described by LTIL differential equations. In this text, we study only this class of systems.

2.3 ZERO-INPUT RESPONSE AND ZERO-STATE RESPONSE

The response of linear, in particular LTIL, systems can always be decomposed into the zero-input response and zero-state response. In this section we shall use a simple example to illustrate this fact and then discuss some general properties of the zero-input response. The Laplace transform in Appendix A is needed for the following discussion.

Consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 3\frac{du(t)}{dt} - u(t) \tag{2.12}$$

Many methods are available to solve this equation. The simplest method is to use the Laplace transform. The application of the Laplace transform to (2.12) yields, using (A.9),

$$\begin{aligned} s^2Y(s) - sy(0^-) - \dot{y}(0^-) + 3[sY(s) - y(0^-)] + 2Y(s) \\ = 3[sU(s) - u(0^-)] - U(s) \end{aligned} \tag{2.13}$$

where $\dot{y}(t) := dy(t)/dt$ and capital letters denote the Laplace transforms of the corresponding lowercase letters.[2] Equation (2.13) is an algebraic equation and can be manipulated using addition, subtraction, multiplication, and division. The grouping of $Y(s)$ and $U(s)$ in (2.13) yields

$$(s^2 + 3s + 2)Y(s) = sy(0^-) + \dot{y}(0^-) + 3y(0^-) - 3u(0^-) + (3s - 1)U(s)$$

which implies

$$Y(s) = \underbrace{\frac{(s + 3)y(0^-) + \dot{y}(0^-) - 3u(0^-)}{s^2 + 3s + 2}}_{\text{Zero-Input Response}} + \underbrace{\frac{3s - 1}{s^2 + 3s + 2}U(s)}_{\text{Zero-State Response}} \tag{2.14}$$

This equation reveals that the solution of (2.12) is partly excited by the input $u(t)$, $t \geq 0$, and partly excited by the initial conditions $y(0^-)$, $\dot{y}(0^-)$, and $u(0^-)$. These initial conditions will be called *the initial state*. The initial state is excited by the input applied before $t = 0$. In some sense, the initial state summarizes the effect of the past input $u(t)$, $t < 0$, on the future output $y(t)$, for $t \geq 0$. If different past inputs $u_1(t)$, $u_2(t)$, . . . , $t \leq 0$, excite the *same* initial state, then their effects on the future output will be identical. Therefore, how the differential equation acquires the initial state at $t = 0$ is immaterial in studying its solution $y(t)$, for $t \geq 0$. We mention that

[2]We use A := B to denote that A, by definition, equals B, and A =: B to denote that B, by definition, equals A.

the initial time $t = 0$ is not the absolute time; it is the instant we start to study the system.

Consider again (2.14). The response can be decomposed into two parts. The first part is excited exclusively by the initial state and is called the *zero-input response*. The second part is excited exclusively by the input and is called the *zero-state response*. In the study of LTIL systems, it is convenient to study the zero-input response and the zero-state response separately. We first study the zero-input response and then the zero-state response.

2.3.1 Zero-Input Response—Characteristic Polynomial

Consider the differential equation in (2.12). If $u(t) = 0$, for $t \geq 0$, then (2.12) reduces to

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0$$

This is called the *homogeneous* equation. We now study its response due to a nonzero initial state. The application of the Laplace transform yields, as in (2.13),

$$s^2Y(s) - sy(0^-) - \dot{y}(0^-) + 3[sY(s) - y(0^-)] + 2Y(s) = 0$$

which implies

$$Y(s) = \frac{(s + 3)y(0^-) + \dot{y}(0^-)}{s^2 + 3s + 2} = \frac{(s + 3)y(0^-) + \dot{y}(0^-)}{(s + 1)(s + 2)} \tag{2.15}$$

This can be expanded as

$$Y(s) = \frac{k_1}{s + 1} + \frac{k_2}{s + 2} \tag{2.16}$$

with

$$k_1 = \left.\frac{(s + 3)y(0^-) + \dot{y}(0^-)}{s + 2}\right|_{s=-1} = 2y(0^-) + \dot{y}(0^-)$$

and

$$k_2 = \left.\frac{(s + 3)y(0^-) + \dot{y}(0^-)}{s + 1}\right|_{s=-2} = -[y(0^-) + \dot{y}(0^-)]$$

Thus the zero-input response is

$$y(t) = k_1e^{-t} + k_2e^{-2t} \tag{2.17}$$

No matter what the initial conditions $y(0^-)$ and $\dot{y}(0^-)$ are, the zero-input response is always a linear combination of the two functions e^{-t} and e^{-2t}. The two functions e^{-t} and e^{-2t} are the inverse Laplace transforms of $1/(s + 1)$ and $1/(s + 2)$. The two roots -1 and -2—or, equivalently, the two roots of the denominator of (2.15)

are called the *modes* of the system. The modes govern the form of the zero-input response of the system.

