Figure 2.20(b) equals

$$I_1(s) = \frac{U_1(s)}{1 + \dfrac{s(3s + 2)}{4s + 2}} \tag{2.47}$$

and the current I_2 equals

$$I_2(s) = \frac{s}{s + 3s + 2} \cdot I_1(s) = \frac{s}{4s + 2 + s(3s + 2)} \cdot U_1(s) \tag{2.48}$$

Thus, the voltage $y_2(t)$ is given by

$$Y_2(s) = I_2(s) \cdot 2 = \frac{2s}{3s^2 + 6s + 2} \cdot U_1(s) \tag{2.49}$$

and the transfer function from u_1 to y_2 of the tandem connection is

$$\frac{Y_2(s)}{U_1(s)} = \frac{2s}{3s^2 + 6s + 2} \tag{2.50}$$

This is different from the product of $G_1(s)$ and $G_2(s)$ or

$$G_1(s)G_2(s) = \frac{s}{s + 1} \cdot \frac{2}{3s + 2} = \frac{2s}{3s^2 + 5s + 2} \tag{2.51}$$

Thus, Equation (2.38) cannot be used for this connection.

The loading of the two networks in Figure 2.20 can be easily explained. The current I_2 in Figure 2.20(a) is zero before the connection; it becomes nonzero after the connection. Thus, the loading occurs. In electrical networks, the loading often can be eliminated by inserting an isolating amplifier, as shown in Figure 2.20(c). The input impedance Z_{in} of an ideal isolating amplifier is infinity and the output impedance Z_{out} is zero. Under these assumptions, I_2 in Figure 2.20(c) remains zero and the transfer function of the connection is $G_2(s)kG_1(s)$ with $k = 1$.

The loading problem must be considered in developing a block diagram for a control system. This problem is considered in detail in the next chapter.

2.6 STATE-VARIABLE EQUATIONS

The transfer function describes only the relationship between the input and output of LTIL systems and is therefore called the *input-output description* or *external description.* In this section we shall develop a different description, called the *state-variable description* or *internal description.* Strictly speaking, this description is the same as the differential equations discussed in Section 2.2. The only difference is that *high-order* differential equations are now written as sets of *first-order* differential equations. In this way, the study can be simplified.

The state-variable description of LTIL systems is of the form

$$\begin{aligned}\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + a_{13}x_3(t) + b_1u(t)\\ \dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + a_{23}x_3(t) + b_2u(t) \qquad (2.52a)\\ \dot{x}_3(t) &= a_{31}x_1(t) + a_{32}x_2(t) + a_{33}x_3(t) + b_3u(t)\end{aligned}$$

$$y(t) = c_1x_1(t) + c_2x_2(t) + c_3x_3(t) + du(t) \qquad (2.52b)$$

where u and y are the input and output; x_i, $i = 1, 2, 3$, are called *the state variables*; a_{ij}, b_i, c_i, and d are constants; and $\dot{x}_i(t) := dx_i(t)/dt$. These equations are more often written in matrix form as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \qquad \text{(State equation)} \qquad (2.53a)$$

$$y(t) = \mathbf{c}\mathbf{x}(t) + du(t) \qquad \text{(Output equation)} \qquad (2.53b)$$

with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad (2.54a)$$

and

$$\mathbf{c} = [c_1 \quad c_2 \quad c_3] \qquad (2.54b)$$

The vector $\mathbf{x}$ is called the *state vector* or simply the *state*. If $\mathbf{x}$ has n state variables or is an $n \times 1$ vector, then $\mathbf{A}$ is an $n \times n$ square matrix, $\mathbf{b}$ is an $n \times 1$ column vector, $\mathbf{c}$ is a $1 \times n$ row vector, and d is a 1×1 scalar. $\mathbf{A}$ is sometimes called the *system matrix* and d the *direct transmission part*. Equation (2.53a) describes the relationship between the input and state, and is called the *state equation*. The state equation in (2.53a) consists of three first-order differential equations and is said to have *dimension 3*. The equation in (2.53b) relates the input, state, and output, and is called the *output equation*. It is an algebraic equation; it does not involve differentiation of $\mathbf{x}$. Thus if $\mathbf{x}(t)$ and $u(t)$ are known, the output $y(t)$ can be obtained simply by multiplication and addition.

