac{-0.2(s - 10)}{(s + 1)(s + 2)}$

Which type of zeros, closer to or farther away from the origin, has a larger effect on responses?

2.19. Find the poles and zeros of the following transfer functions:

a. $G(s) = \dfrac{2(s^2 - 9)(s + 1)}{(s + 3)^2(s + 2)(s - 1)^2}$

b. $G(s) = \dfrac{10(s^2 - s + 1)}{s^4 + 2s^3 + s + 2}$

c. $G(s) = \dfrac{2s^2 + 8s + 8}{(s + 1)(s^2 + 2s + 2)}$

2.20. a. Consider a system with transfer function $G(s) = (s - 2)/s(s + 1)$. Show that if $u(t) = e^t$, $t \geq 0$, then the response of the system can be decomposed as

$$\text{Total response} = \text{Response due to the poles of } G(s) + \text{Response due to the poles of } U(s)$$

b. Does the decomposition hold if $u(t) = 1$, for $t \geq 0$?

c. Does the decomposition hold if $u(t) = e^{2t}$, for $t \geq 0$?

d. Show that the decomposition is valid if $U(s)$ and $G(s)$ have no common pole and if there are no pole-zero cancellations between $U(s)$ and $G(s)$.

2.21. Show that the network in Figure 2.18 is described by the differential equation in (2.42).

2.22. Consider the simplified model of an aircraft shown in Figure P2.22. It is assumed that the aircraft is dynamically equivalent at the pitched angle θ_0, elevator angle u_0, altitude h_0, and cruising speed v_0. It is assumed that small deviations of θ and u from θ_0 and u_0 generate forces $f_1 = k_1\theta$ and $f_2 = k_2u$, as shown in the figure. Let m be the mass of the aircraft; I, the moment of

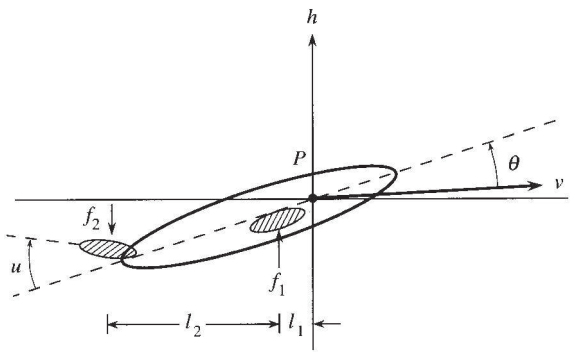


Figure P2.22

inertia about the center of gravity P; $b\dot{\theta}$, the aerodynamic damping and h, the deviation of the altitude from h_0. Show that the transfer function from u to h is, by neglecting the effect of I,

$$G(s) = \frac{k_1k_2l_2 - k_2bs}{ms^2(bs + k_1l_1)}$$

2.23. Consider a cart with a stick hinged on top of it, as shown in Figure P2.23. This could be a model of a space booster on takeoff. If the angular displacement θ of the stick is small, then the system can be described by

$$\ddot{\theta} = \theta + u$$

$$\ddot{y} = \beta\theta - u$$

where β is a constant, and u and y are expressed in appropriate units. Find the transfer functions from u to θ and from u to y. Is the system completely characterized by the transfer function from u to y? By the one from u to θ? Which transfer function can be used to study the system?

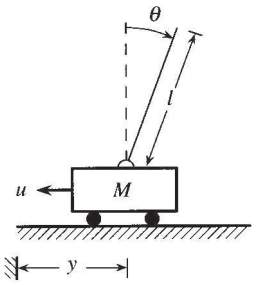


Figure P2.23

2.24. Show that the two tanks shown in Figure P2.24(a) can be represented by the block diagram shown in Figure P2.24(b). Is there any loading problem in the block diagram? The transfer function of the two tanks shown in Figure 2.12 is computed in Exercise 2.4.2. Is it possible to represent the two tanks in Figure 2.12 by two blocks as in Figure P2.24(b) on page 66? Give your reasons.

2.25. Find state-variable equations to describe the systems in Problems 2.1 and 2.2.

2.26. Find state-variable equations to describe the systems in Figure P2.3.

2.27. Find state-variable equations to describe the networks in Figure P2.5.

2.28. The soft landing phase of a lunar module descending on the moon can be modeled as shown in Figure P2.28. It is assumed that the thrust generated is proportional to $\dot{m}$, where m is the mass of the module. The system can be described by $m\ddot{y} = -k\dot{m} - mg$, where g is the gravity constant on the lunar surface. Define the state variables of the system as $x_1 = y$, $x_2 = \dot{y}$, $x_3 = m$, and $u = \dot{m}$. Find a state-variable equation to describe the system. Is it a time-invariant equation?

