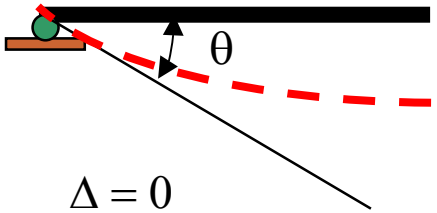
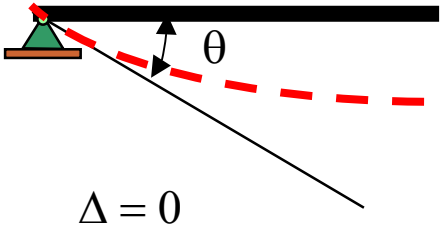
DEFLECTIONS: GEOMETRIC METHODS

- Deflection Diagrams and the Elastic Curve
- Elastic-Beam Theory
- Moment-Area Method
- Conjugate Beam Method

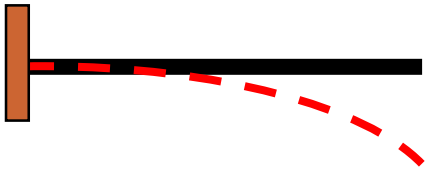
Deflection Diagrams and the Elastic Curve



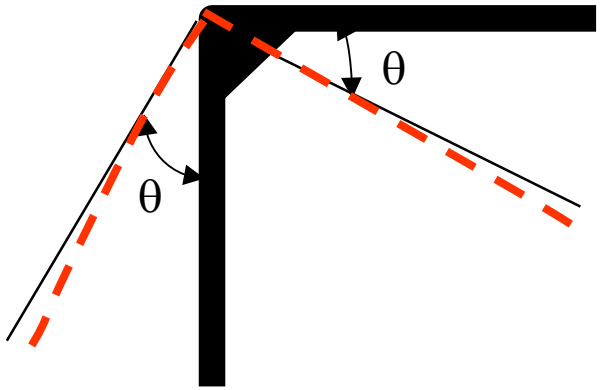
$\Delta = 0$
roller or rocker



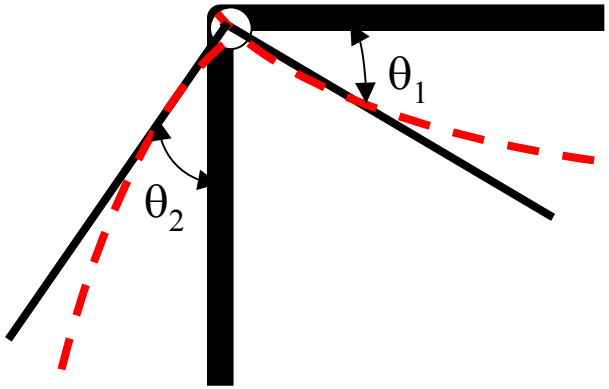
$\Delta = 0$
pin



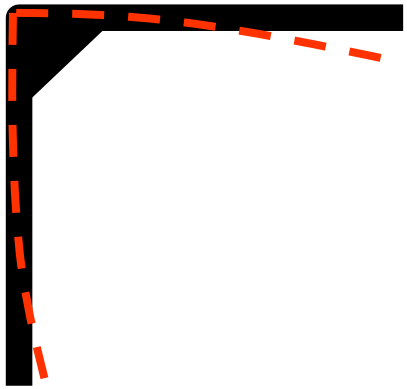
$\Delta = 0$
$\theta = 0$
fixed support

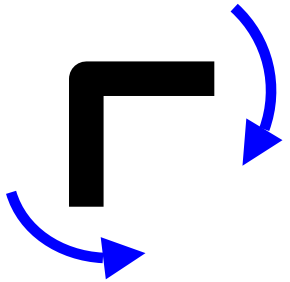


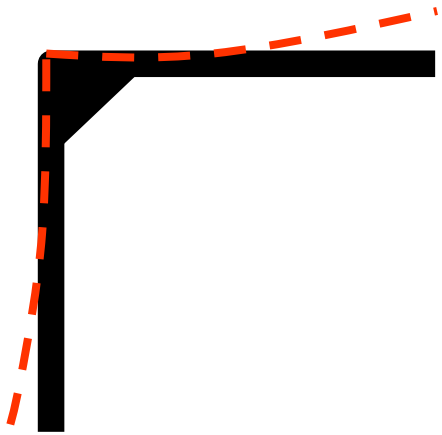
fixed-connected joint

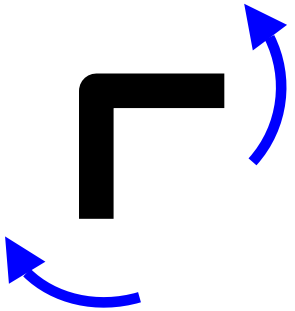


pin-connected joint

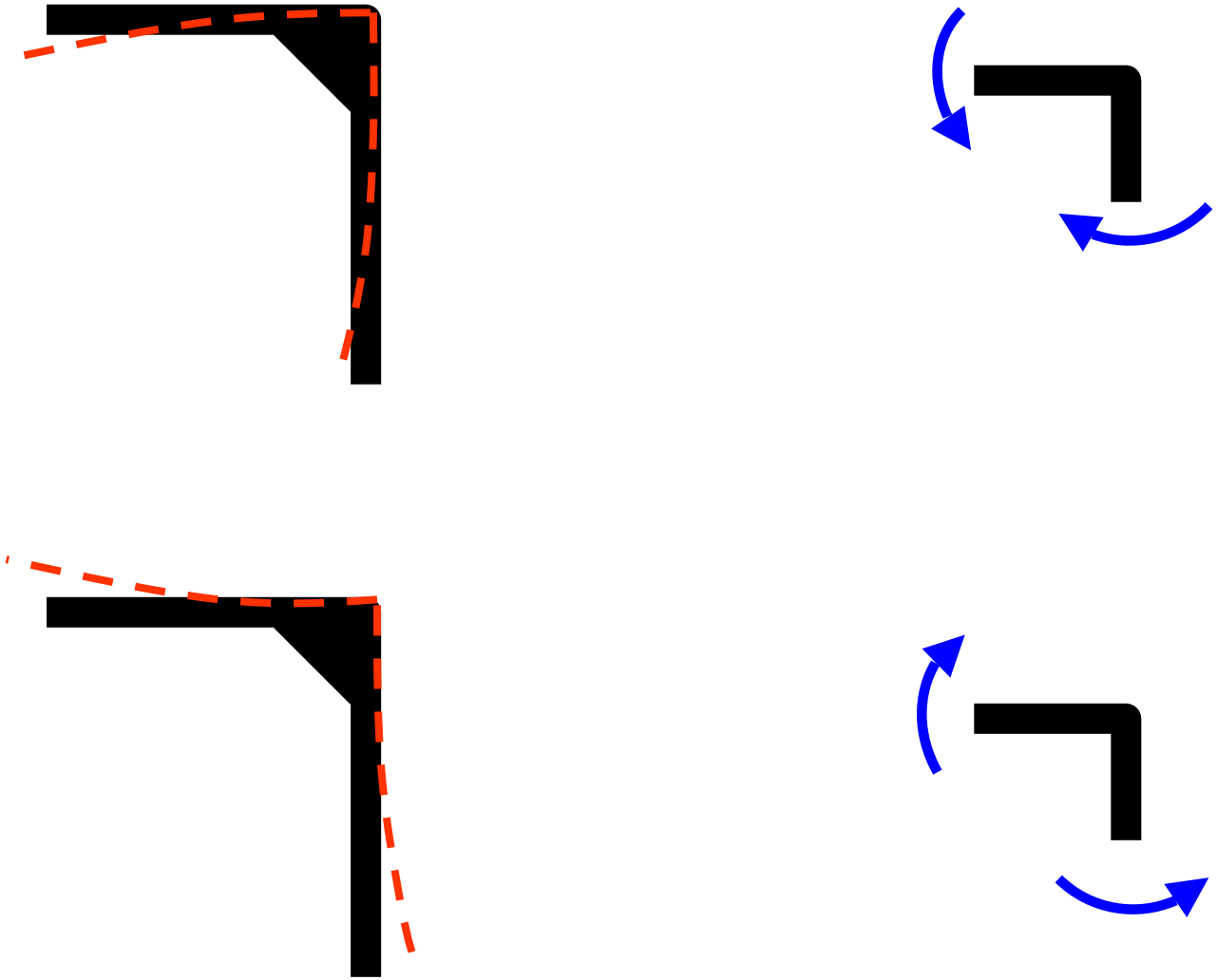




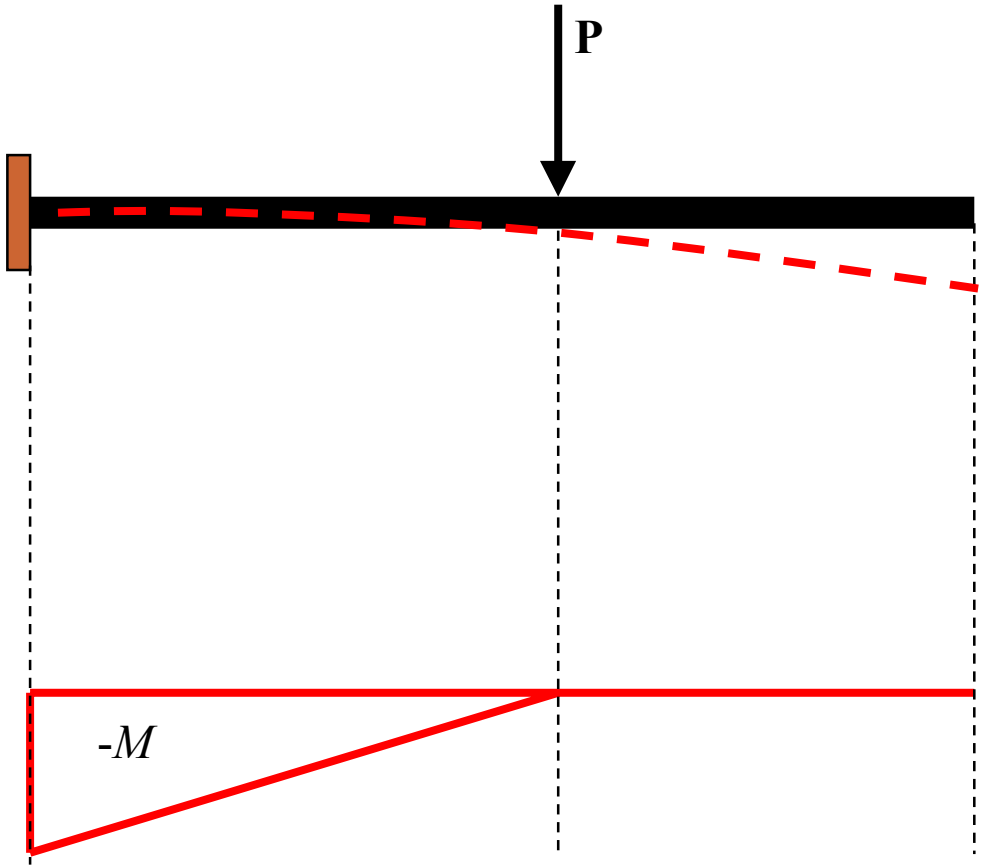


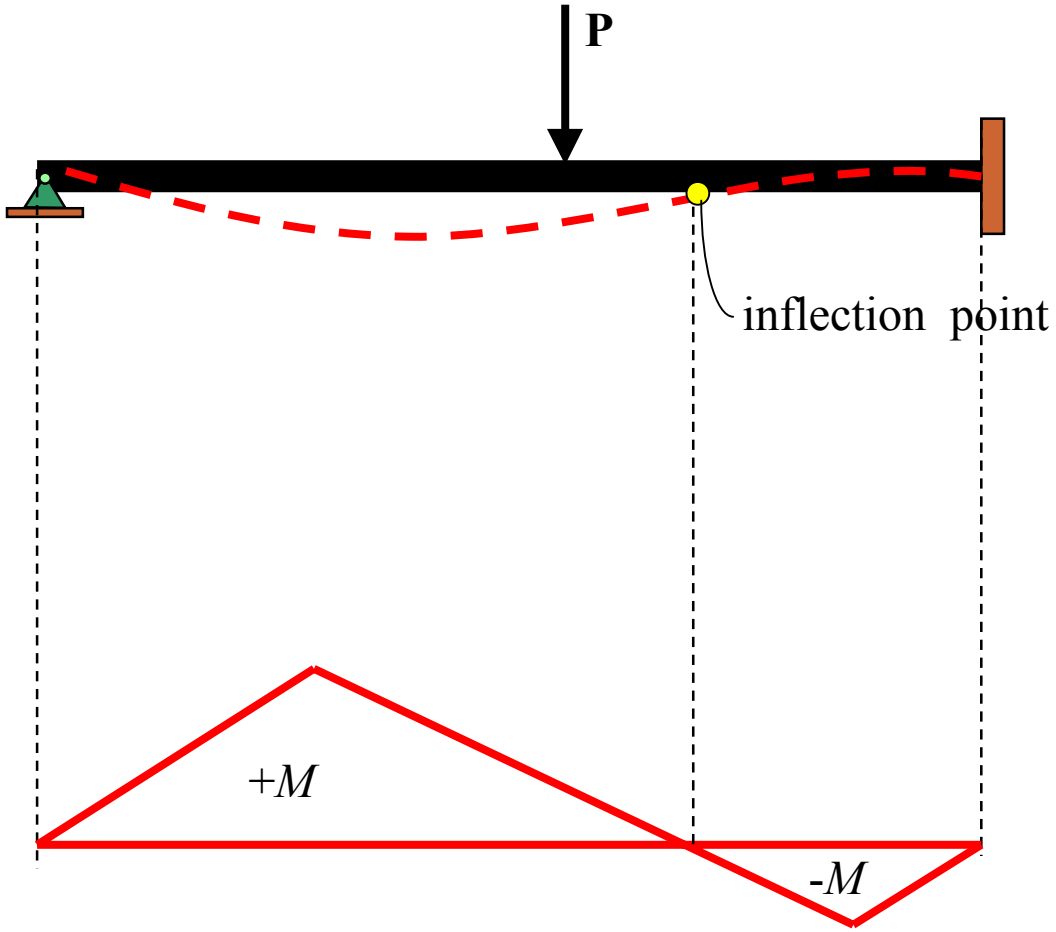


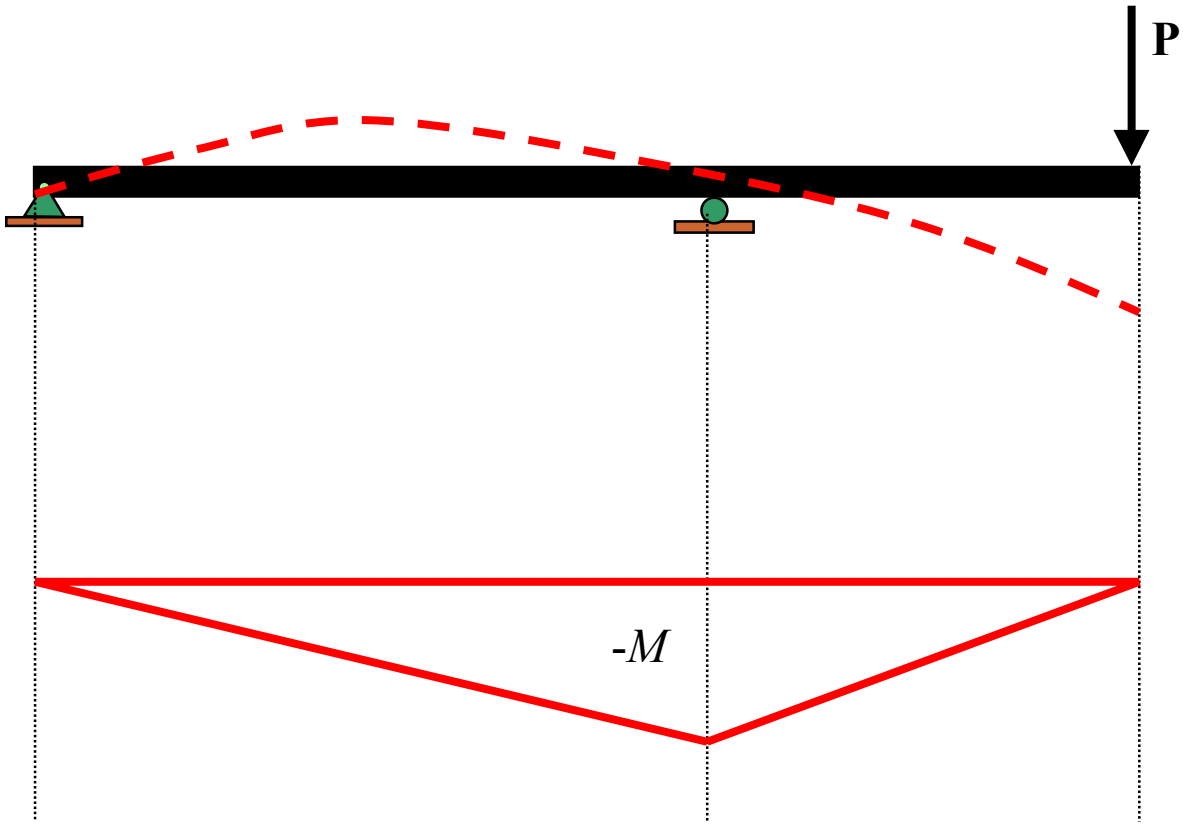