Before proceeding, we remark on the notation. Vectors are denoted by boldface lowercase letters; matrices by boldface capital letters. Scalars are denoted by regular-face lowercase letters. In (2.53), $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{x}$ are boldface because they are either vectors or matrices; u, y, and d are regular face because they are scalars.

This text studies mainly single-variable systems, that is, systems with single input and single output. For multivariable systems, we have two or more inputs and/or two and more outputs. In this case, $u(t)$ and $y(t)$ will be vectors and the orders of $\mathbf{b}$ and $\mathbf{c}$ and d must be modified accordingly. For example, if a system has three inputs and two outputs, then u will be 3×1; y, 2×1; $\mathbf{b}$, $n \times 3$; $\mathbf{c}$; $2 \times n$; and d, 2×3. Otherwise, the form of state-variable equations remains the same.

The transfer function describes only the zero-state response of systems. Thus, when we use the transfer function, the initial state or initial conditions of the system

must be assumed to be zero. In using the state-variable equation, no such assumption is necessary. The equation is applicable even if the initial state is nonzero. The equation describes not only the output but also the state variables. Because the state variables reside within a system and are not necessarily accessible from the input and output terminals, the state-variable equation is also called the *internal description*. Thus the state-variable description is more general than the transfer function description. However, if a system is completely characterized by its transfer function (Section 2.5), the two descriptions are essentially the same. This will be discussed further in a later section.

In developing state-variable equations for LTIL systems, we must first choose state variables. State variables are associated with energy. For example, the potential and kinetic energy of a mass are stored in its position and velocity. Thus, position and velocity can be chosen as state variables for a mass. For RLC networks, capacitors and inductors are energy storage elements because they can store energy in their electric and magnetic fields. Therefore, all capacitor voltages and inductor currents are generally chosen as state variables for RLC networks. Resistors will not store energy; all energy is dissipated as heat. Therefore, resistor voltages or currents are not state-variables. With this brief discussion, we are ready to develop state-variable equations to describe LTIL systems.

Example 2.6.1 (Mechanical system)

Consider the mechanical system shown in Figure 2.5(a). As discussed in (2.4), its differential equation description is

$$m\ddot{y}(t) + k_1\dot{y}(t) + k_2y(t) = u(t) \tag{2.55}$$

where $u(t)$ is the applied force (input), $y(t)$ is the displacement (output), $\dot{y}(t) := dy(t)/dt$, and $\ddot{y}(t) := d^2y(t)/dt^2$.

The potential energy and kinetic energy of a mass are stored in its position and velocity; therefore the position and velocity will be chosen as state variables. Define

$$x_1(t) := y(t) \tag{2.56a}$$

and

$$x_2(t) := \dot{y}(t) \tag{2.56b}$$

Then we have

$$\dot{x}_1(t) = \dot{y}(t) = x_2(t)$$

This relation follows from the definition of $x_1(t)$ and $x_2(t)$ and is independent of the system. Taking the derivative of $x_2(t)$ yields

$$\dot{x}_2(t) = \ddot{y}(t)$$

which becomes, after the substitution of (2.55) and (2.56),

$$\dot{x}_2(t) = \frac{1}{m}[-k_2 y(t) - k_1 \dot{y}(t) + u(t)]$$
$$= -\frac{k_2}{m} x_1(t) - \frac{k_1}{m} x_2(t) + \frac{1}{m} u(t)$$

These equations can be arranged in matrix form as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_2}{m} & -\frac{k_1}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t) \tag{2.57a}$$

$$y(t) = [1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2.57b}$$

This is a state-variable equation of dimension 2. Note that the direct transmission d in (2.53) is zero for this example.

We discuss in the following a procedure for developing state-variable equations for RLC networks that contain a voltage or current source.

Procedure for Developing State-Variable Equations for RLC Networks

Step 1: Assign all capacitor voltages and inductor currents as state variables. Write down capacitor currents and inductor voltages as shown in Figure 2.8. Note that currents flow from high to low potential.

Step 2: Use Kirchhoff's voltage and/or current laws to express every resistor's voltage and current in terms of state variables and, if necessary, the input.

Step 3: Use Kirchhoff's voltage and/or current laws to develop a state equation.

This procedure offers only a general guide. For a more detailed and specific procedure, see Reference [15]. We use an example to illustrate the procedure.

