and three poles, -2, -1, and -1. The pole -2 is called a *simple* pole and the pole -1 is called a *repeated* pole with multiplicity 2.[4]

From this example, we see that if polynomials $N(s)$ and $D(s)$ have no common factors,[5] then all roots of $N(s)$, and all roots of $D(s)$ are, respectively, the zeros and poles of $G(s) = N(s)/D(s)$. If $N(s)$ and $D(s)$ have no common factor, they are said to be *coprime* and $G(s) = N(s)/D(s)$ is said to be *irreducible*. Unless stated otherwise, every transfer function will be assumed to be irreducible.

We now discuss the computation of the zero-state response. The zero-state response of a system is governed by $Y(s) = G(s)U(s)$. To compute $Y(s)$, we first compute the Laplace transform of $u(t)$. We then multiply $G(s)$ and $U(s)$ to yield $Y(s)$. The inverse Laplace transform of $Y(s)$ yields the zero-state response. This is illustrated by an example.

Example 2.4.3

Find the zero-state response of (2.25) due to $u(t) = 1$, for $t \geq 0$. This is called the unit-step response of (2.25). The Laplace transform of $u(t)$ is $1/s$. Thus we have

$$Y(s) = G(s)U(s) = \frac{3s - 1}{(s + 1)(s + 2)} \cdot \frac{1}{s} \tag{2.33}$$

To compute its inverse Laplace transform, we carry out the partial fraction expansion as

$$Y(s) = \frac{3s - 1}{(s + 1)(s + 2)s} = \frac{k_1}{s + 1} + \frac{k_2}{s + 2} + \frac{k_3}{s}$$

where

$$k_1 = Y(s) \cdot (s + 1)\Big|_{s=-1} = \frac{3s - 1}{(s + 2)s}\Big|_{s=-1} = \frac{-4}{(1)(-1)} = 4$$

$$k_2 = Y(s) \cdot (s + 2)\Big|_{s=-2} = \frac{3s - 1}{(s + 1)s}\Big|_{s=-2} = \frac{-7}{(-1)(-2)} = -3.5$$

and

$$k_3 = Y(s) \cdot s\Big|_{s=0} = \frac{3s - 1}{(s + 1)(s + 2)}\Big|_{s=0} = \frac{-1}{2} = -0.5$$

[4]If s is very large, (2.32) reduces to $G(s) = 1/s^2$ and $G(\infty) = 0$. Thus ∞ can be considered as a repeated zero with multiplicity 2. Unless stated otherwise, we consider only finite poles and zeros.

[5]Any two polynomials, such as $4s + 2$ and $6s + 2$, have a constant as a common factor. Such a common factor, a polynomial of degree 0, is called a *trivial common factor*. We consider only nontrivial common factors—that is, common factors of degree 1 or higher.

Using Table A.1, the zero-state response is

$$y(t) = \underbrace{4e^{-t} - 3.5e^{-2t}}_{\substack{\text{Due to the} \\ \text{Poles of } G(s)}} \underbrace{- 0.5}_{\substack{\text{Due to the} \\ \text{Poles of } U(s)}} \tag{2.34}$$

for $t \geq 0$. Thus, the use of the Laplace transform to compute the zero-state response is simple and straightforward.

This example reveals an important fact of the zero-state response. We see from (2.34) that the response consists of three terms. Two are the inverse Laplace transforms of $1/(s + 2)$ and $1/(s + 1)$, which are the poles of the system. The remaining term is due to the step input. In fact, for any $u(t)$, the response of (2.33) is generally of the form

$$y(t) = k_1 e^{-t} + k_2 e^{-2t} + (\text{terms due to the poles of } U(s)) \tag{2.35}$$

(see Problem 2.20). Thus the poles of $G(s)$ determine the basic form of the zero-state response.

Exercise 2.4.6

Find the zero-state response of $1/(s + 1)$ due to e^{-2t}, $t \geq 0$.

[**Answer:** $y(t) = e^{-t} - e^{-2t}$.]

If k_1 in (2.35) is zero, the corresponding pole -1 is not excited. A similar remark applies to k_2. Because both k_1 and k_2 in (2.34) are nonzero, both poles of $G(s)$ in (2.33) are excited by the step input. For other inputs, the two poles may not always be excited. This is illustrated by an example.