fixed-connected joint

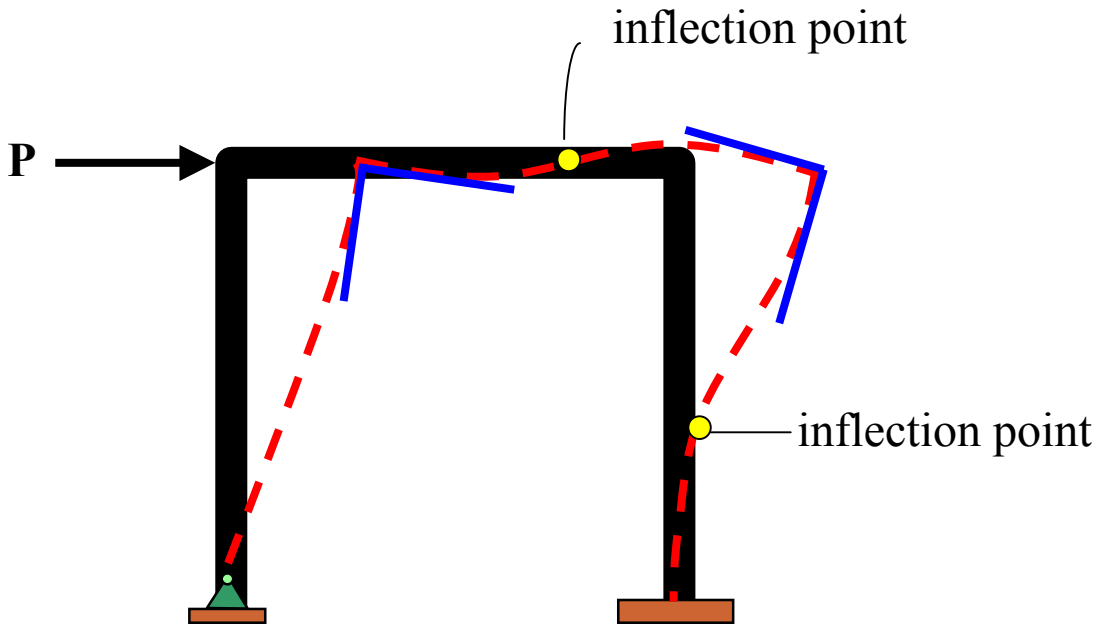


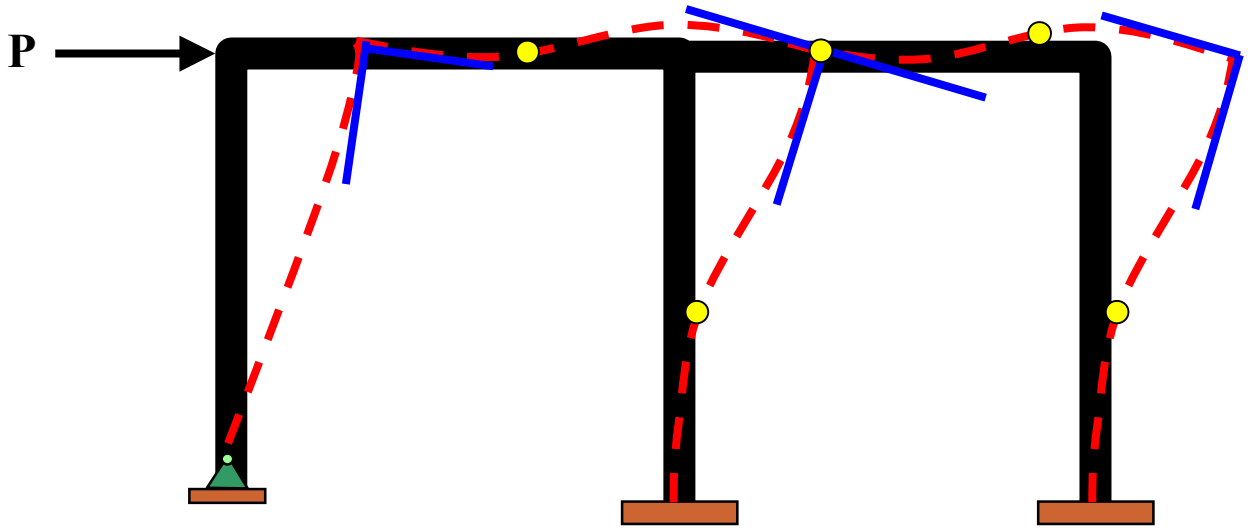
fixed-connected joint


P
-M

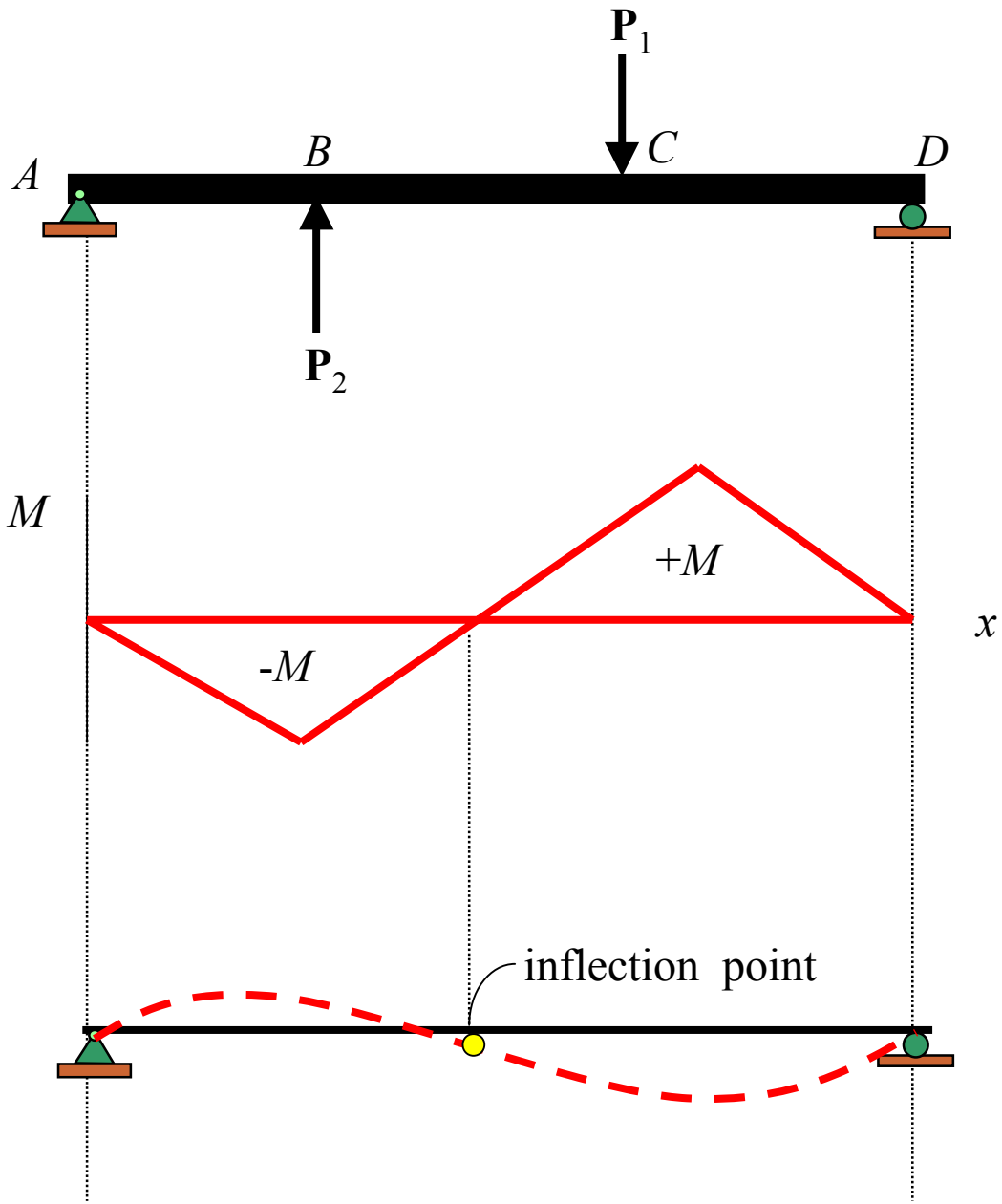

P
inflection point
+M
-M

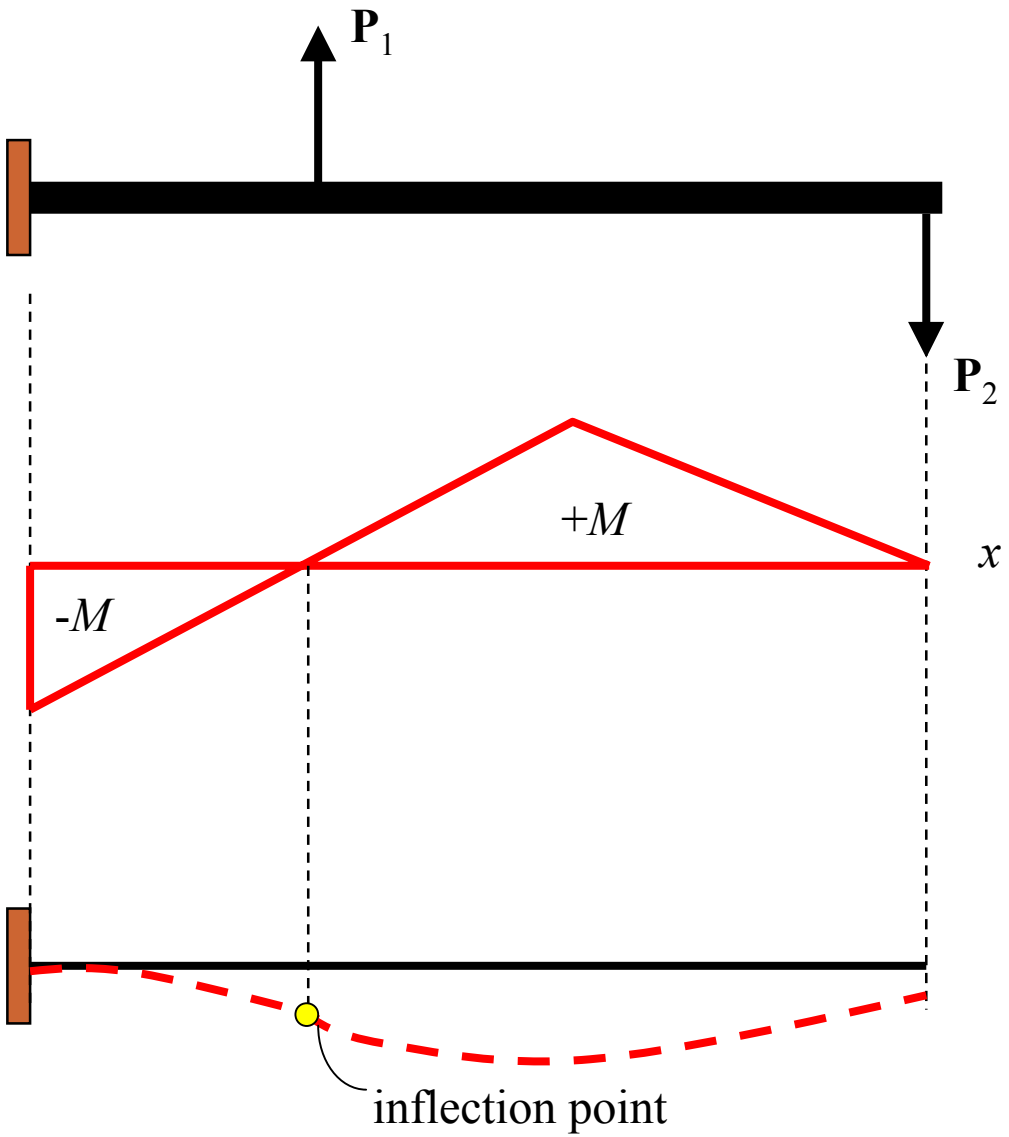

P
-M


inflection point
P
inflection point

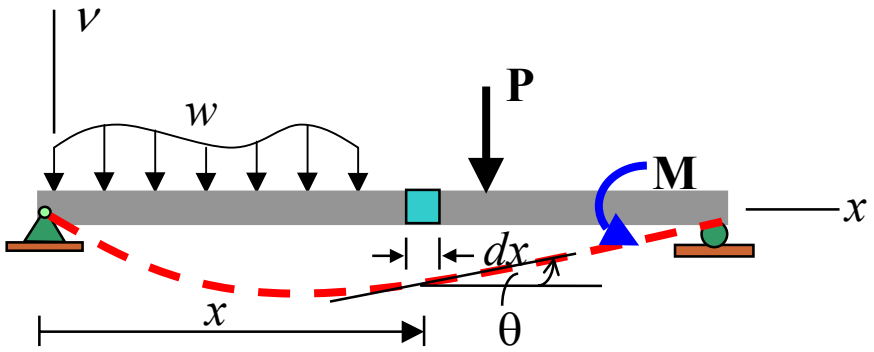


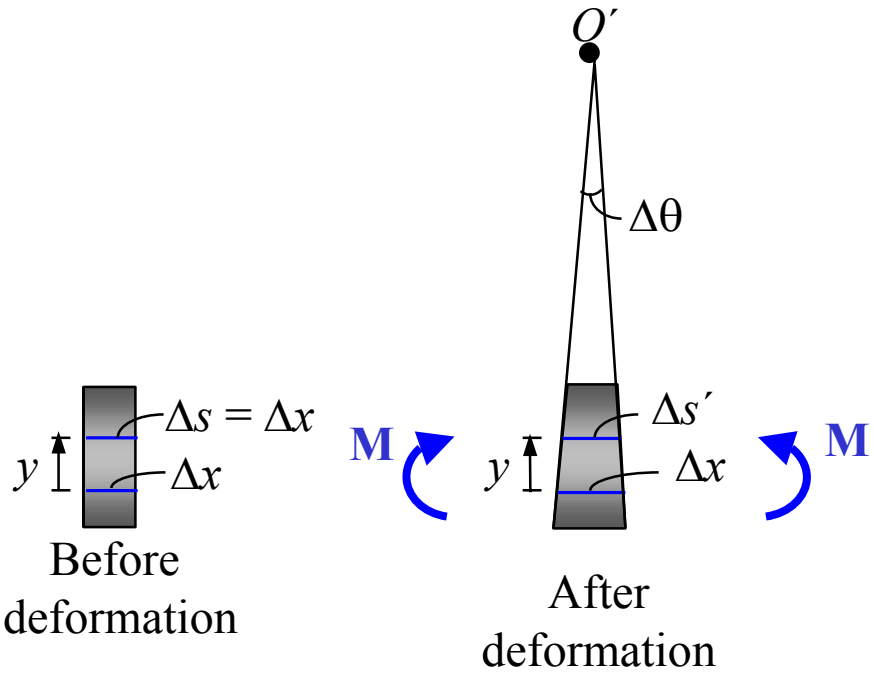
inflection point


P1
A
B
C
D
P2
M
+M
-M
x
inflection point

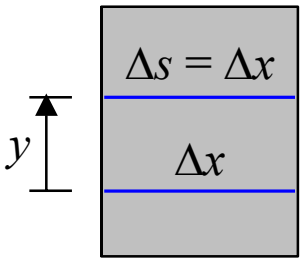

P1
P2
+M
-M
x
inflection point

Elastic-Beam Theory

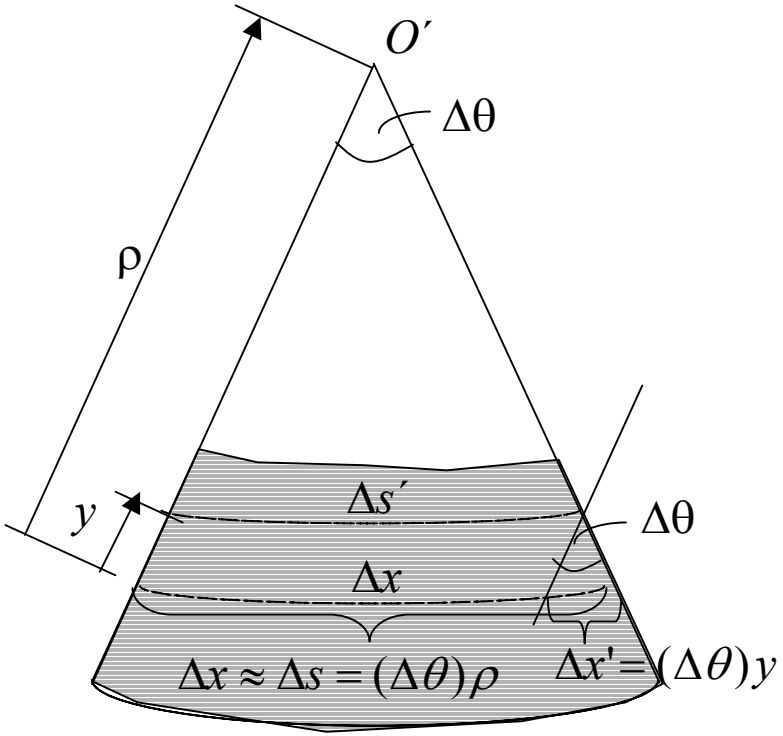




- **Moment-curvature relationship**



Before deformation

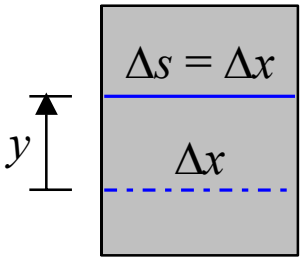


After deformation

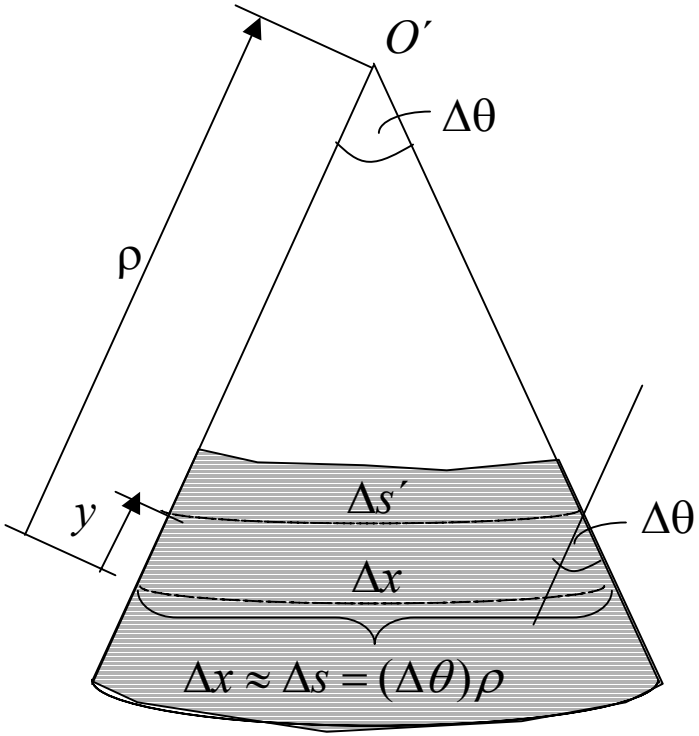
$$\frac{M}{EI} = \frac{1}{\rho} \qquad \text{------(8-1)}$$

Curvature

Radius of Curvature



Before deformation



After deformation