We now extend the preceding discussion to the general case. Consider the nth order LTIL differential equation

$$\begin{aligned} &a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y^{(1)}(t) + a_0 y(t) \\ &= b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_1 u^{(1)}(t) + b_0 u(t) \end{aligned} \tag{2.18}$$

where

$$y^{(i)}(t) := \frac{d^i}{dt^i} y(t), \quad u^{(i)}(t) := \frac{d^i}{dt^i} u(t)$$

and $\dot{y}(t) := y^{(1)}(t)$, $\ddot{y}(t) := y^{(2)}(t)$. We define

$$D(p) := a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0 \tag{2.19a}$$

and

$$N(p) := b_m p^m + b_{m-1} p^{m-1} + \cdots + b_1 p + b_0 \tag{2.19b}$$

where the variable p is the differentiator d/dt defined by

$$py(t) := \frac{d}{dt} y(t) \qquad p^2 y(t) := \frac{d^2}{dt^2} y(t) \qquad p^3 y(t) := \frac{d^3}{dt^3} y(t) \tag{2.20}$$

and so forth. Using this notation, (2.18) can be written as

$$D(p)y(t) = N(p)u(t) \tag{2.21}$$

In the study of the zero-input response, we assume $u(t) \equiv 0$. Then (2.21) reduces to

$$D(p)y(t) = 0 \tag{2.22}$$

This is the homogeneous equation. Its solution is excited exclusively by initial conditions. The application of the Laplace transform to (2.22) yields, as in (2.15),

$$Y(s) = \frac{I(s)}{D(s)}$$

where $D(s)$ is defined in (2.19a) with p replaced by s and $I(s)$ is a polynomial of s depending on initial conditions. We call $D(s)$ the *characteristic polynomial* of (2.21) because it governs the *free, unforced,* or *natural* response of (2.21). The roots of the polynomial $D(s)$ are called the *modes.*[3] For example, if

$$D(s) = (s - 2)(s + 1)^2(s + 2 - j3)(s + 2 + j3)$$

then the modes are $2, -1, -1, -2 + j3$, and $-2 - j3$. The root 2 and the complex roots $-2 \pm j3$ are simple modes and the root -1 is a repeated mode with multi-

[3]In the literature, they are also called the *natural frequencies.* However the ω_n in $D(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$ is also called the natural frequency in this and some other control texts. To avoid possible confusion, we call the roots of $D(s)$ the modes.

plicity 2. Thus for any initial conditions, $Y(s)$ can be expanded as

$$Y(s) = \frac{k_1}{s-2} + \frac{k_2}{s+2-j3} + \frac{k_3}{s+2+j3} + \frac{c_1}{s+1} + \frac{c_2}{(s+1)^2}$$

and its zero-input response is, using Table A.1,

$$y(t) = k_1 e^{2t} + k_2 e^{-(2-j3)t} + k_3 e^{-(2+j3)t} + c_1 e^{-t} + c_2 t e^{-t}$$

This is the general form of the zero-input response and is determined by the modes of the system.

Exercise 2.3.1

Find the zero-input responses of (2.12) due to $y(0^-) = 1$, $\dot{y}(0^-) = -1$, and $y(0^-) = \dot{y}(0^-) = 1$.

[**Answers:** $y(t) = e^{-t}$; $y(t) = 3e^{-t} - 2e^{-2t}$.]

Exercise 2.3.2

Find the modes and the general form of the zero-input responses of

$$D(p)y(t) = N(p)u(t) \tag{2.23}$$

where

$$D(p) = p^2(p-2)^2(p^2+4p+8) \quad \text{and} \quad N(p) = 3p^2 - 10$$

[**Answers:** 0, 0, 2, 2, $-2+j2$, and $-2-j2$; $k_1 + k_2 t + k_3 e^{2t} + k_4 t e^{2t} + k_5 e^{-(2-j2)t} + k_6 e^{-(2+j2)t}$]

2.4 ZERO-STATE RESPONSE—TRANSFER FUNCTION

Consider the differential equation in (2.12) or

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 3\frac{du}{dt} - u(t) \tag{2.24}$$

The response of (2.24) is partly excited by the initial conditions and partly excited by the input $u(t)$. If all initial conditions equal zero, the response is excited exclusively by the input and is called the *zero-state response*. In the Laplace transform domain, the zero-state response of (2.24) is governed by, setting all initial conditions in (2.14) to zero,

$$Y(s) = \frac{3s-1}{s^2+3s+2}U(s) =: G(s)U(s) \tag{2.25}$$

where the rational function $G(s) = (3s - 1)/(s^2 + 3s + 2)$ is called the *transfer function*. It is the ratio of the Laplace transforms of the output and input when all initial conditions are zero or

$$G(s) = \left.\frac{Y(s)}{U(s)}\right|_{\text{Initial conditions}=0} = \left.\frac{\mathcal{L}[\text{Output}]}{\mathcal{L}[\text{Input}]}\right|_{\text{Initial conditions}=0} \tag{2.26}$$

The transfer function describes only the zero-state responses of LTIL systems.

