

Figure 2.13 Network.

as shown in Figure 2.13(c). Hence the current $I(s)$ shown in Figure 2.13 is given by

$$I(s) = \frac{U(s)}{1 + \dfrac{3s + 2}{6s^2 + 4s + 1}} = \frac{6s^2 + 4s + 1}{6s^2 + 7s + 3} U(s)$$

Thus the transfer function from u to i is

$$G(s) = \frac{I(s)}{U(s)} = \frac{6s^2 + 4s + 1}{6s^2 + 7s + 3}$$

Exercise 2.4.3

Find the transfer functions from u to y of the networks in Figures 2.10 and 2.11 using the concept of impedances.

2.4.1 Proper Transfer Functions

Consider the rational function

$$G(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are two polynomials with real coefficients. We use deg to denote the degree of a polynomial. If

$$\deg N(s) > \deg D(s)$$

$G(s)$ is called an *improper* rational function. For example, the rational functions

$$\frac{s^2 + 1}{s + 1} \qquad s \qquad s^2 + 1 \qquad \text{and} \qquad \frac{s^{10}}{s^9 + s^8 + s^7 - 10}$$

are all improper. If

$$\deg N(s) \leq \deg D(s)$$

$G(s)$ is called a *proper* rational function. It is *strictly proper* if deg $N(s) <$ deg $D(s)$; *biproper* if deg $N(s) =$ deg $D(s)$. Thus proper rational functions include both strictly proper and biproper rational functions. If $G(s)$ is biproper, so is $G^{-1}(s) = D(s)/N(s)$. This is the reason for calling it *bi*proper.

Exercise 2.4.4

Classify the following rational functions

$$s^2 + 1 \qquad 2 \qquad \frac{1}{s+1} \qquad \frac{s^2-1}{s+1} \qquad \frac{s-1}{s+1}.$$

How many of them are proper rational functions?

[**Answers:** Improper, biproper, strictly proper, improper, biproper; 3.]

The properness of a rational function $G(s)$ can also be determined from the value of $G(s)$ at $s = \infty$. It is clear that $G(s)$ is improper if $G(\infty) = \pm\infty$, proper if $G(\infty)$ is a finite nonzero or zero constant, biproper if $G(\infty)$ is finite and nonzero, and strictly proper if $G(\infty) = 0$.

The transfer functions we will encounter in this text are mostly proper rational functions. The reason is twofold. First, improper transfer functions are difficult, if not impossible, to build in practice, as will be discussed in Chapter 5. Second, improper transfer functions will amplify high-frequency noise, as will be explained in the following.

Signals are used to carry information. However, they are often corrupted by noise during processing, transmission, or transformation. For example, an angular position can be transformed into an electrical voltage by using the wirewound potentiometer shown in Figure 2.14. The potentiometer consists of a finite number of turns of wiring, hence the contact point moves from turn to turn. Because of brush

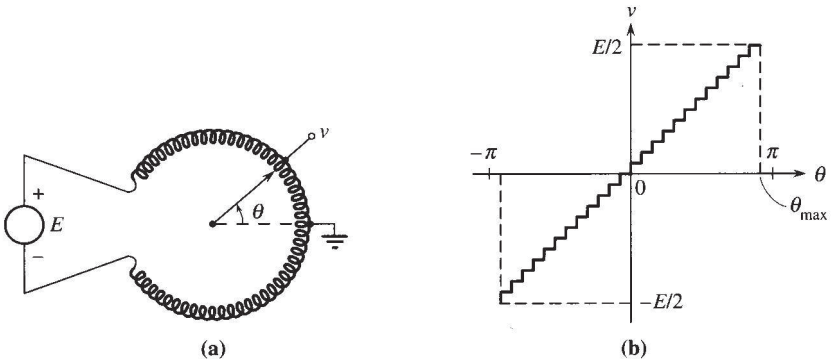


Figure 2.14 Potentiometer and its characteristic.