$$dx = \rho d\theta$$

$$d\theta = \frac{1}{\rho} dx$$

$$\theta = \int \frac{M}{EI} dx \qquad \text{------(8-2)}$$

If we chose the υ axis positive upward, and if we can express the curvature ($1/\rho$) in terms of x and , we can then determine the elastic curve for the beam.

$$\frac{1}{\rho} = \frac{d^2\upsilon / dx^2}{[1+(d\upsilon / dx)^2]^{3/2}}$$

Therefore,

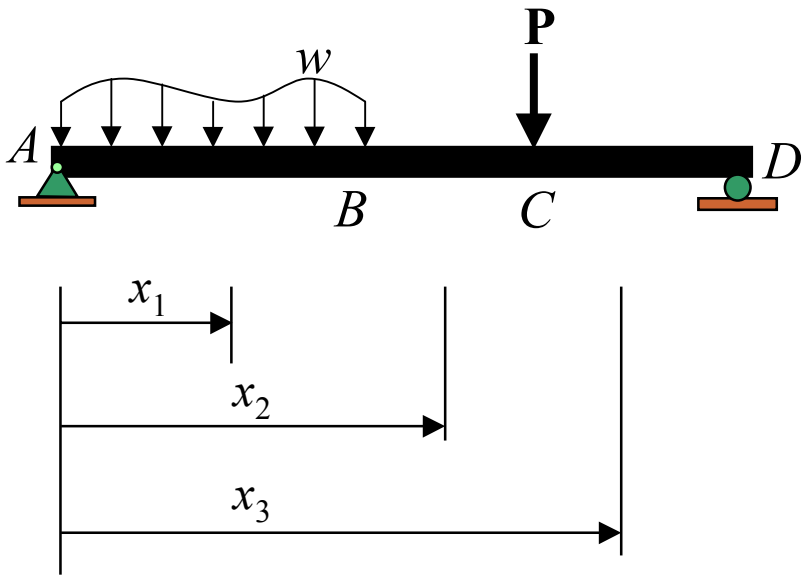
$$\frac{M}{EI} = \frac{d^2\upsilon / dx^2}{[1+(d\upsilon / dx)^2]^{3/2}}$$

$$(\frac{d\upsilon}{dx})^2 \approx 0$$

$$\boxed{\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}} \quad \text{------(8-3)*}$$

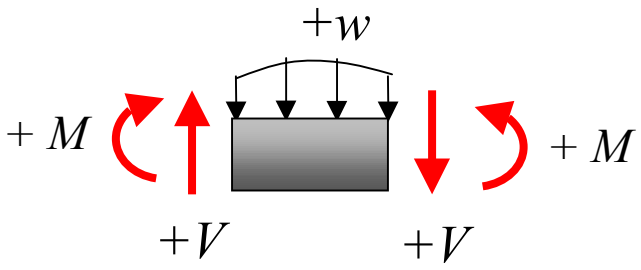
* Small deflection theory

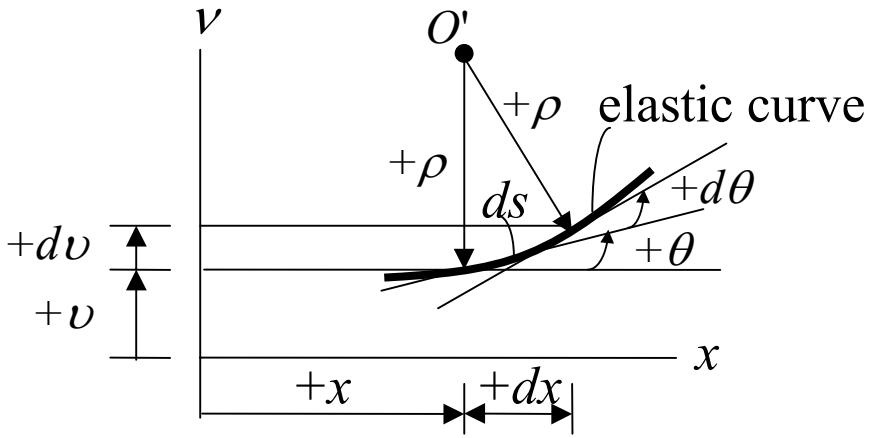
The Double Integration Method



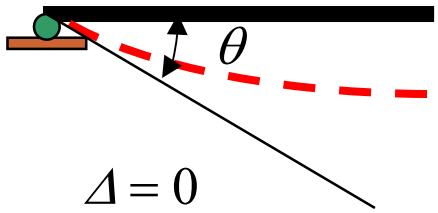
$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI} \qquad \text{------(8-3)}$$