Example 2.4.4

Consider the system in (2.33). Find a bounded input $u(t)$ so that the pole -1 will not be excited. If $U(s) = s + 1$, then

$$Y(s) = G(s)U(s) = \frac{3s - 1}{(s + 1)(s + 2)} \cdot (s + 1)$$

$$= \frac{3s - 1}{s + 2} = \frac{3(s + 2) - 7}{s + 2} = 3 - \frac{7}{s + 2}$$

which implies

$$y(t) = 3\delta(t) - 7e^{-2t}$$

This response does not contain e^{-t}, thus the pole -1 is not excited. Therefore if we introduce a zero in $U(s)$ to cancel a pole, then the pole will not be excited by the input $u(t)$.

If $U(s)$ is biproper or improper, as is the case for $U(s) = s + 1$, then its inverse Laplace transform $u(t)$ will contain an impulse and its derivatives and is not bounded. In order for $u(t)$ to be bounded, we choose, rather arbitrarily, $U(s) = (s + 1)/s(s + 3)$, a strictly proper rational function. Its inverse Laplace transform is

$$u(t) = \frac{1}{3} + \frac{2}{3}e^{-3t}$$

for $t \geq 0$ and is bounded. The application of this input to (2.33) yields

$$Y(s) = \frac{3s - 1}{(s + 2)(s + 1)} \cdot \frac{s + 1}{s(s + 3)} = \frac{3s - 1}{(s + 2)(s + 3)s}$$
$$= \frac{7}{2(s + 2)} - \frac{10}{3(s + 3)} - \frac{1}{6s}$$

which implies

$$y(t) = \frac{7}{2}e^{-2t} - \frac{10}{3}e^{-3t} - \frac{1}{6}$$

for $t \geq 0$. The second and third terms are due to the input, the first term is due to the pole -2. The term e^{-1} does not appear in $y(t)$, thus the pole -1 is not excited by the input. Similarly, we can show that the input $(s + 2)/s(s + 1)$ or $(s + 2)/(s + 3)^2$ will not excite the pole -2 and the input $(s + 2)(s + 1)/s(s + 3)^2$ will not excite either pole.

From this example, we see that whether or not a pole will be excited depends on whether $u(t)$ or $U(s)$ has a zero to cancel it. The Laplace transforms of the unit-step function and $\sin \omega_0 t$ are

$$\frac{1}{s} \quad \text{and} \quad \frac{\omega_0}{s^2 + \omega_0^2}$$

They have no zero. Therefore, either input will excite all poles of every LTIL system.

The preceding discussion can be extended to the general case. Consider, for example,

$$Y(s) = G(s)U(s) := \frac{(s + 10)(s + 2)(s - 1)^2}{s^3(s - 2)^2(s + 2 - j2)(s + 2 + j2)} U(s)$$

The transfer function $G(s)$ has poles at 0, 0, 0, 2, 2, and $-2 \pm j2$. The complex poles $-2 \pm j2$ are simple poles, the poles 0 and 2 are repeated poles with multiplicities 3 and 2. If $G(s)$ and $U(s)$ have no pole in common, then the zero-state

response of the system due to $U(s)$ is of the form

$$y(t) = k_1 + k_2 t + k_3 t^2 + k_4 e^{2t} + k_5 t e^{2t} + k_6 e^{-(2-j2)t} + k_7 e^{-(2+j2)t} + (\text{Terms due to the poles of } U(s)) \tag{2.36}$$

(see Problem 2.20.) Thus the poles of $G(s)$ determine the basic form of the response.

Do the zeros of $G(s)$ play any role in the zero-state response? Certainly, they do. They affect the values of k_i. Different sets of k_i yield drastically different responses, as is illustrated by the following example.

Example 2.4.5

Consider

$$G_1(s) = \frac{2}{(s+1)(s+1+j)(s+1-j)}$$

$$G_2(s) = \frac{0.2(s+10)}{(s+1)(s+1+j)(s+1-j)}$$

$$G_3(s) = \frac{-0.2(s-10)}{(s+1)(s+1+j)(s+1-j)}$$

$$G_4(s) = \frac{10(s^2+0.1s+0.2)}{(s+1)(s+1+j)(s+1-j)}$$

The transfer function $G_1(s)$ has no zero, $G_2(s)$ and $G_3(s)$ have one zero, and $G_4(s)$ has a pair of complex conjugate zeros at $-0.05 \pm 0.444j$. They all have the same