jumps, wire irregularities, or variations of contact resistance, unwanted spurious voltage will be generated. Thus the output voltage $v(t)$ of the potentiometer will not be exactly proportional to the angular displacement $\theta(t)$, but rather will be of the form

$$v(t) = k\theta(t) + n(t) \tag{2.27}$$

where k is a constant and $n(t)$ is noise. Therefore, in general, every signal is of the form

$$v(t) = i(t) + n(t) \tag{2.28}$$

where $i(t)$ denotes information and $n(t)$ denotes noise. Clearly in order for $v(t)$ to be useful, we require

$$v(t) \approx i(t)$$

and for any system to be designed, we require

$$\begin{aligned} &\text{Response of the system due to } v(t) \\ &\approx \text{response of the system due to } i(t) \end{aligned} \tag{2.29}$$

where $\approx$ denotes "roughly equal to." If the response of a system excited by $v(t)$ is drastically different from that excited by $i(t)$, the system is generally useless in practice. Now we show that if the transfer function of a system is improper and if the noise is of high frequency, then the system is useless. Rather than discussing the general case, we study a system with transfer function s and a system with transfer function $1/s$. A system with transfer function s is called a *differentiator* because it performs differentiation in the time domain. A system with transfer function $1/s$ is called an *integrator* because it performs integration in the time domain. The former has an improper transfer function, the latter has a strictly proper transfer function. We shall show that the differentiator will *amplify* high-frequency noise; whereas the integrator will *suppress* high-frequency noise. For convenience of discussion, we assume

$$i(t) = \sin 2t \qquad n(t) = 0.01 \sin 1000t$$

and

$$v(t) = i(t) + n(t) = \sin 2t + 0.01 \sin 1000t \tag{2.30}$$

The magnitude of the noise is very small, so we have $v(t) \approx i(t)$. If we apply this signal to a differentiator, then the output is

$$\frac{dv(t)}{dt} = 2 \cos 2t + 0.01 \times 1000 \cos 1000t = 2 \cos 2t + 10 \cos 1000t$$

Because the amplitude of the noise term is five times larger than that of the information, we do not have $dv(t)/dt \approx di(t)/dt$ as shown in Figure 2.15. Thus a differentiator—and, more generally, systems with improper transfer functions—cannot be used if a signal contains high-frequency noise.

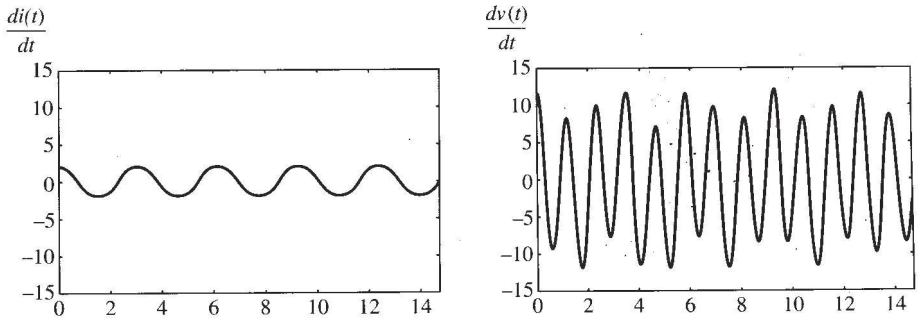


Figure 2.15 Responses of differentiator.

If we apply (2.30) to an integrator, then the output is

$$\int_0^t v(\tau)d\tau = -\frac{1}{2}\cos 2t - \frac{0.01}{1000}\cos 1000t$$

The output due to the noise term is practical zero and

$$\int_0^t v(\tau)d\tau \approx \int_0^t i(\tau)d\tau$$

Thus we conclude that an integrator—and, more generally, systems with strictly proper transfer functions—will suppress high-frequency noise.

In practice, we often encounter high-frequency noise. As was discussed earlier, wirewound potentiometers will generate unwanted high-frequency noise. Thermal noise and shot noise, which are of high-frequency compared to control signals, are always present in electrical systems. Because of splashing and turbulent flow of incoming liquid, the measured liquid level of a tank will consist of high-frequency noise. In order not to amplify high-frequency noise, most systems and devices used in practice have proper transfer functions.