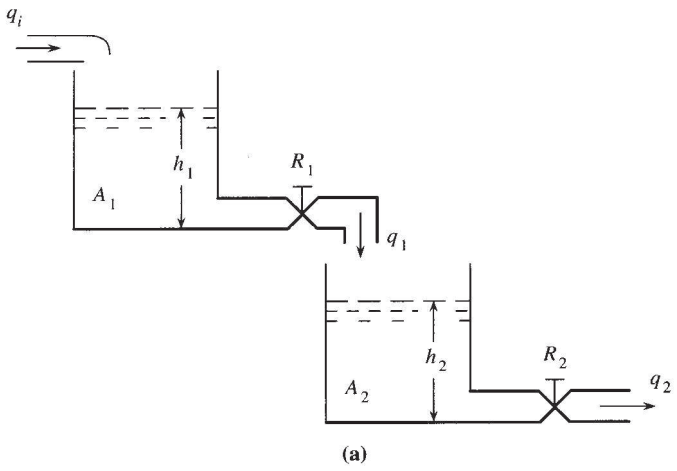


Figure P2.24

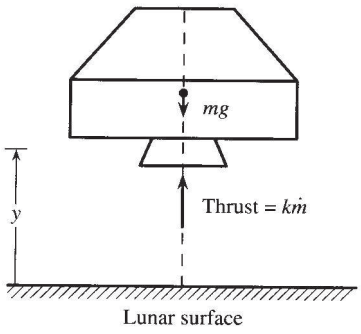


Figure P2.28

2.29. Use the Laplace transform to find the response of

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u$$

$$y = [4 \quad 5]\mathbf{x}$$

due to $\mathbf{x}(0) = [1 \quad 2]'$ and $u(t) = 1$ for $t \geq 0$.

2.30. Use the infinite series in (2.67) to compute $e^{\mathbf{A}t}$ with

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

Verify your results by using (2.74).

2.31. Compute the transfer functions of the following state-variable equations:

a. $\dot{\mathbf{x}} = -x + u \qquad y = 2x$

b. $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$y = [2 \quad -2]\mathbf{x}$

c. $\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} u$

$y = [1 \quad 0]\mathbf{x}$

d. $\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} u$

$y = [1 \quad 0 \quad 0]\mathbf{x}$

Do they have the same transfer functions? Which are minimal equations?

2.32. Find the characteristic polynomials for the matrices

$$\begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

2.33. Find the outputs of the equations in Problem 2.31 due to a unit-step input and zero initial state.

2.34. Compute the transfer functions of the state-variable equations in Problems 2.25(b), 2.26(a), and 2.27(a). Are they minimal state-variable equations?

2.35. Compute $y(k)$ for $k = 0, 1, 2, 3, 4$, for the discrete-time state-variable equation

$$\mathbf{x}(k+1) = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u(k)$$

$$y(k) = [1 \quad 0]\mathbf{x}(k)$$

due to $\mathbf{x}(0) = [1 \quad -2]'$ and $u(k) = 1$ for all $k \geq 0$.

2.36. Discretize the continuous-time equation with sampling period 0.1 second

$$\dot{\mathbf{x}} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} u$$

$$y = [1 \quad 0]\mathbf{x}$$

2.37. a. The characteristic polynomial of

$$\mathbf{A} = \begin{bmatrix} -6 & -3.5 \\ 6 & 4 \end{bmatrix}$$

is computed in (2.76) as

$$\Delta(s) = s^2 + 2s - 3$$

Verify

$$\Delta(\mathbf{A}) = \mathbf{A}^2 + 2\mathbf{A} - 3\mathbf{I} = \mathbf{0}$$

This is called the *Cayley-Hamilton Theorem.* It states that every square matrix meets its own characteristic polynomial.

b. Show that $\mathbf{A}^2, \mathbf{A}^3, \ldots,$ can be expressed as a linear combination of $\mathbf{I}$ and $\mathbf{A}$. In general, for a square matrix $\mathbf{A}$ of order n, $\mathbf{A}^k$ for $k \geq n$ can be expressed as a linear combination of $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^{n-1}$.

2.38. Show that if (2.67) is used, $\tilde{\mathbf{b}}$ in (2.89c) can be expressed as

$$\tilde{\mathbf{b}} = \left(\int_0^T e^{\mathbf{A}\tau} d\tau \right) \mathbf{b} = T \left(\mathbf{I} + \frac{T}{2!}\mathbf{A} + \frac{T^2}{3!}\mathbf{A}^2 + \frac{T^3}{4!}\mathbf{A}^3 + \cdots \right) \mathbf{b}$$

This can be used to compute $\tilde{\mathbf{b}}$ on a digital computer if it converges rapidly or if T is small.