- **Sign Convention**



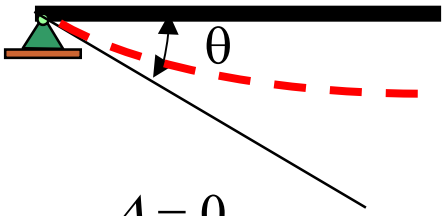


- **Boundary condition**



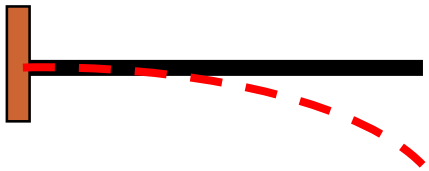
$\Delta = 0$

roller or rocker



$\Delta = 0$

pin

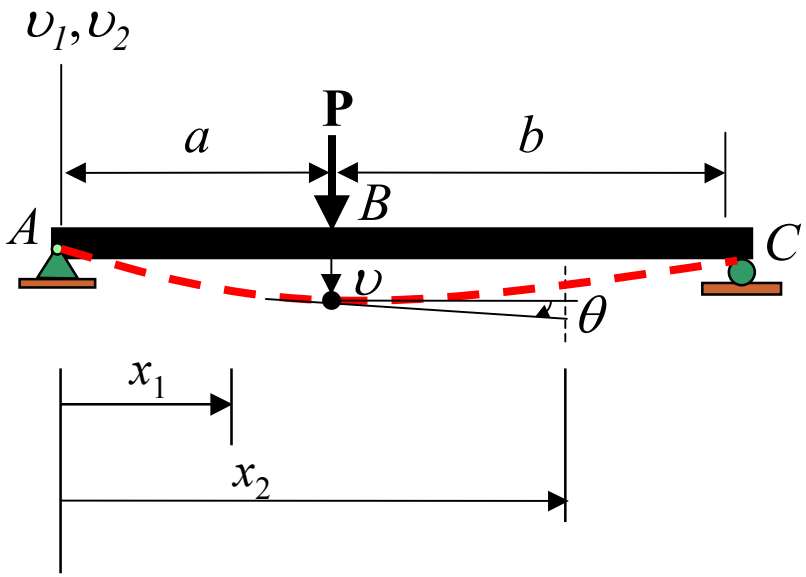


$\Delta = 0$

$\theta = 0$

fixed support

- **Continuity Conditions**



at point B, $x_1 = x_2 = a$,

$$\theta_1(a) = \theta_2(a)$$

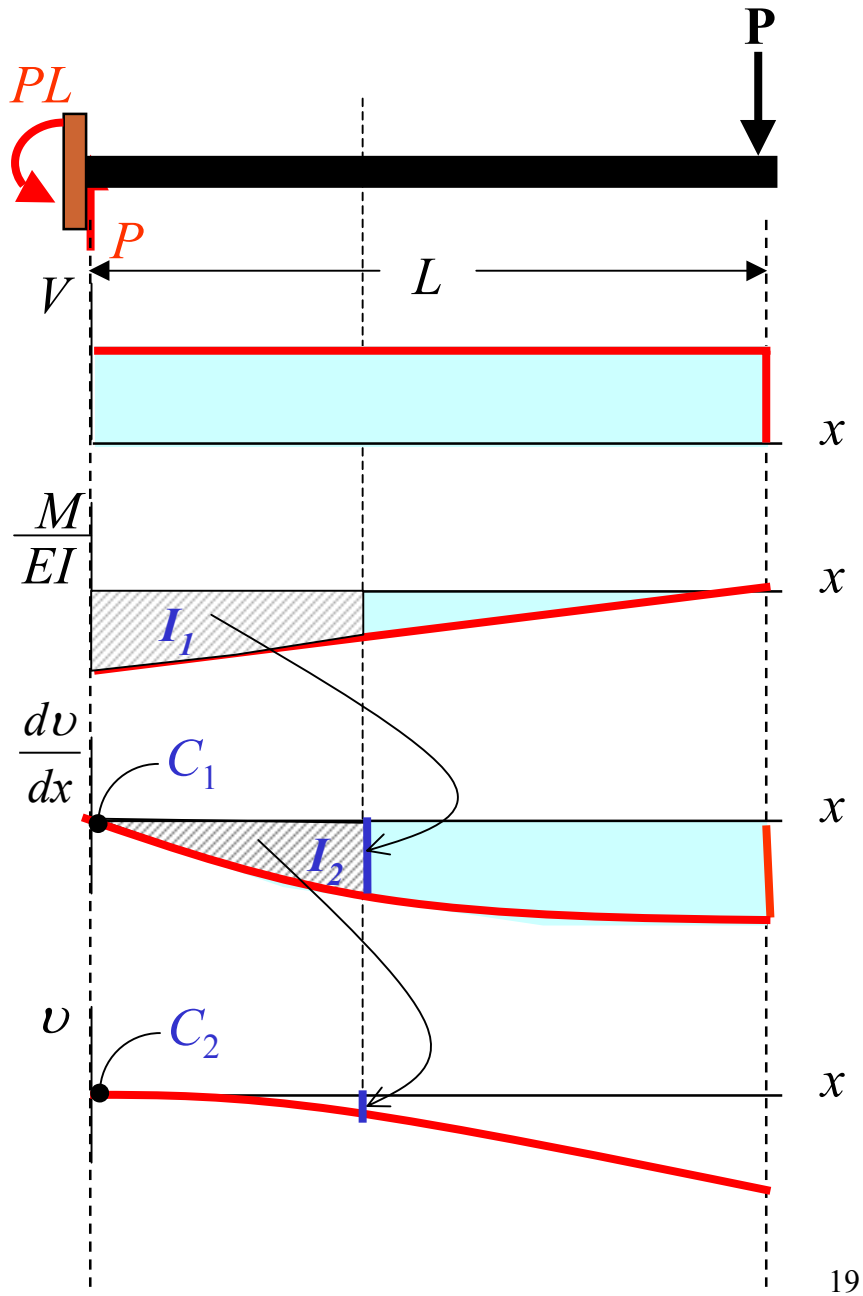
$$\upsilon_1(a) = \upsilon_2(a)$$

integration Constants

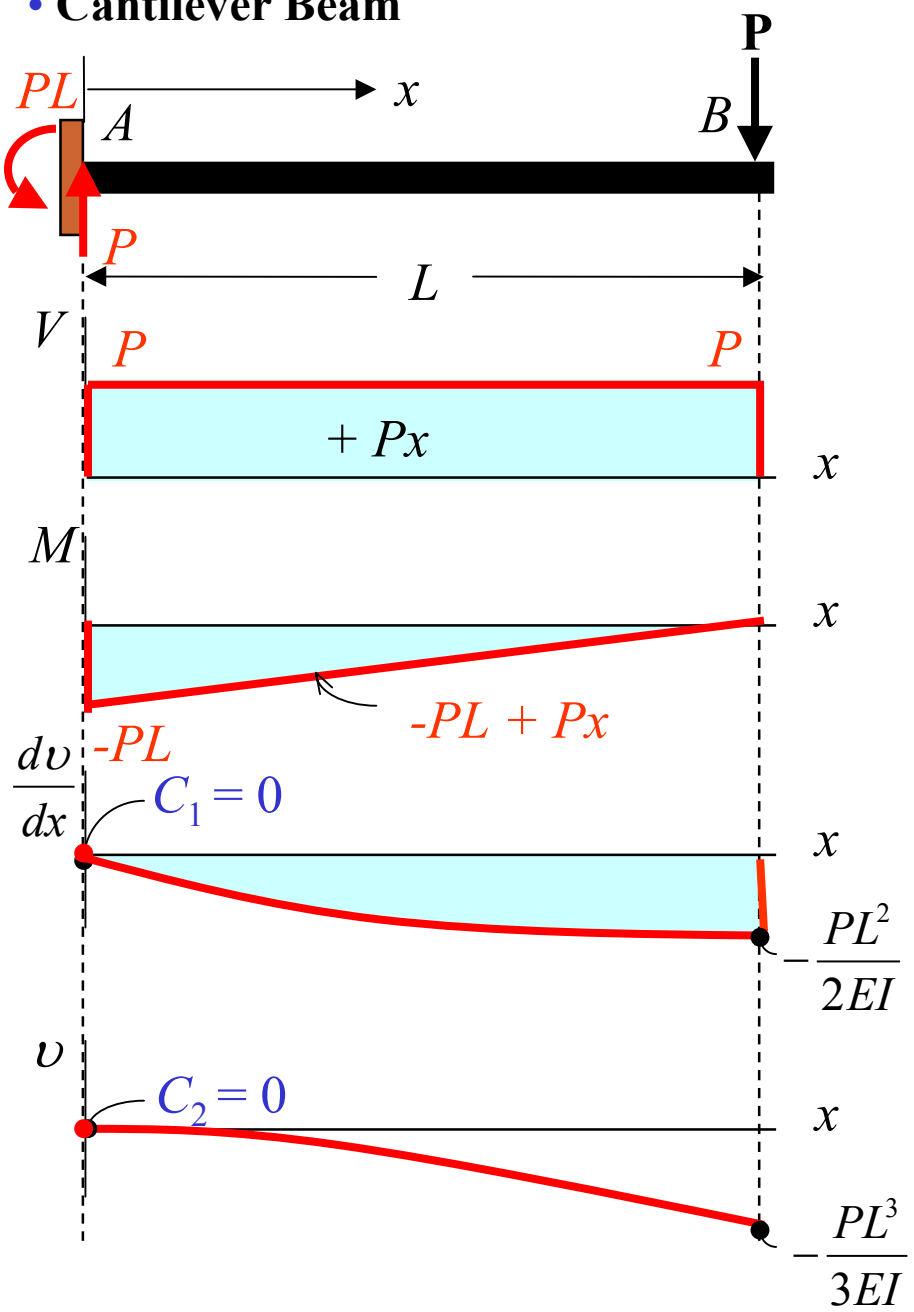
$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}$$

Slope (θ) — $\dfrac{d\upsilon}{dx} = \displaystyle\int \frac{M}{EI}\,dx = (I_1) + C_1$

Deflection (Δ) — $\upsilon = \displaystyle\int (I_1 + c_1)dx = I_2 + C_2$



• **Cantilever Beam**



Moment-curvature relationship

$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}$$

$$\frac{d^2\upsilon}{dx^2} = \frac{1}{EI}(-PL + Px)$$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}(-PLx + \frac{Px^2}{2}) + C_1 \quad ---(1)$$

$$\upsilon = \frac{1}{EI}(-\frac{PLx^2}{2} + \frac{Px^3}{6}) + C_1x + C_2 \quad ---(2)$$

Boundary Condition

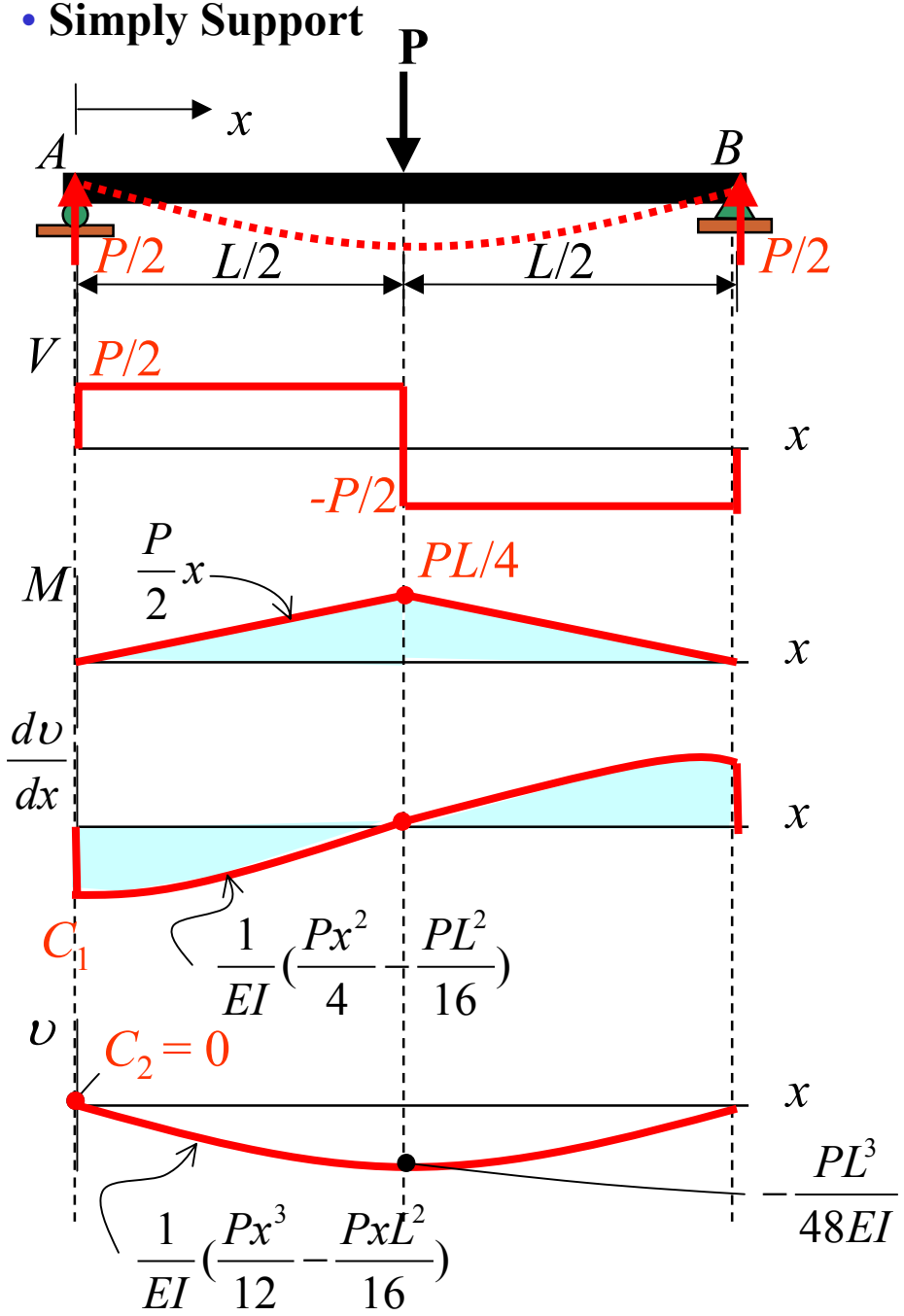
- $(\frac{d\upsilon}{dx})_{x=0} = 0$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}(-PLx + \frac{Px^2}{2}) + C_1 \quad ---> C_1 = 0$$

- $(\upsilon)_{x=0} = 0$

$$\upsilon = \frac{1}{EI}(-\frac{PLx^2}{2} + \frac{Px^3}{6}) + C_1x + C_2 \quad -> C_2 = 0$$

- **Simply Support**



Moment-curvature relationship

$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}$$

$$\frac{d^2\upsilon}{dx^2} = \frac{1}{EI}\left(\frac{Px}{2}\right)$$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}\left(\frac{Px^2}{4}\right) + C_1 \quad ---(1)$$

$$\upsilon = \frac{1}{EI}\left(\frac{Px^3}{12}\right) + C_1 x + C_2 \quad ---(2)$$