Example 2.6.2 (Electrical system)

Consider the RLC network shown in Figure 2.21. It consists of one resistor, one capacitor, and one inductor. The input $u(t)$ is a voltage source and the voltage across the 3-H inductor is chosen as the output.

Step 1: The capacitor voltage $x_1(t)$ and the inductor currents $x_2(t)$ are chosen as state variables. The capacitor current is $2\dot{x}_1(t)$, and the inductor voltage is $3\dot{x}_2(t)$.

Step 2: The current passing through the 4-Ω resistor clearly equals $x_2(t)$. Thus, the voltage across the resistor is $4x_2(t)$. The polarity of the voltage must be specified, otherwise confusion may occur.

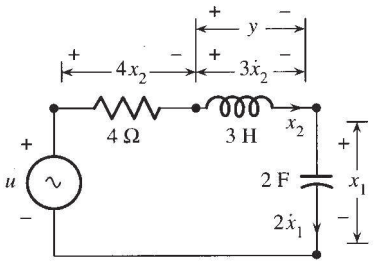


Figure 2.21 Network.

Step 3: From Figure 2.21, we see that the capacitor current $2\dot{x}_1$ equals $x_2(t)$, which implies

$$\dot{x}_1(t) = \frac{1}{2}x_2(t) \tag{2.58}$$

The voltage across the inductor is, using Kirchhoff's voltage law,

$$3\dot{x}_2(t) = u(t) - x_1(t) - 4x_2(t)$$

or

$$\dot{x}_2(t) = -\frac{1}{3}x_1(t) - \frac{4}{3}x_2(t) + \frac{1}{3}u(t) \tag{2.59}$$

Equations (2.58) and (2.59) can be arranged in matrix form as

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & \frac{1}{2}\\ -\frac{1}{3} & -\frac{4}{3}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{3}\end{bmatrix}u(t) \tag{2.60a}$$

This is the state equation of the network. The output $y(t)$ is

$$y(t) = 3\dot{x}_2$$

which is not in the form of $\mathbf{cx} + du$. However the substitution of (2.59) yields

$$\begin{aligned} y(t) &= -x_1(t) - 4x_2(t) + u(t)\\ &= [-1 \quad -4]\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + u(t)\end{aligned} \tag{2.60b}$$

This is the output equation.

Exercise 2.6.1

Find state-variable descriptions of the networks shown in Figure 2.22 with the state variables and outputs chosen as shown.

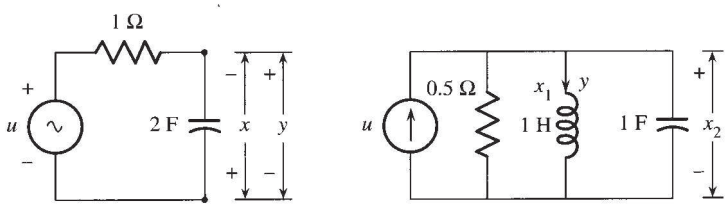


Figure 2.22 Networks.

[**Answers:** (a) $\dot{x} = -0.5x - 0.5u$, $y = -x$ (b) $\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$, $y = [1 \quad 0]\mathbf{x}$.]

2.7 SOLUTIONS OF STATE EQUATIONS—LAPLACE TRANSFORM METHOD

Consider the n-dimensional state-variable equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \tag{2.61a}$$

$$y(t) = \mathbf{c}\mathbf{x}(t) + du(t) \tag{2.61b}$$

where $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$, and d are respectively $n \times n$, $n \times 1$, $1 \times n$, and 1×1 matrices. In analysis, we are interested in the output $y(t)$ excited by an input $u(t)$ and some initial state $\mathbf{x}(0)$. This problem can be solved directly in the time domain or indirectly by using the Laplace transform. We study the latter in this section. The time-domain method will be studied in the next subsection.