Example 2.4.1 (Mechanical system)

Consider the mechanical system studied in Figure 2.5(a). As derived in (2.4), it is described by

$$m\frac{d^2y(t)}{dt^2} + k_1\frac{dy(t)}{dt} + k_2y(t) = u(t)$$

The application of the Laplace transform yields, assuming zero initial conditions,

$$ms^2Y(s) + k_1sY(s) + k_2Y(s) = U(s)$$

or

$$(ms^2 + k_1s + k_2)Y(s) = U(s)$$

Thus the transfer function from u to y of the mechanical system is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + k_1s + k_2}$$

This example reveals that the transfer function of a system can be readily obtained from its differential-equation description. For example, if a system is described by the differential equation

$$D(p)y(t) = N(p)u(t)$$

where $D(p)$ and $N(p)$ are defined as in (2.19), then the transfer function of the system is

$$G(s) = \frac{N(s)}{D(s)}$$

Exercise 2.4.1

Find the transfer functions from u to y of the networks shown in Figures 2.10 and 2.11.

[**Answers:** $1/(6s + 2), 1/(LCs^2 + RCs + 1), (CR_2s + 1)/(C(R_1 + R_2)s + 1)$.]

Exercise 2.4.2 (Industrial process)

Find the transfer function from q_i to q_2 of the system in Figure 2.12.

[**Answer:** $G(s) = 1/(A_1A_2R_1R_2s^2 + (A_1(R_1 + R_2) + A_2R_2)s + 1)$.]

RLC Networks

Although the transfer function of an RLC network can be obtained from its differential-equation description, it is generally simpler to compute it by using the concept of the Laplacian impedance or, simply, the impedance. If all initial conditions are zero, the application of the Laplace transforms to (2.6) yields

$$V(s) = RI(s) \qquad (\textit{resistor})$$

$$V(s) = \frac{1}{Cs} I(s) \qquad (\textit{capacitor})$$

and

$$V(s) = LsI(s) \qquad (\textit{inductor})$$

These relationships can be written as $V(s) = Z(s)I(s)$, and $Z(s)$ is called the (Laplacian) *impedance*. Thus the impedances of the resistor, capacitor, and inductor are respectively R, $1/Cs$, and Ls. If we consider $I(s)$ the input and $V(s)$ the output, then the impedance is a special case of the transfer function defined in (2.26). Whenever impedances are used, all initial conditions are implicitly assumed to be zero.

The manipulation involving impedances is purely algebraic, identical to the manipulation of resistances. For example, the resistance of the series connection of two resistances R_1 and R_2 is $R_1 + R_2$; the resistance of the parallel connection of R_1 and R_2 is $R_1R_2/(R_1 + R_2)$. Similarly, the impedance of the series connection of two impedances $Z_1(s)$ and $Z_2(s)$ is $Z_1(s) + Z_2(s)$; the impedance of the parallel connection of $Z_1(s)$ and $Z_2(s)$ is $Z_1(s)Z_2(s)/(Z_1(s) + Z_2(s))$. The only difference is that now we are dealing with rational functions, rather than real numbers as in the resistive case.

Example 2.4.2

Compute the transfer function from u to i of the network shown in Figure 2.13(a). Its equivalent network using impedances is shown in Figure 2.13(b). The impedance of the parallel connection of $1/2s$ and $3s + 2$ is

$$\frac{\frac{1}{2s}(3s + 2)}{\frac{1}{2s} + (3s + 2)} = \frac{3s + 2}{6s^2 + 4s + 1}$$

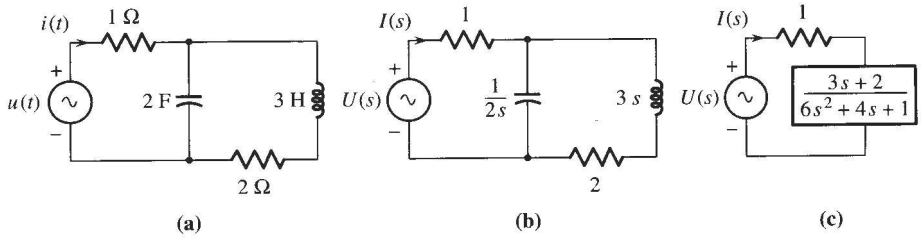


Figure 2.13 Network.

as shown in Figure 2.13(c). Hence the current $I(s)$ shown in Figure 2.13 is given by

$$I(s) = \frac{U(s)}{1 + \dfrac{3s + 2}{6s^2 + 4s + 1}} = \frac{6s^2 + 4s + 1}{6s^2 + 7s + 3} U(s)$$

Thus the transfer function from u to i is

$$G(s) = \frac{I(s)}{U(s)} = \frac{6s^2 + 4s + 1}{6s^2 + 7s + 3}$$

Exercise 2.4.3

Find the transfer functions from u to y of the networks in Figures 2.10 and 2.11 using the concept of impedances.

2.4.1 Proper Transfer Functions

Consider the rational function

$$G(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are two polynomials with real coefficients. We use deg to denote the degree of a polynomial. If

$$\deg N(s) > \deg D(s)$$

$G(s)$ is called an *improper* rational function. For example, the rational functions

$$\frac{s^2 + 1}{s + 1} \qquad s \qquad s^2 + 1 \qquad \text{and} \qquad \frac{s^{10}}{s^9 + s^8 + s^7 - 10}$$

are all improper. If

$$\deg N(s) \leq \deg D(s)$$