Boundary Condition

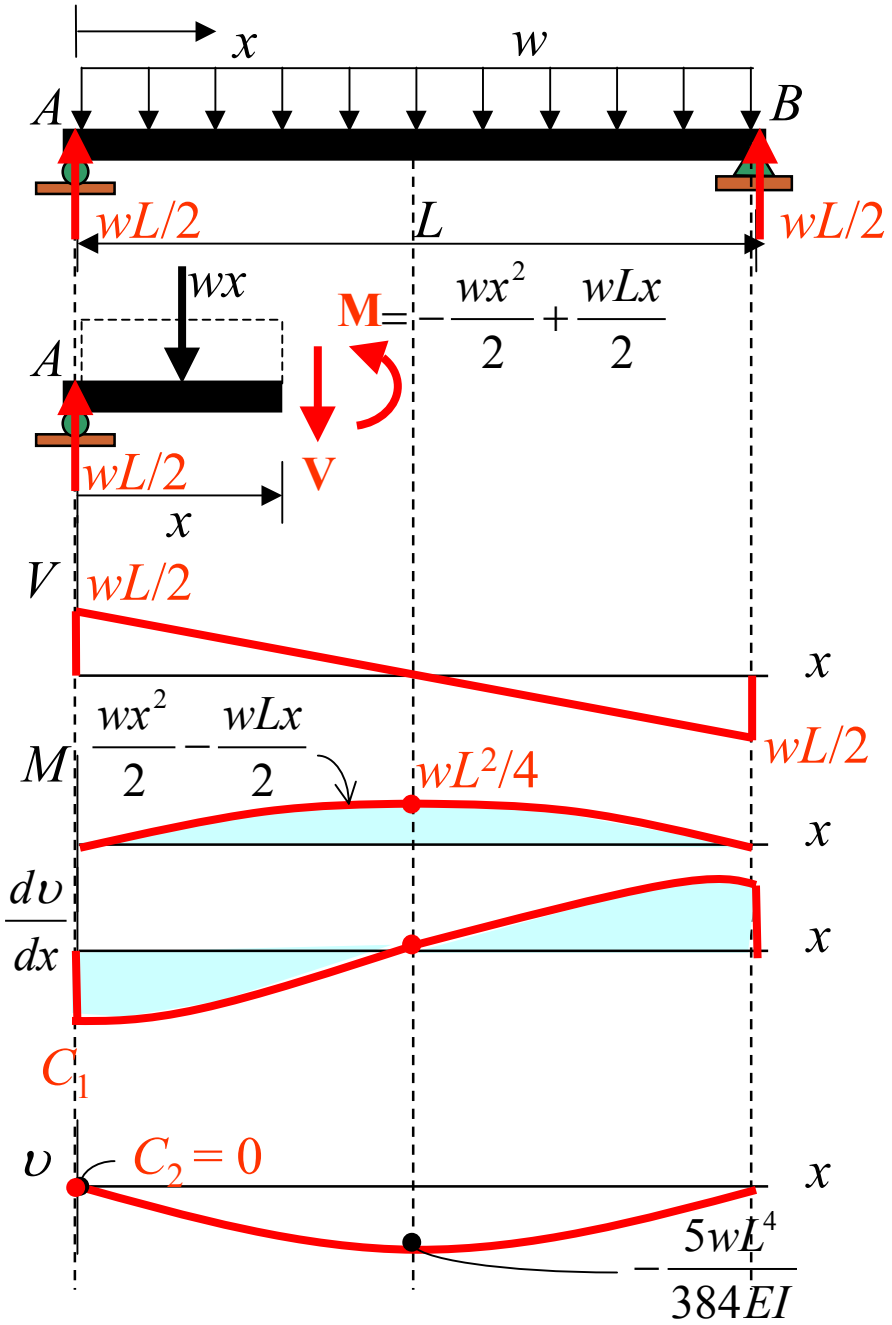
- $(\upsilon)_{x=0} = 0$

$$\upsilon = \frac{1}{EI}\left(\frac{Px^3}{12}\right) + C_1 x + C_2 \text{ (0, 0, 0)} \dashrightarrow C_2 = 0$$

- $\left(\frac{d\upsilon}{dx}\right)_{x=L/2} = 0$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}\left(\frac{Px^2}{4}\right) + C_1 \text{ (0, } L/2\text{)} \dashrightarrow C_1 = -\frac{PL^2}{16EI}$$

- **Simply Support**



Moment-curvature relationship

$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}$$

$$\frac{d^2\upsilon}{dx^2} = \frac{1}{EI}\left(-\frac{wx^2}{2} + \frac{wLx}{2}\right)$$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}\left(-\frac{wx^3}{6} + \frac{wLx^2}{4}\right) + C_1 \quad ---(1)$$

$$\upsilon = \frac{1}{EI}\left(-\frac{wx^4}{24} + \frac{wLx^3}{12}\right) + C_1 x + C_2 \quad ---(2)$$

Boundary Condition

- $(\upsilon)_{x=0} = 0$

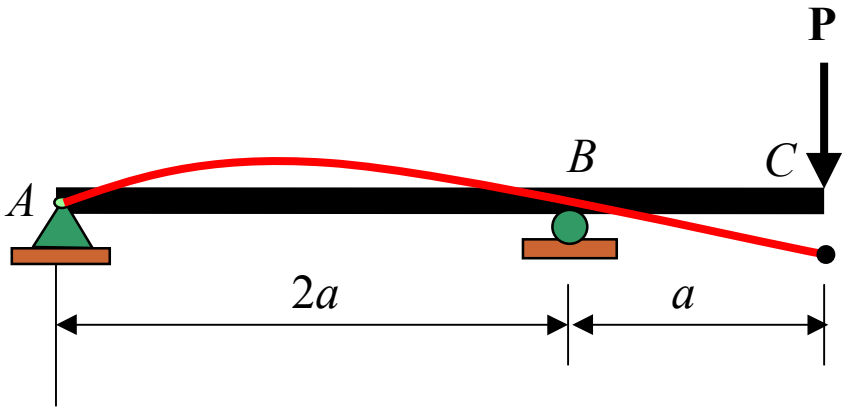
$$\upsilon = \frac{1}{EI}\left(-\frac{wx^4}{24} + \frac{wLx^3}{12}\right) + C_1 x + C_2 \dashrightarrow C_2 = 0$$

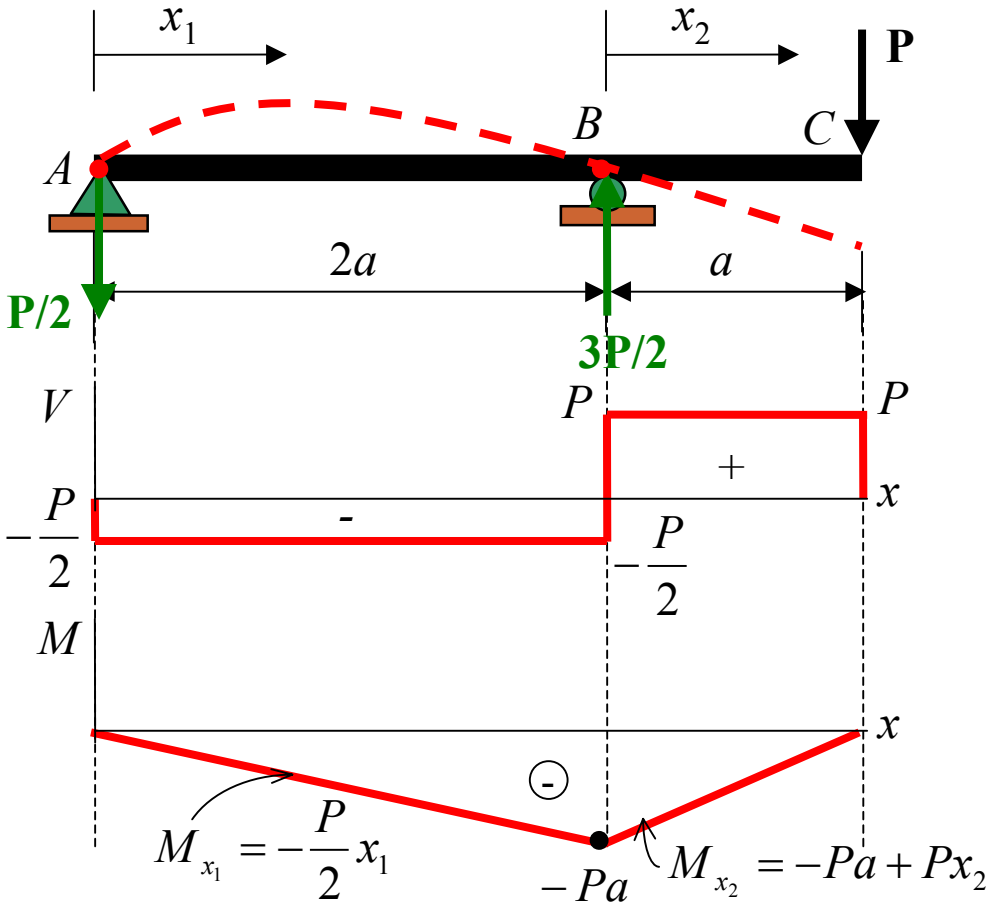
- $\upsilon_{x=L} = 0$

$$\upsilon = \frac{1}{EI}\left(-\frac{wx^4}{24} + \frac{wLx^3}{12}\right) + C_1 x \dashrightarrow C_1 = -\frac{wL^3}{24EI}$$

Example 8-1

The beam shown is subjected to a load **P** at its end. Determine the equation of the elastic curve and the slope and displacement at *C*. *EI* is constant.





- **Double Integration of x_1**

$$\frac{d^2\upsilon_1}{dx_1^2} = \frac{M_{x_1}}{EI} = -\frac{Px_1}{2EI}$$

$$\frac{d\upsilon_1}{dx_1} = \int \left(\frac{M_{x_1}}{EI}\right)dx \quad = -\frac{Px_1^2}{4EI} + C_1 \quad ---(1)$$

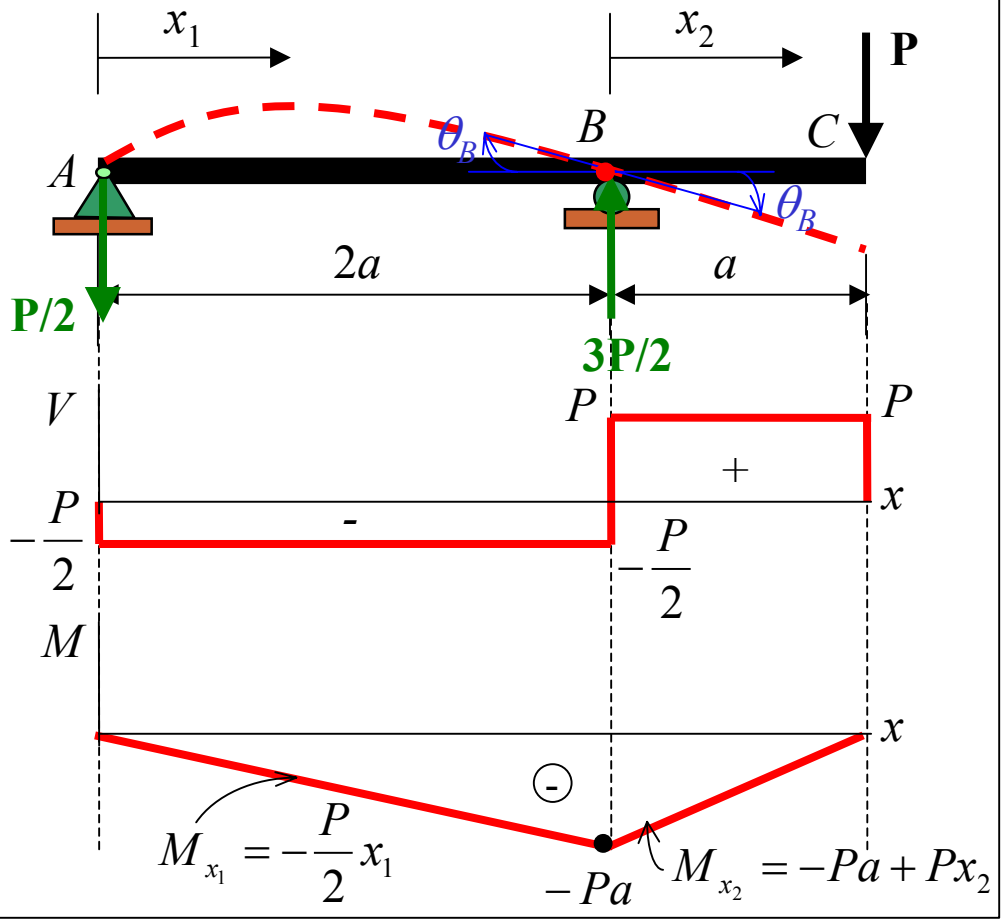
$$\upsilon_1 = \int \left(\int \frac{M_{x_1}}{EI}\right)dx)dx$$

$$\upsilon_1 = -\frac{Px_1^3}{12EI} + C_1x_1 + C_2 \quad ---(2)$$

Boundary conditions

$$\upsilon_1\Big|_{x_1=0} = 0; \qquad C_2 = 0$$

$$\upsilon_1\Big|_{x_1=2a} = 0; \qquad C_1 = \frac{Pa^2}{3EI}$$



- **Double Integration of x_2**

$$\frac{d^2\upsilon_2}{dx_2^2} = \frac{M_{x2}}{EI} = -\frac{Pa}{EI} + \frac{Px_2}{EI}$$

$$\frac{d\upsilon_2}{dx_2} = \int (\frac{M_{x_2}}{EI})dx = -\frac{Pax_2}{EI} + \frac{Px_2^2}{2EI} + C_3 \quad --(3)$$

$$\upsilon_2 = \int (\int \frac{M_{x_2}}{EI} dx)dx$$

$$\upsilon_2 = -\frac{Pax_2^2}{2EI} + \frac{Px_2^3}{6EI} + C_3x_2 + C_4 \quad ---(4)$$

Boundary conditions

$$\upsilon_2\Big|_{x_2=0} = 0; \qquad C_4 = 0$$

$$\frac{d\upsilon_1}{dx_1}\Big|_{x_1=2a} = \frac{d\upsilon_2}{dx_2}\Big|_{x_2=0}; \quad C_3 = -\frac{2Pa^2}{3EI}$$

- **Deflection and Slope at *C***

$$\upsilon_C = \upsilon_2\Big|_{x_2=a} = -\frac{Pa^3}{2EI} + \frac{Pa^3}{6EI} - \frac{2Pa^3}{3EI} = \frac{-Pa^3}{EI} \Leftarrow$$

$$\theta_C = \frac{d\upsilon_2}{dx_2}\Big|_{x_2=a} = -\frac{Pa^2}{EI} + \frac{Pa^2}{2EI} - \frac{2Pa^2}{3EI} = -\frac{7Pa^2}{6EI} \Leftarrow$$