Exercise 2.4.5

Consider

$$v(t) = i(t) + n(t) = \cos 2t + 0.01 \cos 0.001t$$

Note that the frequency of the noise $n(t)$ is much smaller than that of the information $i(t)$. Do we have $v(t) \approx i(t)$? Do we have $dv(t)/dt \approx di(t)/dt$? Is it true that a differentiator amplifies any type of noise?

[**Answers:** Yes, yes, no.]

2.4.2 Poles and Zeros

The zero-state response of a system is governed by its transfer function. Before computing the response, we introduce the concepts of poles and zeros. Consider a proper rational transfer function

$$G(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials with real coefficients and $\deg N(s) \leq \deg D(s)$.

□ Definition

A finite real or complex number λ is a *pole* of $G(s)$ if $|G(\lambda)| = \infty$, where $|\cdot|$ denotes the absolute value. It is a *zero* of $G(s)$ if $G(\lambda) = 0$. ■

Consider the transfer function

$$G(s) = \frac{N(s)}{D(s)} = \frac{2(s^3 + 3s^2 - s - 3)}{(s - 1)(s + 2)(s + 1)^3} \tag{2.31}$$

We have

$$G(-2) = \frac{N(-2)}{D(-2)} = \frac{2[(-2)^3 + 3(-2)^2 - (-2) - 3]}{(-3) \cdot 0 \cdot (-1)} = \frac{6}{0} = \infty$$

Therefore -2 is a pole of $G(s)$ by definition. Clearly -2 is a root of $D(s)$. Does this imply every root of $D(s)$ is a pole of $G(s)$? To answer this, we check $s = 1$, which is also a root of $D(s)$. We compute $G(1)$:

$$G(1) = \frac{N(1)}{D(1)} = \frac{2(1 + 3 - 1 - 3)}{0 \cdot 3 \cdot 8} = \frac{0}{0}$$

It is not defined. However l'Hôpital's rule implies

$$G(1) = \left.\frac{N(s)}{D(s)}\right|_{s=1} = \left.\frac{N'(s)}{D'(s)}\right|_{s=1}$$

$$= \left.\frac{2(3s^2 + 6s - 1)}{5s^4 + 16s^3 + 12s^2 - 4s - 5}\right|_{s=1} = \frac{16}{24} \neq \infty$$

Thus $s = 1$ is not a pole of $G(s)$. Therefore not every root of $D(s)$ is a pole of $G(s)$.

Now we factor $N(s)$ in (2.31) and then cancel the common factors between $N(s)$ and $D(s)$ to yield

$$G(s) = \frac{2(s + 3)(s - 1)(s + 1)}{(s - 1)(s + 2)(s + 1)^3} = \frac{2(s + 3)}{(s + 2)(s + 1)^2} \tag{2.32}$$

We see immediately that $s = 1$ is not a pole of $G(s)$. Clearly $G(s)$ has one zero, -3,

and three poles, -2, -1, and -1. The pole -2 is called a *simple* pole and the pole -1 is called a *repeated* pole with multiplicity 2.[4]

From this example, we see that if polynomials $N(s)$ and $D(s)$ have no common factors,[5] then all roots of $N(s)$, and all roots of $D(s)$ are, respectively, the zeros and poles of $G(s) = N(s)/D(s)$. If $N(s)$ and $D(s)$ have no common factor, they are said to be *coprime* and $G(s) = N(s)/D(s)$ is said to be *irreducible*. Unless stated otherwise, every transfer function will be assumed to be irreducible.

We now discuss the computation of the zero-state response. The zero-state response of a system is governed by $Y(s) = G(s)U(s)$. To compute $Y(s)$, we first compute the Laplace transform of $u(t)$. We then multiply $G(s)$ and $U(s)$ to yield $Y(s)$. The inverse Laplace transform of $Y(s)$ yields the zero-state response. This is illustrated by an example.