The application of the Laplace transform to (2.61a) yields

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{b}U(s)$$

where $\mathbf{X}(s) = \mathcal{L}[\mathbf{x}(t)$ and $U(s) = \mathcal{L}[u(t)]$. This is an algebraic equation and can be arranged as

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{b}U(s) \tag{2.62}$$

where $\mathbf{I}$ is a unit matrix of the same order as $\mathbf{A}$. Note that without introducing $\mathbf{I}$, $(s - \mathbf{A})$ is not defined, for s is a scalar and $\mathbf{A}$ is an $n \times n$ matrix. The premultiplication of $(s\mathbf{I} - \mathbf{A})^{-1}$ to (2.62) yields

$$\mathbf{X}(s) = \underbrace{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)}_{\substack{\text{Zero-Input}\\ \text{Response}}} + \underbrace{(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}U(s)}_{\substack{\text{Zero-State}\\ \text{Response}}} \tag{2.63}$$

This response consists of the zero-input response (the response due to nonzero $\mathbf{x}(0)$) and the zero-state response (the response due to nonzero $u(t)$). If we substitute $\mathbf{X}(s)$ into the Laplace transform of (2.61b), then we will obtain

$$Y(s) = \underbrace{\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)}_{\substack{\text{Zero-Input}\\ \text{Response}}} + \underbrace{[\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d]U(s)}_{\substack{\text{Zero-State}\\ \text{Response}}} \tag{2.64}$$

This is the output in the Laplace transform domain. Its inverse Laplace transform yields the time response. Thus, the response of state-variable equations can be easily obtained using the Laplace transform.

Example 2.7.1

Consider the state-variable equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -6 & -3.5 \\ 6 & 4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}u \tag{2.65a}$$

$$y = [4 \quad 5]x \tag{2.65b}$$

Find the output due to a unit-step input and the initial state $\mathbf{x}(0) = [-2 \quad 1]'$, where the prime denotes the transpose of a matrix or a vector. First we compute

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -6 & -3.5 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} s + 6 & 3.5 \\ -6 & s - 4 \end{bmatrix}$$

Its inverse is

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s + 6 & 3.5 \\ -6 & s - 4 \end{bmatrix}^{-1}$$

$$= \frac{1}{(s + 6)(s - 4) + 6 \times 3.5}\begin{bmatrix} s - 4 & -3.5 \\ 6 & s + 6 \end{bmatrix}$$

Thus we have

$$\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) = [4 \quad 5]\frac{1}{s^2 + 2s - 3}\begin{bmatrix} s - 4 & -3.5 \\ 6 & s + 6 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2 + 2s - 3}[4s + 14 \quad 5s + 16]\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \frac{-3s - 12}{s^2 + 2s - 3}$$

and

$$\begin{aligned} G(s) &:= \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \\ &= [4 \quad 5]\frac{1}{s^2 + 2s - 3}\begin{bmatrix} s - 4 & -3.5 \\ 6 & s + 6 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{s + 2}{s^2 + 2s - 3} \end{aligned} \tag{2.66}$$

Thus, using (2.64) and $U(s) = 1/s$, the output is given by

$$Y(s) = \frac{-3s - 12}{s^2 + 2s - 3} + \frac{s + 2}{s^2 + 2s - 3} \cdot \frac{1}{s} = \frac{-3s^2 - 11s + 2}{(s - 1)(s + 3)s}$$

which can be expanded by partial fraction expansion as

$$Y(s) = \frac{-3}{s - 1} + \frac{2/3}{s + 3} - \frac{2/3}{s}$$

Thus the output is

$$y(t) = -3e^t + \frac{2}{3}e^{-3t} - \frac{2}{3}$$

for $t \geq 0$.

This example shows that the response of state-variable equations can indeed be readily obtained by using the Laplace transform.

2.7.1 Time-Domain Solutions

The solution of state-variable equations can also be obtained directly in the time domain without using the Laplace transform. Following

$$e^{at} = 1 + ta + \frac{t^2}{2!}a^2 + \cdots + \frac{t^n}{n!}a^n + \cdots$$

we define

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots + \frac{t^n}{n!}\mathbf{A}^n + \cdots \tag{2.67}$$

where $n! = n \cdot (n - 1) \cdots 2 \cdot 1$, $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AAA}$, and so forth. If $\mathbf{A}$ is an $n \times n$ matrix, then $e^{\mathbf{A}t}$ is also an $n \times n$ matrix. If $t = 0$, then (2.67) reduces to

$$e^{\mathbf{0}} = \mathbf{I} \tag{2.68}$$

As in $e^{a(t-\tau)} = e^{at}e^{-a\tau}$, we have

$$e^{\mathbf{A}(t-\tau)} = e^{\mathbf{A}t}e^{-\mathbf{A}\tau} \tag{2.69}$$

which becomes, if $\tau = t$,

$$e^{\mathbf{A}t}e^{-\mathbf{A}t} = e^{\mathbf{A}\cdot 0} = e^{\mathbf{0}} = \mathbf{I} \tag{2.70}$$