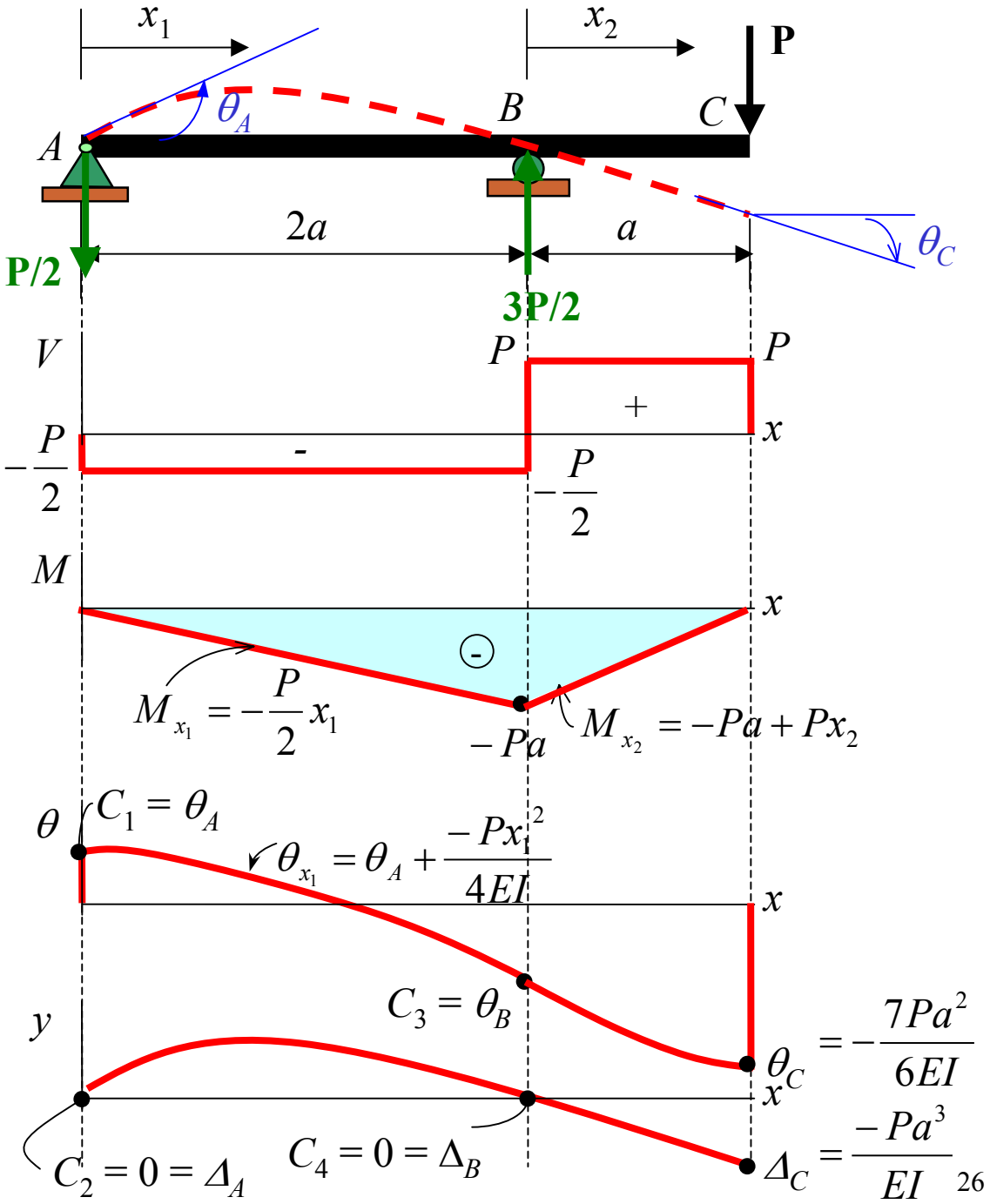
- **Conclusion**

$$\frac{d\upsilon_1}{dx_1} = -\frac{Px_1^{\,2}}{4EI} + C_1 \qquad C_1 = \frac{Pa^2}{3EI}$$

$$\upsilon_1 = -\frac{Px_1^{\,3}}{12EI} + C_1 x_1 + C_2 \qquad C_2 = 0$$

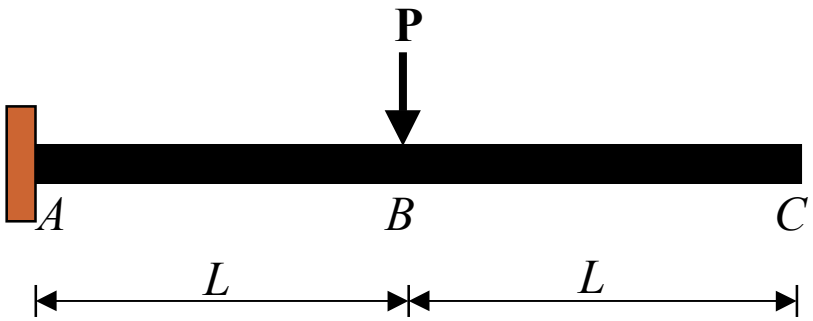
$$\frac{d\upsilon_2}{dx_2} = -\frac{Pax_2}{EI} + \frac{Px_2^{\,2}}{2EI} + C_3 \qquad C_3 = -\frac{2Pa^2}{3EI}$$

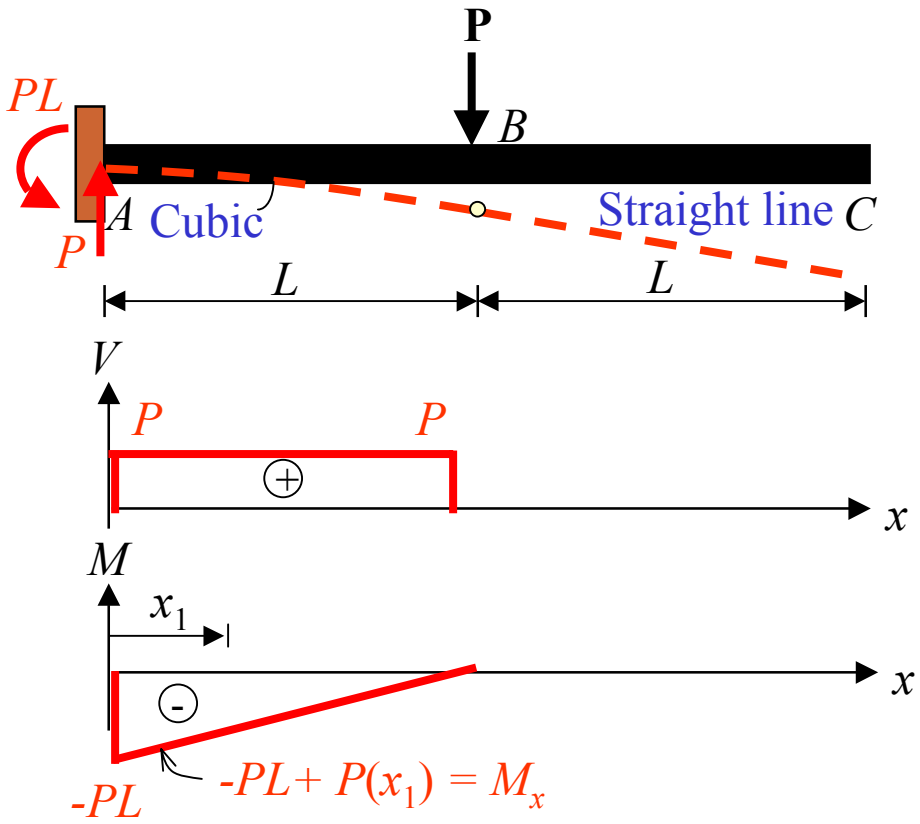
$$\upsilon_2 = -\frac{Pax_2^{\,2}}{2EI} + \frac{Px_2^{\,3}}{6EI} + C_3 x_2 + C_4 \qquad C_4 = 0$$

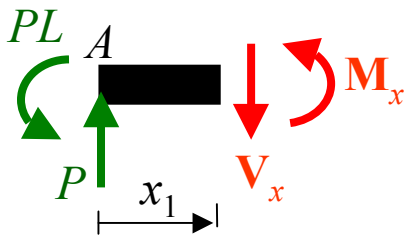


Example 8-2

The cantilevered beam shown is subjected to a vertical load **P** at its end. Determine the equation of the elastic curve. *EI* is constant.







- **Moment-curvature relationship**

$$\frac{d^2\upsilon}{dx^2} = \frac{M}{EI}$$

$$\frac{d^2\upsilon}{dx^2} = \frac{1}{EI}(-PL + Px)$$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}(-PLx + \frac{Px^2}{2}) + C_1$$

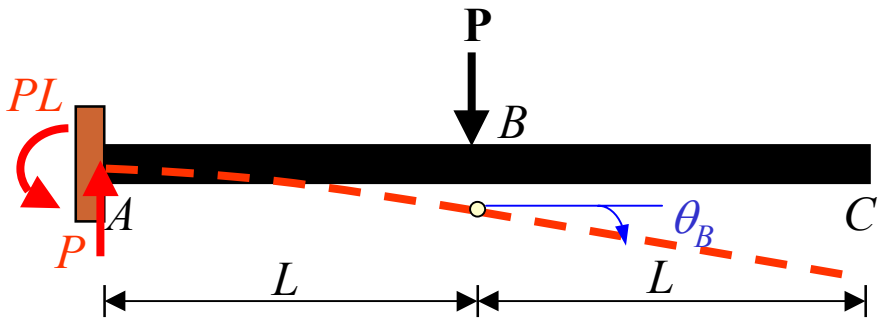
$$\upsilon = \frac{1}{EI}(-\frac{PLx^2}{2} + \frac{Px^3}{6}) + C_1x + C_2$$

Boundary condition :

- $(\frac{d\upsilon}{dx})_{x=0} = 0 \quad ---> C_1 = 0$
- $(\upsilon)_{x=0} = 0 \quad ---> C_2 = 0$

$$\frac{d\upsilon}{dx} = \frac{1}{EI}(-PLx + \frac{Px^2}{2}) \quad ---(1)$$

$$\upsilon = \frac{1}{EI}(-\frac{PLx^2}{2} + \frac{Px^3}{6}) \quad ---(2)$$



$$\frac{d\upsilon}{dx} = \frac{1}{EI}(-PLx + \frac{Px^2}{2}) \quad ---(1)$$

$$\upsilon = \frac{1}{EI}(-\frac{PLx^2}{2} + \frac{Px^3}{6}) \quad ---(2)$$

- **Elastic Curve Equation ; x : 0 - L**

$$(\upsilon)_{x=L} = \frac{1}{EI}(-\frac{PL^3}{2} + \frac{PL^3}{6}) = -\frac{PL^3}{3EI} \quad ,\downarrow \Delta_B$$

$$(\frac{d\upsilon}{dx})_{x=L} = \frac{1}{EI}(-PL^2 + \frac{PL^2}{2}) = -\frac{PL^2}{2EI} \quad \theta_B$$

- **Elastic Curve Equation ; x : L - $2L$**

$$\upsilon_{BC} = \upsilon_B + \theta_B(x - L)$$

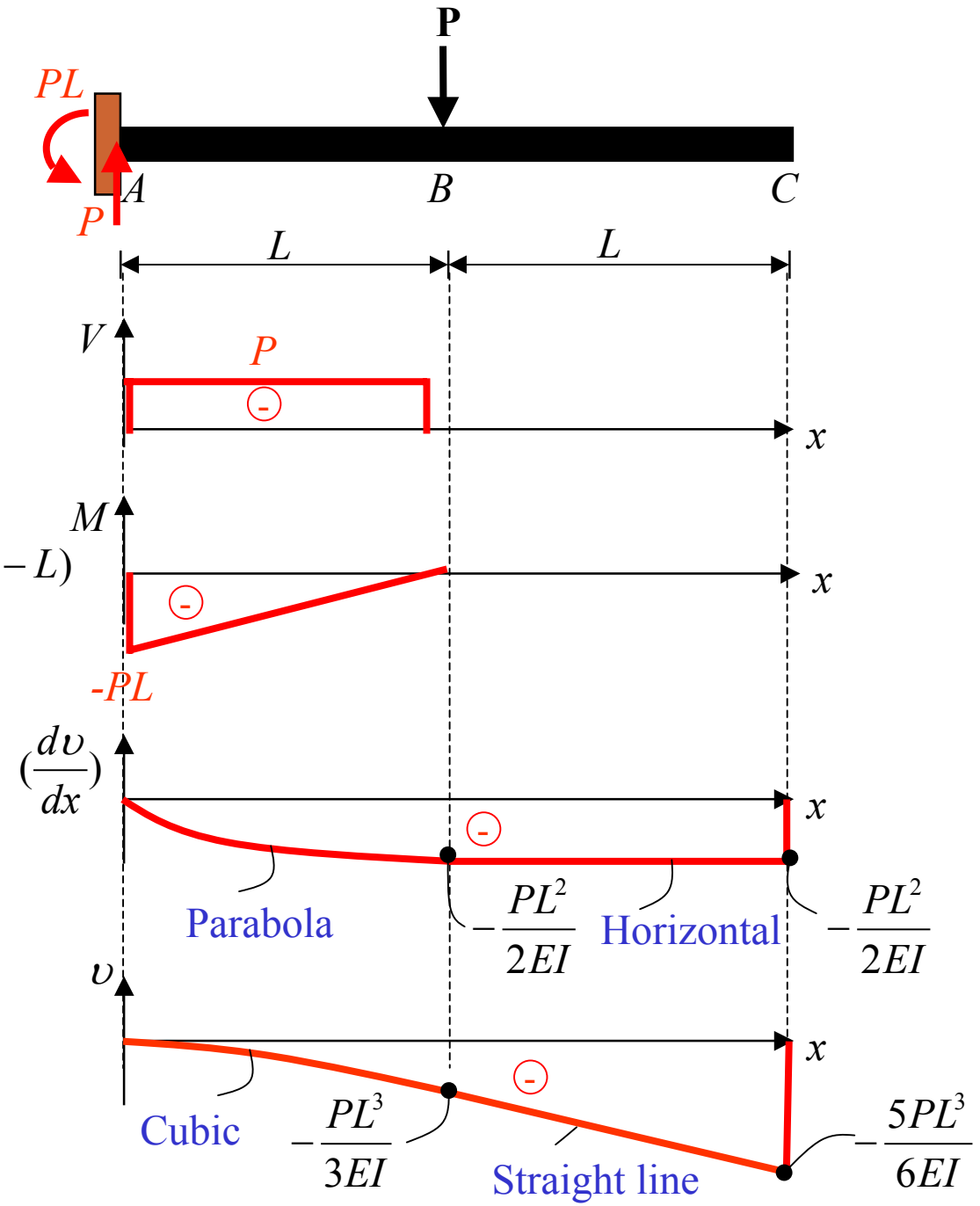
$$= (-\frac{PL^3}{3EI}) + (\frac{PL^2}{2EI})(x - L)$$

- **Conclusion**

$$(\frac{d\upsilon}{dx})_{x=L} = -\frac{PL^2}{2EI}$$

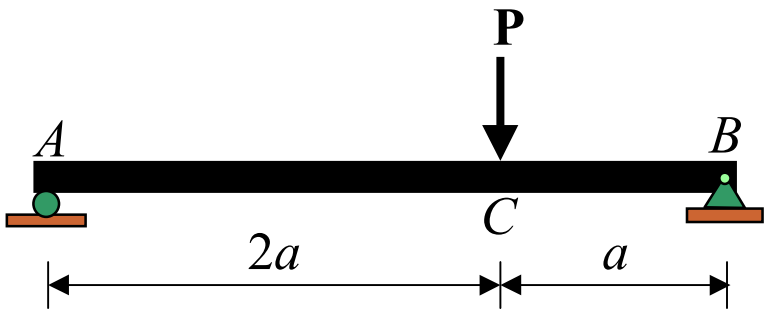
$$(\upsilon)_{x=L} = -\frac{PL^3}{3EI}$$