Example 2.4.3

Find the zero-state response of (2.25) due to $u(t) = 1$, for $t \geq 0$. This is called the unit-step response of (2.25). The Laplace transform of $u(t)$ is $1/s$. Thus we have

$$Y(s) = G(s)U(s) = \frac{3s - 1}{(s + 1)(s + 2)} \cdot \frac{1}{s} \tag{2.33}$$

To compute its inverse Laplace transform, we carry out the partial fraction expansion as

$$Y(s) = \frac{3s - 1}{(s + 1)(s + 2)s} = \frac{k_1}{s + 1} + \frac{k_2}{s + 2} + \frac{k_3}{s}$$

where

$$k_1 = Y(s) \cdot (s + 1)\Big|_{s=-1} = \frac{3s - 1}{(s + 2)s}\Big|_{s=-1} = \frac{-4}{(1)(-1)} = 4$$

$$k_2 = Y(s) \cdot (s + 2)\Big|_{s=-2} = \frac{3s - 1}{(s + 1)s}\Big|_{s=-2} = \frac{-7}{(-1)(-2)} = -3.5$$

and

$$k_3 = Y(s) \cdot s\Big|_{s=0} = \frac{3s - 1}{(s + 1)(s + 2)}\Big|_{s=0} = \frac{-1}{2} = -0.5$$

[4]If s is very large, (2.32) reduces to $G(s) = 1/s^2$ and $G(\infty) = 0$. Thus ∞ can be considered as a repeated zero with multiplicity 2. Unless stated otherwise, we consider only finite poles and zeros.

[5]Any two polynomials, such as $4s + 2$ and $6s + 2$, have a constant as a common factor. Such a common factor, a polynomial of degree 0, is called a *trivial common factor*. We consider only nontrivial common factors—that is, common factors of degree 1 or higher.

Using Table A.1, the zero-state response is

$$y(t) = \underbrace{4e^{-t} - 3.5e^{-2t}}_{\substack{\text{Due to the} \\ \text{Poles of } G(s)}} \underbrace{- 0.5}_{\substack{\text{Due to the} \\ \text{Poles of } U(s)}} \tag{2.34}$$

for $t \geq 0$. Thus, the use of the Laplace transform to compute the zero-state response is simple and straightforward.

This example reveals an important fact of the zero-state response. We see from (2.34) that the response consists of three terms. Two are the inverse Laplace transforms of $1/(s + 2)$ and $1/(s + 1)$, which are the poles of the system. The remaining term is due to the step input. In fact, for any $u(t)$, the response of (2.33) is generally of the form

$$y(t) = k_1 e^{-t} + k_2 e^{-2t} + (\text{terms due to the poles of } U(s)) \tag{2.35}$$

(see Problem 2.20). Thus the poles of $G(s)$ determine the basic form of the zero-state response.

Exercise 2.4.6

Find the zero-state response of $1/(s + 1)$ due to e^{-2t}, $t \geq 0$.

[**Answer:** $y(t) = e^{-t} - e^{-2t}$.]

If k_1 in (2.35) is zero, the corresponding pole -1 is not excited. A similar remark applies to k_2. Because both k_1 and k_2 in (2.34) are nonzero, both poles of $G(s)$ in (2.33) are excited by the step input. For other inputs, the two poles may not always be excited. This is illustrated by an example.

Example 2.4.4

Consider the system in (2.33). Find a bounded input $u(t)$ so that the pole -1 will not be excited. If $U(s) = s + 1$, then

$$Y(s) = G(s)U(s) = \frac{3s - 1}{(s + 1)(s + 2)} \cdot (s + 1)$$

$$= \frac{3s - 1}{s + 2} = \frac{3(s + 2) - 7}{s + 2} = 3 - \frac{7}{s + 2}$$

which implies

$$y(t) = 3\delta(t) - 7e^{-2t}$$

This response does not contain e^{-t}, thus the pole -1 is not excited. Therefore if we introduce a zero in $U(s)$ to cancel a pole, then the pole will not be excited by the input $u(t)$.