The differentiation of (2.67) yields

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{0} + \mathbf{A} + \frac{2t}{2!}\mathbf{A}^2 + \frac{3t^2}{3!}\mathbf{A}^3 + \cdots$$

$$= \mathbf{A}(\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots) = (\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots)\mathbf{A}$$

which implies

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A} \tag{2.71}$$

Now using (2.67) through (2.71) we can show that the solutions of (2.61) due to $u(t)$ and $\mathbf{x}(0)$ are given by

$$\mathbf{x}(t) = \underbrace{e^{\mathbf{A}t}\mathbf{x}(0)}_{\text{Zero-Input Response}} + \underbrace{e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau}_{\text{Zero-State Response}} \tag{2.72}$$

and

$$y(t) = \overbrace{ce^{\mathbf{A}t}\mathbf{x}(0)}^{} + \overbrace{ce^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau + du(t)}^{} \tag{2.73}$$

To show that (2.72) is the solution of (2.61a), we must show that it meets the initial condition and the state equation. Indeed, at $t = 0$, (2.72) reduces to $\mathbf{x}(0) = \mathbf{I}\mathbf{x}(0) + \mathbf{I}\cdot 0 = \mathbf{x}(0)$. Thus it meets the initial condition. The differentiation of (2.72) yields

$$\dot{\mathbf{x}}(t) = \frac{d}{dt}\left[e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau\right]$$

$$= \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) + \frac{d}{dt}(e^{\mathbf{A}t})\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau) + e^{\mathbf{A}t}\frac{d}{dt}\left(\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau\right)$$

$$= \mathbf{A}\left[e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)d\tau\right] + e^{\mathbf{A}t}\left[e^{-\mathbf{A}\tau}\mathbf{b}u(\tau)\right]_{\tau=t}$$

$$= \mathbf{A}\mathbf{x}(t) + e^{\mathbf{A}t}e^{-\mathbf{A}t}\mathbf{b}u(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$$

This shows that (2.72) is the solution of (2.61a). The substitution of (2.72) into (2.61b) yields immediately (2.73). If $u = 0$, (2.61a) reduces to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

This is called the *homogeneous* equation. Its solution due to the initial state $\mathbf{x}(0)$ is, from (2.72),

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

If $u(t) = 0$ or, equivalently, $U(s) = 0$, then (2.63) reduces to

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$$

A comparison of the preceding two equations yields

$$\mathcal{L}(e^{\mathbf{A}t}) = (s\mathbf{I} - \mathbf{A})^{-1} \quad \text{or} \quad e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] \tag{2.74}$$

Thus $e^{\mathbf{A}t}$ equals the inverse Laplace transform of $(s\mathbf{I} - \mathbf{A})^{-1}$.

To use (2.72), we must first compute $e^{\mathbf{A}t}$. The computation of $e^{\mathbf{A}t}$ by using the infinite power series in (2.67) is not simple by hand. An alternative method is to use the Laplace transform in (2.74). Computer computation of (2.67) and (2.72) will be discussed in Section 2.9.

2.7.2 TRANSFER FUNCTION AND CHARACTERISTIC POLYNOMIAL

In this section, we discuss the relationships between state-variable equations and transfer functions. The transfer function of a system is, by definition, the ratio of the Laplace transforms of the output and input under the assumption that all initial conditions are zero. The Laplace transform of the state-variable equation in (2.61) was computed in (2.64). If $\mathbf{x}(0) = 0$, then (2.64) reduces to

$$Y(s) = [\mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d]U(s)$$

Thus the transfer function of the state-variable equation in (2.61) is

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d \tag{2.75}$$

This formula was used in (2.66) to compute the transfer function of (2.65). Now we discuss its general property. Let det stand for the determinant and adj for the adjoint of a matrix. See Appendix B. Then (2.75) can be written as

$$G(s) = \mathbf{c}\frac{1}{\det(s\mathbf{I} - \mathbf{A})}[\text{adj } (s\mathbf{I} - \mathbf{A})]\mathbf{b} + d$$

We call $\det(s\mathbf{I} - \mathbf{A})$ the *characteristic polynomial* of $\mathbf{A}$. For example, the characteristic polynomial of the $\mathbf{A}$ in (2.65) is

$$\begin{aligned}\Delta(s) := \det(s\mathbf{I} - \mathbf{A}) &= \det\begin{bmatrix} s + 6 & 3.5 \\ -6 & s - 4 \end{bmatrix} \\ &= (s + 6)(s - 4) - (-3.5 \times 6) = s^2 + 2s - 3\end{aligned} \tag{2.76}$$