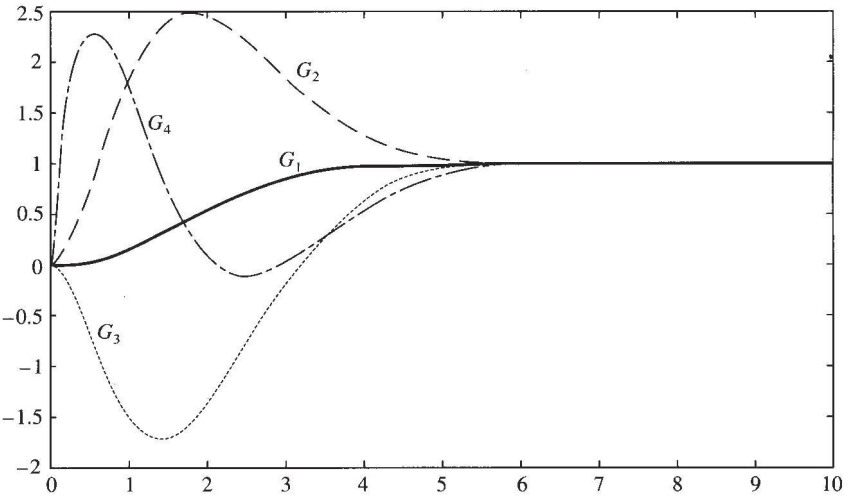


Figure 2.16 Unit-step responses of $G_i(s)$.

set of poles, and their unit-step responses are all of the form

$$y(t) = k_1 e^{-t} + k_2 e^{-(1+j1)t} + k_3 e^{-(1-j1)t} + k_4$$

with k_3 equal to the complex conjugate of k_2. Their responses are shown in Figure 2.16, respectively, with the solid line ($G_1(s)$), dashed line ($G_2(s)$), dotted line ($G_3(s)$), and dash-and-dotted line ($G_4(s)$). They are quite different. In conclusion, even though the poles of $G(s)$ determine the basic form of responses, exact responses are determined by the poles, zeros, and the input. Therefore, the zeros of a transfer function cannot be completely ignored in the analysis and design of control systems.

2.5 BLOCK REPRESENTATION—COMPLETE CHARACTERIZATION

In the analysis and design of control systems, every device is represented by a block as shown in Figure 2.4 or 2.17(a). The block is then represented by its transfer function $G(s)$. If the input is $u(t)$ and the output is $y(t)$, then they are related by

$$Y(s) = G(s)U(s) \tag{2.37}$$

where $Y(s)$ and $U(s)$ are respectively the Laplace transforms of $y(t)$ and $u(t)$. Note that we have mixed the time-domain representation $u(t)$ and $y(t)$ and the Laplace transform representation $G(s)$ in Figure 2.17(a). This convention will be used throughout this text. It is important to know that it is incorrect to write $y(t) = G(s)u(t)$. The correct expression is $Y(s) = G(s)U(s)$.[6]

Equation (2.37) is an algebraic equation. The product of the Laplace transform of the input and the transfer function yields the Laplace transform of the output. The advantage of using this algebraic representation can be seen from the tandem connection of two systems shown in Figure 2.17(b). Suppose the two systems are represented, respectively, by

$$Y_1(s) = G_1(s)U_1(s) \qquad Y_2(s) = G_2(s)U_2(s)$$

In the tandem connection, we have $u_2(t) = y_1(t)$ or $U_2(s) = Y_1(s)$ and

$$Y_2(s) = G_2(s)Y_1(s) = G_2(s)G_1(s)U_1(s) \tag{2.38}$$

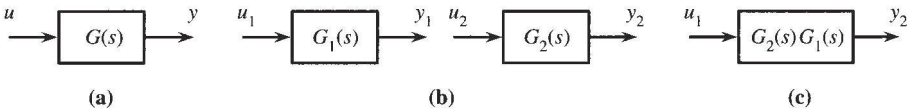


Figure 2.17 (a) A system. (b) Tandem connection of two systems. (c) Reduction of (b).

[6]It can also be expressed as $y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$, where $g(t)$ is the inverse Laplace transform of $G(s)$. See Reference [18]. This form is rarely used in the design of control systems.

Thus the tandem connection can be represented by a single block, as shown in Figure 2.17(c), with transfer function $G(s) := G_2(s)G_1(s)$, the product of the transfer functions of the two subsystems. If we use differential equations, then the differential equation description of the tandem connection will be much more complex. Thus the use of transfer functions can greatly simplify the analysis and design of control systems.

The transfer function describes only the zero-state response of a system. Therefore, whenever we use the transfer function in analysis and design, the zero-input response (the response due to nonzero initial conditions) is completely disregarded. However, can we really disregard the zero-input response? This question is studied in this section.