If $U(s)$ is biproper or improper, as is the case for $U(s) = s + 1$, then its inverse Laplace transform $u(t)$ will contain an impulse and its derivatives and is not bounded. In order for $u(t)$ to be bounded, we choose, rather arbitrarily, $U(s) = (s + 1)/s(s + 3)$, a strictly proper rational function. Its inverse Laplace transform is

$$u(t) = \frac{1}{3} + \frac{2}{3}e^{-3t}$$

for $t \geq 0$ and is bounded. The application of this input to (2.33) yields

$$Y(s) = \frac{3s - 1}{(s + 2)(s + 1)} \cdot \frac{s + 1}{s(s + 3)} = \frac{3s - 1}{(s + 2)(s + 3)s}$$
$$= \frac{7}{2(s + 2)} - \frac{10}{3(s + 3)} - \frac{1}{6s}$$

which implies

$$y(t) = \frac{7}{2}e^{-2t} - \frac{10}{3}e^{-3t} - \frac{1}{6}$$

for $t \geq 0$. The second and third terms are due to the input, the first term is due to the pole -2. The term e^{-1} does not appear in $y(t)$, thus the pole -1 is not excited by the input. Similarly, we can show that the input $(s + 2)/s(s + 1)$ or $(s + 2)/(s + 3)^2$ will not excite the pole -2 and the input $(s + 2)(s + 1)/s(s + 3)^2$ will not excite either pole.

From this example, we see that whether or not a pole will be excited depends on whether $u(t)$ or $U(s)$ has a zero to cancel it. The Laplace transforms of the unit-step function and $\sin \omega_0 t$ are

$$\frac{1}{s} \quad \text{and} \quad \frac{\omega_0}{s^2 + \omega_0^2}$$

They have no zero. Therefore, either input will excite all poles of every LTIL system.

The preceding discussion can be extended to the general case. Consider, for example,

$$Y(s) = G(s)U(s) := \frac{(s + 10)(s + 2)(s - 1)^2}{s^3(s - 2)^2(s + 2 - j2)(s + 2 + j2)} U(s)$$

The transfer function $G(s)$ has poles at 0, 0, 0, 2, 2, and $-2 \pm j2$. The complex poles $-2 \pm j2$ are simple poles, the poles 0 and 2 are repeated poles with multiplicities 3 and 2. If $G(s)$ and $U(s)$ have no pole in common, then the zero-state

response of the system due to $U(s)$ is of the form

$$y(t) = k_1 + k_2 t + k_3 t^2 + k_4 e^{2t} + k_5 t e^{2t} + k_6 e^{-(2-j2)t} + k_7 e^{-(2+j2)t} + (\text{Terms due to the poles of } U(s)) \tag{2.36}$$

(see Problem 2.20.) Thus the poles of $G(s)$ determine the basic form of the response.

Do the zeros of $G(s)$ play any role in the zero-state response? Certainly, they do. They affect the values of k_i. Different sets of k_i yield drastically different responses, as is illustrated by the following example.

Example 2.4.5

Consider

$$G_1(s) = \frac{2}{(s+1)(s+1+j)(s+1-j)}$$

$$G_2(s) = \frac{0.2(s+10)}{(s+1)(s+1+j)(s+1-j)}$$

$$G_3(s) = \frac{-0.2(s-10)}{(s+1)(s+1+j)(s+1-j)}$$

$$G_4(s) = \frac{10(s^2+0.1s+0.2)}{(s+1)(s+1+j)(s+1-j)}$$

The transfer function $G_1(s)$ has no zero, $G_2(s)$ and $G_3(s)$ have one zero, and $G_4(s)$ has a pair of complex conjugate zeros at $-0.05 \pm 0.444j$. They all have the same