We see that if $\mathbf{A}$ is $n \times n$, then its characteristic polynomial is of degree n and has n roots. These n roots are called the *eigenvalues* of $\mathbf{A}$. The characteristic polynomial and eigenvalues of $\mathbf{A}$ are in fact the same as the characteristic polynomial and modes discussed in Section 2.3.1. They govern the zero-input response of the system. For

example, if the characteristic polynomial of **A** is

$$\Delta(s) = s^3(s + 2)(s - 3)$$

then the eigenvalues are 0, 0, 0, -2, and 3, and every entry of $e^{\mathbf{A}t}$ is a linear combination of the time functions 1, t, t^2, e^{-2t}, and e^{3t}. Therefore any zero-input response will be a linear combination of these time functions.

The characteristic polynomial of square matrices often requires tedious computation. However, the characteristic polynomials of the following two matrices

$$\begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix} \tag{2.77}$$

and their transposes are all

$$\Delta(s) = s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 \tag{2.78}$$

This characteristic polynomial can be read out directly from the entries of the matrices in (2.77). Thus, these matrices are called *companion forms* of the polynomial in (2.78).

Now we use an example to discuss the relationship between the eigenvalues of a state-variable equation and the poles of its transfer function.

Example 2.8.1

Consider the state-variable equation in (2.65), repeated in the following

$$\dot{\mathbf{x}} = \begin{bmatrix} -6 & -3.5 \\ 6 & 4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \tag{2.79a}$$

$$y = [4 \quad 5]\mathbf{x} \tag{2.79b}$$

Its characteristic polynomial was computed in (2.76) as

$$\Delta(s) = s^2 + 2s - 3 = (s + 3)(s - 1)$$

Thus its eigenvalues are 1 and -3. The transfer function of (2.79) was computed in (2.66) as

$$G(s) = \frac{s + 2}{s^2 + 2s - 3} = \frac{s + 2}{(s + 3)(s - 1)}$$

Its poles are 1 and -3. The number of the eigenvalues of (2.79) equals the number of the poles of its transfer function.

Example 2.8.2

Consider the state-variable equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -6 & -3.5 \\ 6 & 4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \tag{2.80a}$$

$$y = [-2 \quad -1]\mathbf{x} \tag{2.80b}$$

Equation (2.80a) is identical to (2.79a); (2.80b), however, is different from (2.79b). The characteristic polynomial of (2.80) is

$$\Delta(s) = s^2 + 2s - 3$$

Its eigenvalues are -3 and 1. The transfer function of (2.80) is, replacing [4 5] in (2.66) by $[-2 \quad -1]$,

$$\begin{aligned} G(s) &= [-2 \quad -1] \frac{1}{s^2 + 2s - 3} \begin{bmatrix} s - 4 & -3.5 \\ 6 & s + 6 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{1}{s^2 + 2s - 3} [-2 \quad -1] \begin{bmatrix} -s + 0.5 \\ s \end{bmatrix} = \frac{2s - 1 - s}{s^2 + 2s - 3} \\ &= \frac{s - 1}{(s - 1)(s + 3)} \end{aligned} \tag{2.81}$$

Thus the transfer function of (2.80) is $G(s) = 1/(s + 3)$. It has only one pole at -3, one less than the number of the eigenvalues of (2.80).

A state-variable equation is called *minimal* if the number of its eigenvalues or the dimension of the equation equals the number of the poles of its transfer function. The state-variable equation in (2.79) is such an equation. Every eigenvalue of a minimal state-variable equation will appear as a pole of its transfer function and there is no essential difference between using the state-variable equation or its transfer function to study the system. Thus, the state-variable equation is said to be *completely characterized* by its transfer function. As will be discussed in Chapter 11, every minimal state-variable equation has the properties of controllability and observability. If a state-variable equation is not minimal, such as the one in (2.80), then some eigenvalues will not appear as poles of its transfer function and the transfer function is said to have *missing poles*. In this case, the state-variable equation is not completely characterized by its transfer function and we cannot use the transfer function to carry out analysis and design. This situation is similar to the one in Section 2.5.