Consider a linear time-invariant lumped (LTIL) system described by the differential equation

$$D(p)y(t) = N(p)u(t) \tag{2.39}$$

where

$$D(p) = a_n p^n + a_{n-1}p^{n-1} + \cdots + a_1 p + a_0$$
$$N(p) = b_m p^m + b_{m-1}p^{m-1} + \cdots + b_1 p + b_0$$

and the variable p is the differentiator defined in (2.19) and (2.20). Then the zero-input response of the system is described by

$$D(p)y(t) = 0 \tag{2.40}$$

and the response is dictated by the roots of $D(s)$, called *the modes of the system* (Section 2.3.1). The zero-state response of the system is described by the transfer function

$$G(s) := \frac{N(s)}{D(s)}$$

and the basic form of its response is governed by the poles of $G(s)$ (Section 2.4.2). The poles of $G(s)$ are defined as the roots of $D(s)$ after canceling the common factors of $N(s)$ and $D(s)$. Thus if $D(s)$ and $N(s)$ have no common factors, then

$$\text{The set of the poles} = \text{The set of the modes} \tag{2.41}$$

In this case, the system is said to be *completely characterized* by its transfer function. If $D(s)$ and $N(s)$ have common factors, say $R(s)$, then the roots of $R(s)$ are nodes of the system but not poles of $G(s)$. In this case the roots of $R(s)$ are called the *missing poles* of the transfer function, and the system is said to be not completely characterized by its transfer function. We use examples to illustrate this concept and discuss its implications.

Example 2.5.1

Consider the system shown in Figure 2.18. The input is a current source. The output y is the voltage across the 2-Ω resistor as shown. The system can be described by the LTIL differential equation

$$\frac{dy(t)}{dt} - 0.75y(t) = \frac{du(t)}{dt} - 0.75u(t) \tag{2.42a}$$

(Problem 2.21). The equation can also be written, using $p = d/dt$, as

$$(p - 0.75)y(t) = (p - 0.75)u(t) \tag{2.42b}$$

The mode of the system is the root of $(s - 0.75)$ or 0.75. Therefore its zero-input response is of the form

$$y(t) = ke^{0.75t} \tag{2.43}$$

where k depends on the initial voltage of the capacitor in Figure 2.18. We see that if the initial voltage is different from zero, then the response will approach infinity as $t \to \infty$.

We will now study its zero-state response. The transfer function of the system is

$$G(s) = \frac{s - 0.75}{s - 0.75} = 1 \tag{2.44}$$

Because of the common factor, the transfer function reduces to 1. Thus the system has no pole and the zero-state response is $y(t) = u(t)$, for all t. This system is *not* completely characterized by its transfer function because the mode 0.75 does not appear as a pole of $G(s)$. In other words, the transfer function has missing pole 0.75.

If we use the transfer function to study the system in Figure 2.18, we would conclude that the system is acceptable. In reality, the system is not acceptable, because if, for any reason, the voltage of the capacitor becomes nonzero, the response

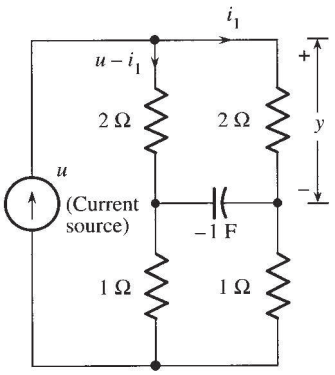


Figure 2.18 Network.

will grow without bound and the system will either become saturated or burn out. Thus the system is of no use in practice.

The existence of a missing pole in Figure 2.18 can easily be explained from the structure of the network. Because of the symmetry of the four resistors, if the initial voltage of the capacitor is zero, its voltage will remain zero no matter what current source is applied. Therefore, the removal of the capacitor will not affect the zero-state response of the system. Thus, the system has a superfluous component as far as the input and output are concerned. These types of systems are not built in practice, except by mistake.

Example 2.5.2

Consider the system described by

$$(p^2 + 2p - 3)y(t) = (p - 2)u(t) \tag{2.45}$$

The zero-input response of the system is governed by

$$(p^2 + 2p - 3)y(t) = 0$$

Its modes are the roots of $(s^2 + 2s - 3) = (s - 1)(s + 3)$ or 1 and -3. Thus the zero-input response due to any initial conditions is of the form

$$y_{zi}(t) = k_1 e^{t} + k_2 e^{-3t}$$

where the subscript zi denotes zero-input. The response of mode 1 approaches infinity and the response of mode -3 approaches zero as $t \to \infty$.