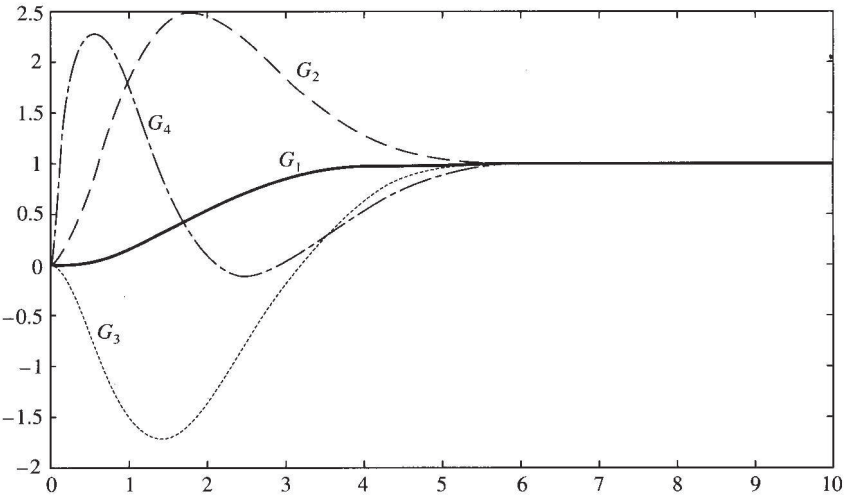


Figure 2.16 Unit-step responses of $G_i(s)$.

set of poles, and their unit-step responses are all of the form

$$y(t) = k_1 e^{-t} + k_2 e^{-(1+j1)t} + k_3 e^{-(1-j1)t} + k_4$$

with k_3 equal to the complex conjugate of k_2. Their responses are shown in Figure 2.16, respectively, with the solid line ($G_1(s)$), dashed line ($G_2(s)$), dotted line ($G_3(s)$), and dash-and-dotted line ($G_4(s)$). They are quite different. In conclusion, even though the poles of $G(s)$ determine the basic form of responses, exact responses are determined by the poles, zeros, and the input. Therefore, the zeros of a transfer function cannot be completely ignored in the analysis and design of control systems.

2.5 BLOCK REPRESENTATION—COMPLETE CHARACTERIZATION

In the analysis and design of control systems, every device is represented by a block as shown in Figure 2.4 or 2.17(a). The block is then represented by its transfer function $G(s)$. If the input is $u(t)$ and the output is $y(t)$, then they are related by

$$Y(s) = G(s)U(s) \tag{2.37}$$

where $Y(s)$ and $U(s)$ are respectively the Laplace transforms of $y(t)$ and $u(t)$. Note that we have mixed the time-domain representation $u(t)$ and $y(t)$ and the Laplace transform representation $G(s)$ in Figure 2.17(a). This convention will be used throughout this text. It is important to know that it is incorrect to write $y(t) = G(s)u(t)$. The correct expression is $Y(s) = G(s)U(s)$.[6]

Equation (2.37) is an algebraic equation. The product of the Laplace transform of the input and the transfer function yields the Laplace transform of the output. The advantage of using this algebraic representation can be seen from the tandem connection of two systems shown in Figure 2.17(b). Suppose the two systems are represented, respectively, by

$$Y_1(s) = G_1(s)U_1(s) \qquad Y_2(s) = G_2(s)U_2(s)$$

In the tandem connection, we have $u_2(t) = y_1(t)$ or $U_2(s) = Y_1(s)$ and

$$Y_2(s) = G_2(s)Y_1(s) = G_2(s)G_1(s)U_1(s) \tag{2.38}$$

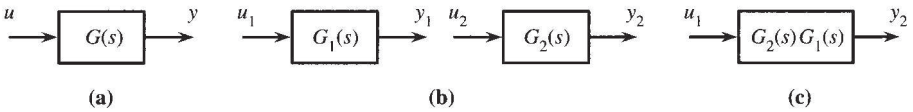


Figure 2.17 (a) A system. (b) Tandem connection of two systems. (c) Reduction of (b).

[6]It can also be expressed as $y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$, where $g(t)$ is the inverse Laplace transform of $G(s)$. See Reference [18]. This form is rarely used in the design of control systems.