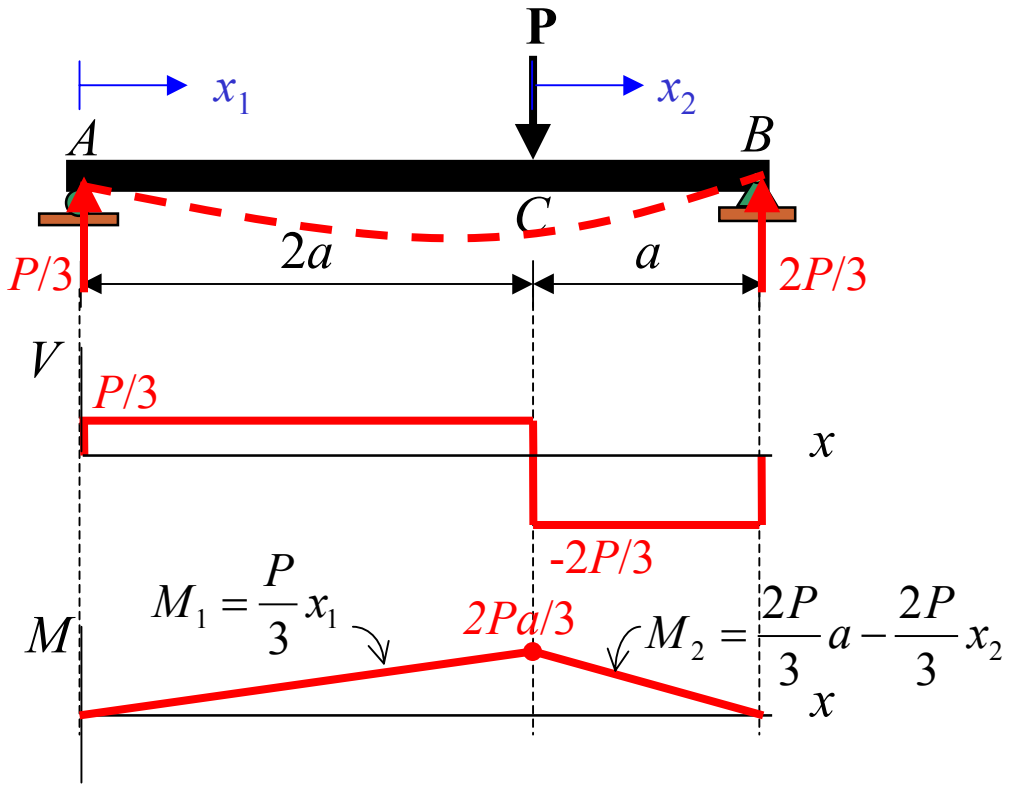
$$(\upsilon)_{x:L-2L} = (-\frac{PL^3}{3EI}) + (\frac{PL^2}{2EI})(x-L)$$



Example 8-3

The simply supported beam shown is subjected to the concentrated force *P*. Determine the maximum deflection of the beam. *EI* is constant.





- **Segment *AC***

$$\frac{d^2\upsilon_1}{dx_1^2} = \frac{M_1}{EI} = \frac{1}{EI}\left(\frac{P}{3}x_1\right)$$

$$\frac{d\upsilon_1}{dx_1} = \frac{1}{EI}\left(\frac{P}{6}x_1^2\right) + C_1 \quad ---(1)$$

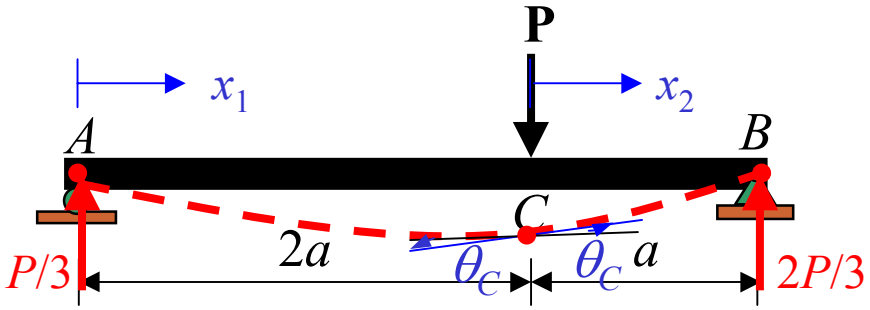
$$\upsilon_1 = \frac{1}{EI}\left(\frac{P}{18}x_1^3\right) + C_1x_1 + C_2 \quad ---(2)$$

- **Segment *AC***

$$\frac{d^2\upsilon_2}{dx_2^2} = \frac{M_2}{EI} = \frac{1}{EI}\left(\frac{2Pa}{3} - \frac{2P}{3}x_2\right)$$

$$\frac{d\upsilon_2}{dx_2} = \frac{1}{EI}\left(\frac{2Pa}{3}x_2 - \frac{2P}{6}x_2^2\right) + C_3 \quad ---(3)$$

$$\upsilon_2 = \frac{1}{EI}\left(\frac{2Pa}{6}x_2^2 - \frac{2P}{18}x_2^3\right) + C_3x_2 + C_4 \quad ---(4)$$



$$\frac{d\upsilon_1}{dx_1} = \frac{1}{EI}\left(\frac{P}{6}x_1^{\,2}\right) + C_1 \quad ---(1)$$

$$\upsilon_1 = \frac{1}{EI}\left(\frac{P}{18}x_1^{\,3}\right) + C_1x_1 + C_2 \quad ---(2)$$

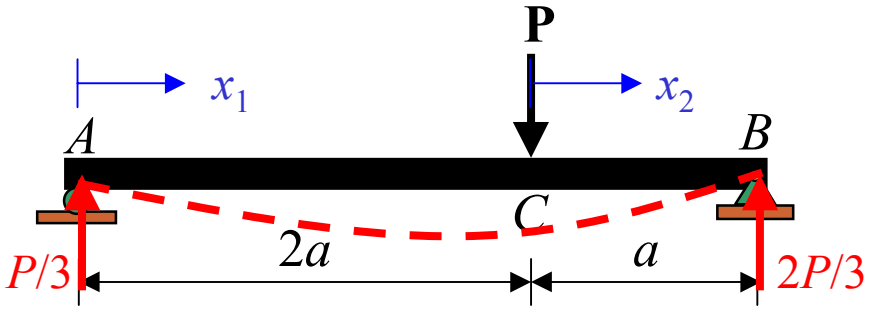
$$\frac{d\upsilon_2}{dx_2} = \frac{1}{EI}\left(\frac{2P}{3}ax_2 - \frac{2P}{6}x_2^{\,2}\right) + C_3 \quad ---(3)$$

$$\upsilon_2 = \frac{1}{EI}\left(\frac{2P}{6}ax_2^{\,2} - \frac{2P}{18}x_2^{\,3}\right) + C_3x_2 + C_4 \quad ---(4)$$

- **Boundary Conditions**
  - $(\upsilon_1)_{x_1=0} = 0$; $\quad 0 = 0 + 0 + C_2 \quad ---(1^*)$
  - $(\upsilon_2)_{x_2=a} = 0$; $\quad 0 = \frac{1}{EI}\left(\frac{2P}{6}a^3 - \frac{2P}{18}a^3\right) + C_3a + C_4 \quad ---(2^*)$
  - $(\upsilon_1)_{x_1=2a} = (\upsilon_2)_{x_2=0}$; $\quad \frac{1}{EI}\left(\frac{P}{18}(2a)^3\right) + C_1(2a) + 0 = 0 - 0 + 0 + C_4 \quad ---(3^*)$
  - $\left.\frac{d\upsilon_1}{dx_1}\right|_{x_1=2a} = \left.\frac{d\upsilon_2}{dx_2}\right|_{x_2=0}$; $\quad \frac{1}{EI}\left(\frac{P}{6}(2a)^2\right) + C_1 = 0 - 0 + C_3 \quad ---(4^*)$

Solving for (1*) to (4*) :

$$C_1 = -\frac{4Pa^2}{9EI} \qquad C_2 = 0 \qquad C_3 = \frac{2Pa^2}{9EI} \qquad C_4 = -\frac{4Pa^3}{9EI}$$



- **Elastic Curve Equation**

$$\frac{d\upsilon_1}{dx_1} = \frac{1}{EI}\left(\frac{P}{6}x_1^{\,2} - \frac{4Pa^2}{9}\right)$$

$$\upsilon_1 = \frac{1}{EI}\left(\frac{P}{18}x_1^{\,3} - \frac{4Pa^2}{9}x_1\right)$$

$$\frac{d\upsilon_2}{dx_2} = \frac{1}{EI}\left(\frac{2Pa}{3}x_2 - \frac{2P}{6}x_2^{\,2} + \frac{2Pa^2}{9}\right)$$

$$\upsilon_2 = \frac{1}{EI}\left(\frac{2P}{6}ax_2^{\,2} - \frac{2P}{18}x_2^{\,3} + \frac{2Pa^2}{9}x_2 - \frac{4Pa^3}{9}\right)$$

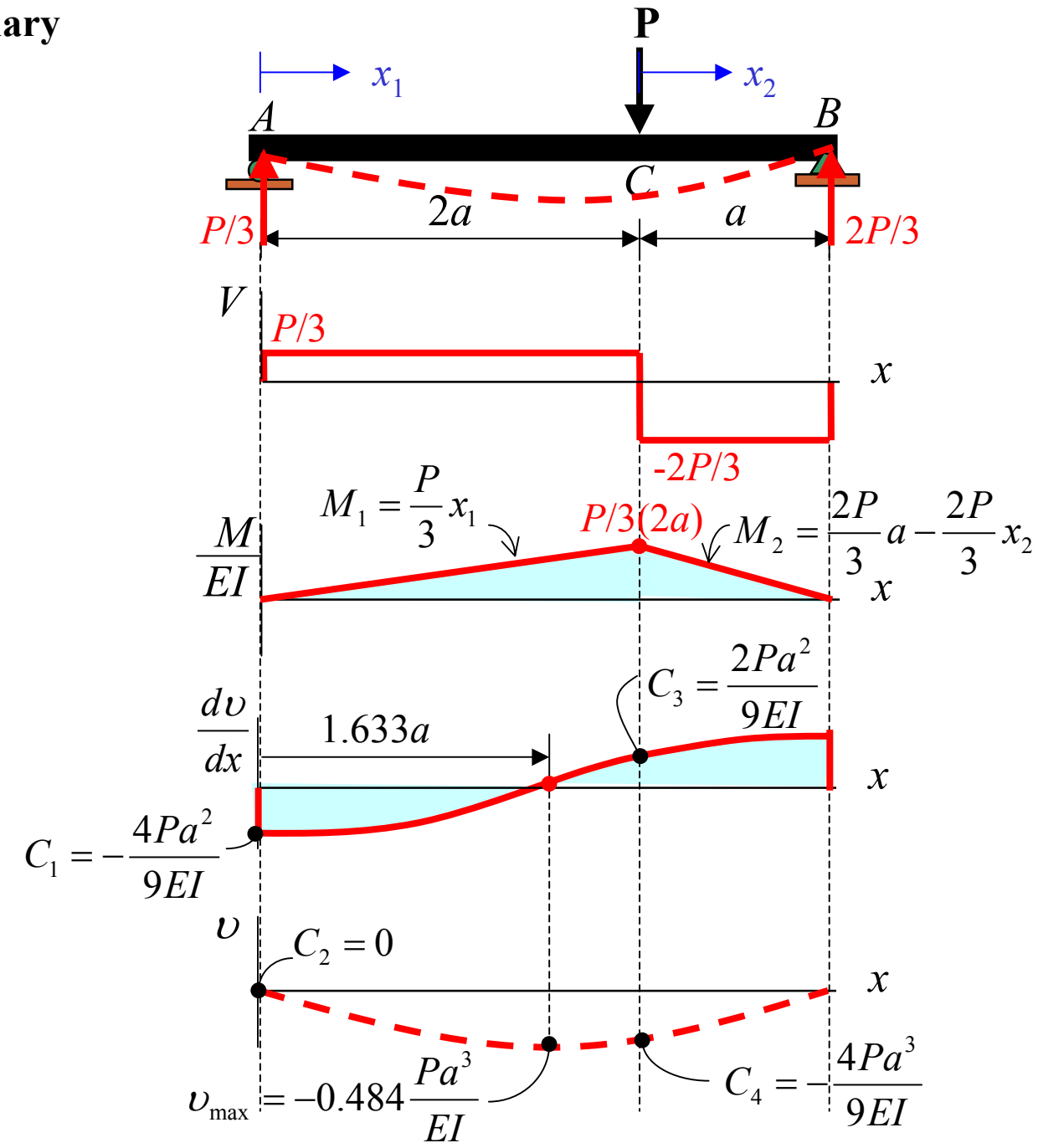
- **Maximum Deflection**

  - $\left.\dfrac{d\upsilon_1}{dx_1}\right|_{\upsilon_{\max}} = 0$; $\dfrac{d\upsilon_1}{dx_1}$ (= 0) $= \dfrac{1}{EI}\left(\dfrac{P}{6}x_1^{\,2} - \dfrac{4Pa^2}{9}\right)$

$$x_1 = 1.633a$$

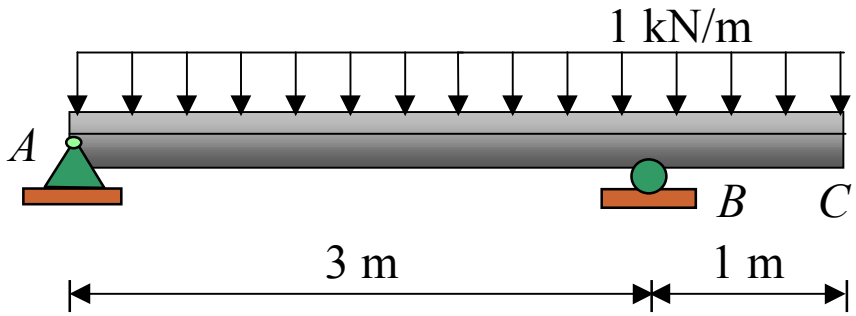
$$\upsilon_{\max}\big|_{x_1=1.633a} = \frac{1}{EI}\left[\frac{P}{18}(1.633a)^3 - \frac{4Pa^2}{9}(1.633a)\right] = -0.484\frac{Pa^3}{EI}$$