The transfer function of the system is

$$G(s) = \frac{s - 2}{s^2 + 2s - 3} = \frac{s - 2}{(s - 1)(s + 3)} \tag{2.46}$$

Thus the zero-state response of the system due to $u(t)$ will generally be of the form

$$y(t) = \bar{k}_1 e^{t} + \bar{k}_2 e^{-3t} + (\text{Terms due to the poles of } U(s))$$

We see that the two modes appear as the poles of $G(s)$. Thus the system has no missing poles and is completely characterized by its transfer function. In this case, the zero-input response due to any initial conditions will appear in the zero-state response, so no essential information is lost in using the transfer function to study the system.

In conclusion, if a system is completely characterized by its transfer function, then the zero-input response will appear essentially as part of the zero-state response. It is therefore permissible to use the transfer function in analysis and design without

considering the response due to nonzero initial conditions. If a system is not completely characterized by its transfer function, care must be exercised in using the transfer function to study the system. This point is discussed further in Chapter 4.

Exercise 2.5.1

Which of the following systems are completely characterized by their transfer functions? If not, find the missing poles.

a. $(p^2 + 2p + 1)y(t) = (p + 1)p\,u(t)$

b. $(p^2 - 3p + 2)y(t) = (p - 1)\,u(t)$

c. $(p^2 - 3p + 2)y(t) = u(t)$

[**Answers:** (a) No, -1 (b) No, 1 (c) Yes.]

2.5.1 The Loading Problem

Most control systems are built by interconnecting a number of subsystems as shown in Figures 1.1(b), 1.6(b), and 1.7(b). In analysis and design, every subsystem will be represented by a block and each block will be represented by a transfer function. Thus, most control systems will consist of a number of blocks interconnected with each other. In connecting two or more blocks, the problem of loading may occur. For example, consider a 10-volt voltage source modeled as shown in Figure 2.19(a). It has an internal resistance of 10 Ω; its output voltage is 10 volts when nothing is connected to its terminals. Now consider a device that can be modeled as a 10-Ω resistor. When we connect the device to the voltage source, the output voltage of the source is not 10 volts, but $10 \cdot 10/(10 + 10) = 5$ volts. If we connect a 20-Ω device to the voltage source, the output is $10 \cdot 20/(10 + 20) = 6.7$ volts. We see that the voltages supplied by the same voltage source will be different when it is connected to different devices. In other words, for different loads, the voltages supplied by the same voltage source will be different. In this case, the connection is said to have *a loading problem.* On the other hand, if the voltage source is designed so that its internal resistance is negligible or zero, as shown in Figure 2.19(b), then

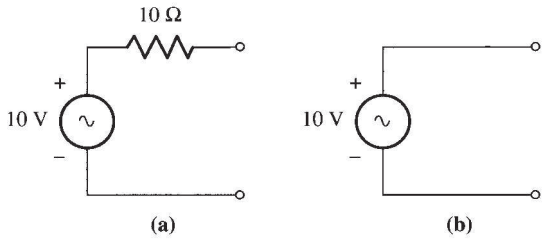


Figure 2.19 Loading problem.

no matter what device is connected to it, the supplied voltage to the device is always 10 volts. In this case, the connection is said to have *no loading problem.*

Roughly speaking, if its transfer function changes after a system is connected to another system, the connection is said to have *a loading effect.* For example, the transfer function from u to y in Figure 2.19(a) is 1 before the system is connected to any device. It becomes $5/10 = 0.5$ when the system is connected a 10-Ω resistor. Thus, the connection has a loading effect. If the tandem connection of two systems has a loading effect, then the transfer function of the tandem connection does not equal the product of the transfer functions of the two subsystems as developed in (2.38). This is illustrated by an example.

Example 2.5.3

Consider the networks shown in Figure 2.20(a). The transfer function from u_1 to y_1 of network M_1 is $G_1(s) = s/(s + 1)$. The transfer function from u_2 to y_2 of network M_2 is $G_2(s) = 2/(2 + 3s)$. Now we connect them together or set $y_1 = u_2$ as shown in Figure 2.20(b) and compute the transfer function from u_1 to y_2. The impedance of the parallel connection of the impedance s and the impedance $(3s + 2)$ is $s(3s + 2)/(s + 3s + 2) = s(3s + 2)/(4s + 2)$. Thus the current I_1 shown in

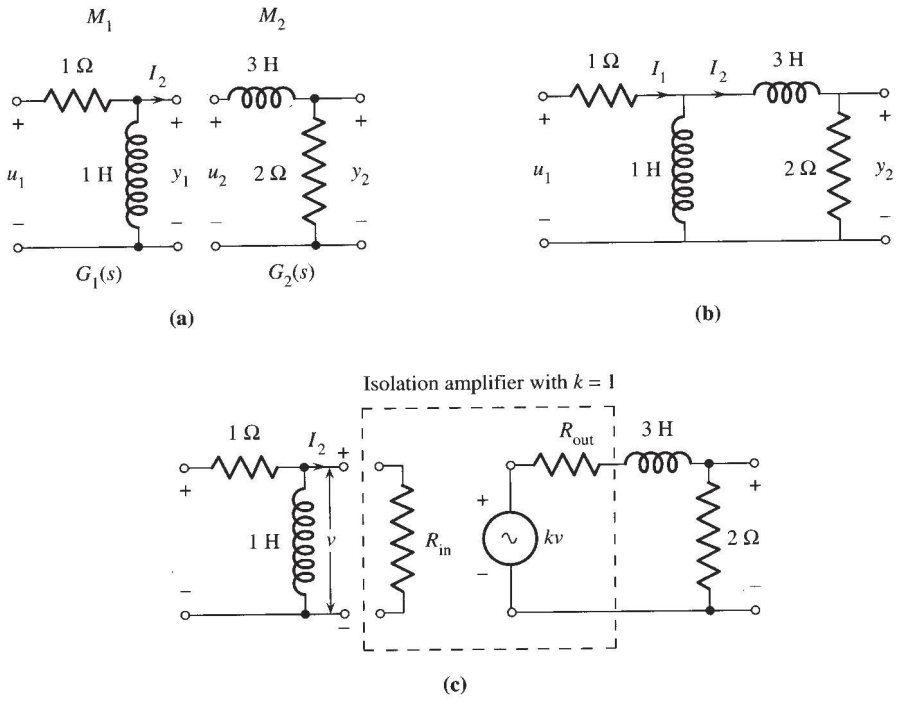


Figure 2.20 Tandem connection of two systems.

Figure 2.20(b) equals

$$I_1(s) = \frac{U_1(s)}{1 + \dfrac{s(3s + 2)}{4s + 2}} \tag{2.47}$$

and the current I_2 equals

$$I_2(s) = \frac{s}{s + 3s + 2} \cdot I_1(s) = \frac{s}{4s + 2 + s(3s + 2)} \cdot U_1(s) \tag{2.48}$$

Thus, the voltage $y_2(t)$ is given by

$$Y_2(s) = I_2(s) \cdot 2 = \frac{2s}{3s^2 + 6s + 2} \cdot U_1(s) \tag{2.49}$$

and the transfer function from u_1 to y_2 of the tandem connection is

$$\frac{Y_2(s)}{U_1(s)} = \frac{2s}{3s^2 + 6s + 2} \tag{2.50}$$

This is different from the product of $G_1(s)$ and $G_2(s)$ or

$$G_1(s)G_2(s) = \frac{s}{s + 1} \cdot \frac{2}{3s + 2} = \frac{2s}{3s^2 + 5s + 2} \tag{2.51}$$

Thus, Equation (2.38) cannot be used for this connection.

The loading of the two networks in Figure 2.20 can be easily explained. The current I_2 in Figure 2.20(a) is zero before the connection; it becomes nonzero after the connection. Thus, the loading occurs. In electrical networks, the loading often can be eliminated by inserting an isolating amplifier, as shown in Figure 2.20(c). The input impedance Z_{in} of an ideal isolating amplifier is infinity and the output impedance Z_{out} is zero. Under these assumptions, I_2 in Figure 2.20(c) remains zero and the transfer function of the connection is $G_2(s)kG_1(s)$ with $k = 1$.

The loading problem must be considered in developing a block diagram for a control system. This problem is considered in detail in the next chapter.

2.6 STATE-VARIABLE EQUATIONS

The transfer function describes only the relationship between the input and output of LTIL systems and is therefore called the *input-output description* or *external description.* In this section we shall develop a different description, called the *state-variable description* or *internal description.* Strictly speaking, this description is the same as the differential equations discussed in Section 2.2. The only difference is that *high-order* differential equations are now written as sets of *first-order* differential equations. In this way, the study can be simplified.