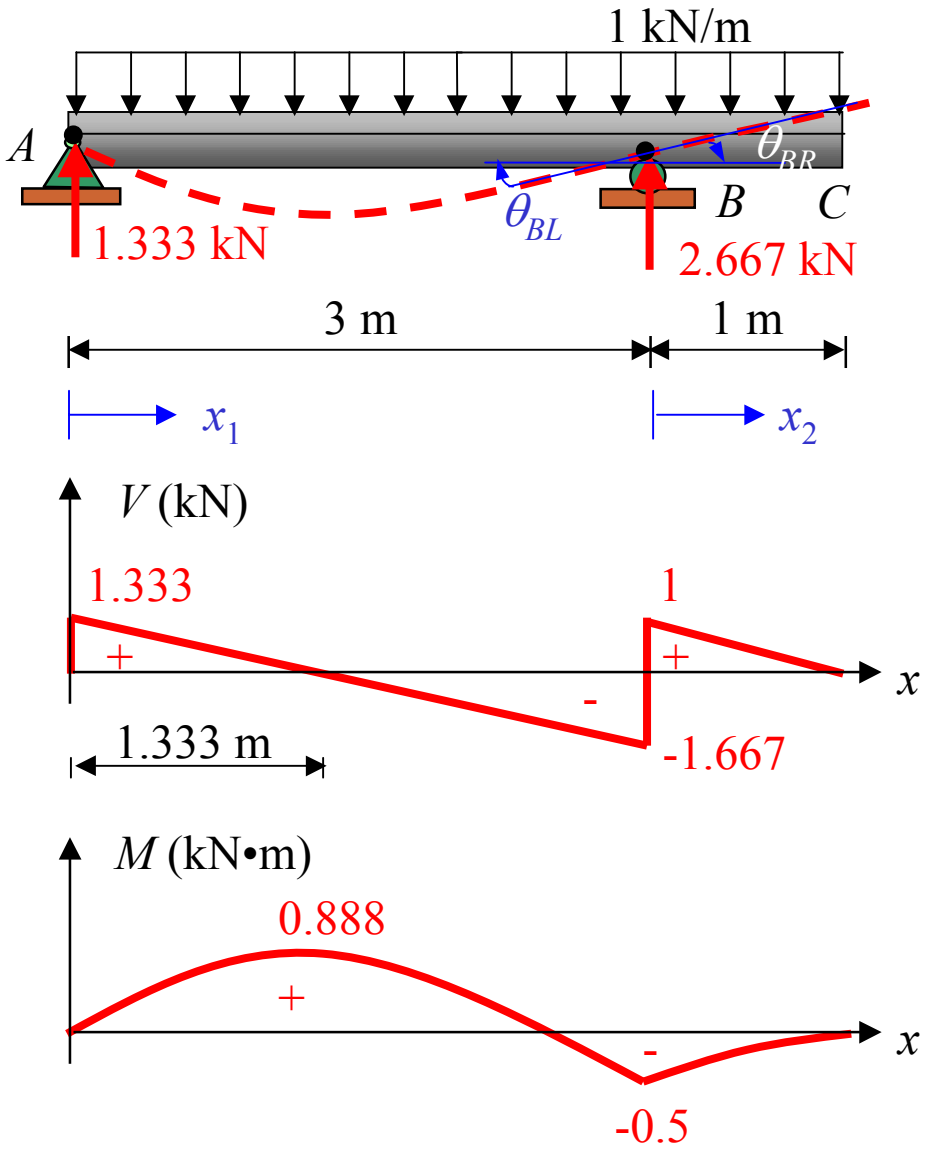
- **Summary**


P
x_1
x_2
A
B
C
2a
a
P/3
2P/3
V
P/3
x
-2P/3
$M_1 = \frac{P}{3}x_1$
P/3(2a)
$M_2 = \frac{2P}{3}a - \frac{2P}{3}x_2$
$\frac{M}{EI}$
x
$C_3 = \frac{2Pa^2}{9EI}$
$\frac{d\upsilon}{dx}$
1.633a
x
$C_1 = -\frac{4Pa^2}{9EI}$
υ
$C_2 = 0$
x
$\upsilon_{max} = -0.484\frac{Pa^3}{EI}$
$C_4 = -\frac{4Pa^3}{9EI}$

Example 8-4

From the beam shown determine the slope at supports , the maximum deflection of the beam and the deflection at *C*. Take $E = 10$ GPa and $I_{NA} = 15(10^6)$ mm^4.





- **Boundary condition**

1) **Deflection** at support is zero

$$(a)\ \upsilon_1|_{x_1=0} = 0$$

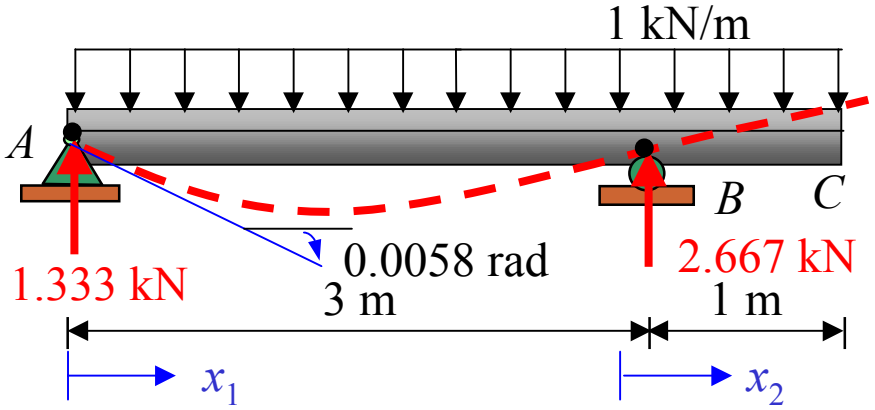
$$(b)\ \upsilon_1|_{x_1=3} = 0$$

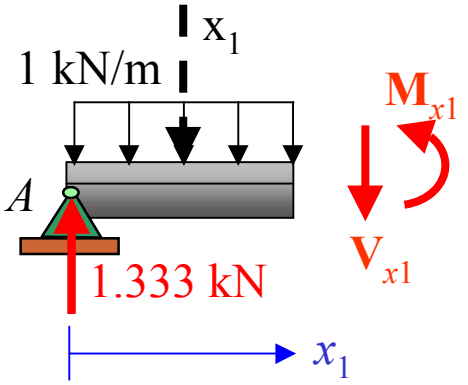
$$(c)\ \upsilon_2|_{x_2=0} = 0$$

2) The left of slope and the right of **slope at *B*** are equivalent

$$(d)\ \left.\frac{d\upsilon_1}{dx_1}\right|_{x_1=3} = \left.\frac{d\upsilon_2}{dx_2}\right|_{x_2=0}$$

Segment *AB* $E = 10$ GPa and $I_{NA} = 15(10^6)$ mm^4.





$+\circlearrowleft \Sigma M_{x1} = 0:$

$$M_{x_1} = \frac{4x_1}{3} - \frac{1x_1^{\,2}}{2}$$

- *Moment Curvature*

$$\frac{d^2\upsilon}{dx_1^{\,2}} = \frac{1}{EI}\left(\frac{4x_1}{3} - \frac{1x_1^{\,2}}{2}\right)$$

$$\frac{d\upsilon_1}{dx_1} = \frac{1}{EI}\left(\frac{2x_1^{\,2}}{3} - \frac{x_1^{\,3}}{6}\right) + C_1 \quad ---(1)$$

$$\upsilon_1 = \frac{1}{EI}\left(\frac{2x_1^{\,3}}{9} - \frac{x_1^{\,4}}{24}\right) + C_1 x_1 + C_2 \quad ---(2)$$

- *Boundary Condition*

$(a)\quad \upsilon_1\big|_{x_1=0} = 0 \quad : \quad C_2 = 0$

$(b)\quad \upsilon_1\big|_{x_1=3} = 0 \quad : \quad C_1 = -\frac{7}{8EI}$

- *The Slope at A*

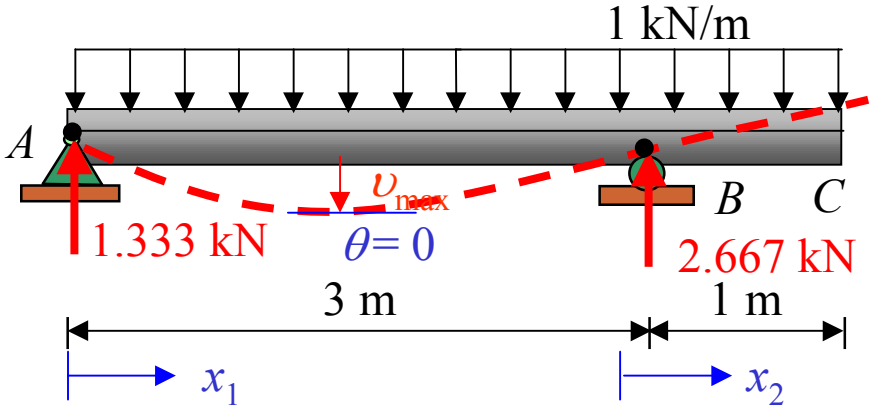
from (1) : *at* $x_1 = 0$,

$$\theta_A = \left(\frac{d\upsilon_1}{dx_1}\right)\Big|_{x_1=0} = \frac{1}{EI}\left(\frac{2\overset{0}{\cancel{x_1}^{\,2}}}{3} - \frac{\overset{0}{\cancel{x_1}^{\,3}}}{6}\right) + C_1$$

$$\theta_A = C_1 = -\frac{7}{8EI}$$

$$\theta_A = -\frac{7}{8(10\times10^6\ kPa)(15\times10^{-6}\ m^4)}$$

$$= -0.005833\ rad \Leftarrow$$



$E = 10$ GPa and $I_{NA} = 15(10^6)$ mm^4.

$$\frac{d\upsilon_1}{dx_1} = \frac{1}{EI}\left(\frac{2x_1^{\,2}}{3} - \frac{x_1^{\,3}}{6}\right) - \frac{7}{8EI} \quad ---(1)$$

$$\upsilon_1 = \frac{1}{EI}\left(\frac{2x_1^{\,3}}{9} - \frac{x_1^{\,4}}{24}\right) - \frac{7}{8EI}x_1 \quad ---(2)$$

- *The Maximum Displacement*

υ_{max} *when* $\frac{d\upsilon_1}{dx_1} = 0$; *substute* $\frac{d\upsilon_1}{dx_1} = 0$ *in* (1)

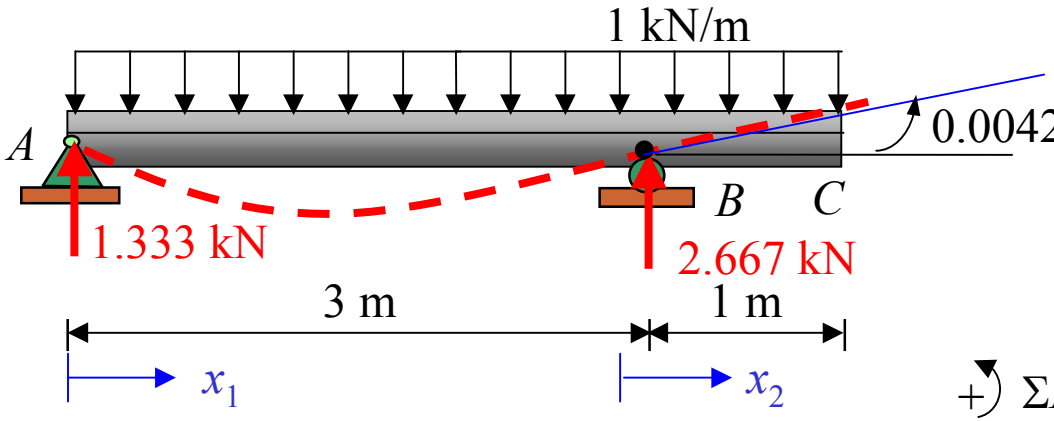
$$0 = \frac{1}{(10\times10^{+6})(15\times10^{-6})}\left(\frac{2x_1^{\,2}}{3} - \frac{x_1^{\,3}}{6}\right) - \frac{7}{8[(10\times10^{+6})(15\times10^{-6})]}$$

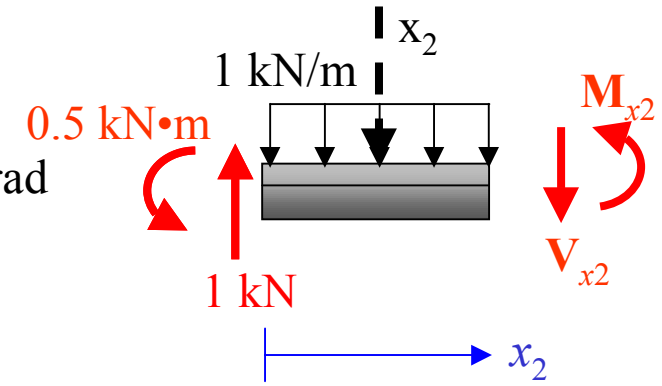
$$x_1 = 1.427 \text{ m}$$

Substitute $x_1 = 1.427$ m, $E = 10\text{x}10^6$ kPa and $I = 15(10^{-6})$ m^4 in (2)

$$\upsilon_{max} = -5.171 \text{ mm} \quad \Leftarrow$$

Segment *BC*





$$+\circlearrowleft \Sigma M_{x2} = 0:\ M_{x_2} = \frac{-1}{2} + x_2 - \frac{x_2^{\,2}}{2}$$

- *Moment Curvature*

$$\frac{d^2\upsilon_2}{dx_2^{\,2}} = \frac{1}{EI}\left(\frac{-1}{2} + x_2 - \frac{x_2^{\,2}}{2}\right)$$

$$\frac{d\upsilon_2}{dx_2} = \frac{1}{EI}\left(\frac{-x_2}{2} + \frac{x_2^{\,2}}{2} - \frac{x_2^{\,3}}{6}\right) + C_3$$

$$\upsilon_2 = \frac{1}{EI}\left(\frac{-x_2^{\,2}}{4} + \frac{x_2^{\,3}}{6} - \frac{x_2^{\,4}}{24}\right) + C_3 x_2 + C_4$$

- *Boundary Condition*

$$(c)\quad \upsilon_2\big|_{x_2=0} = 0 \ :\ C_4 = 0$$

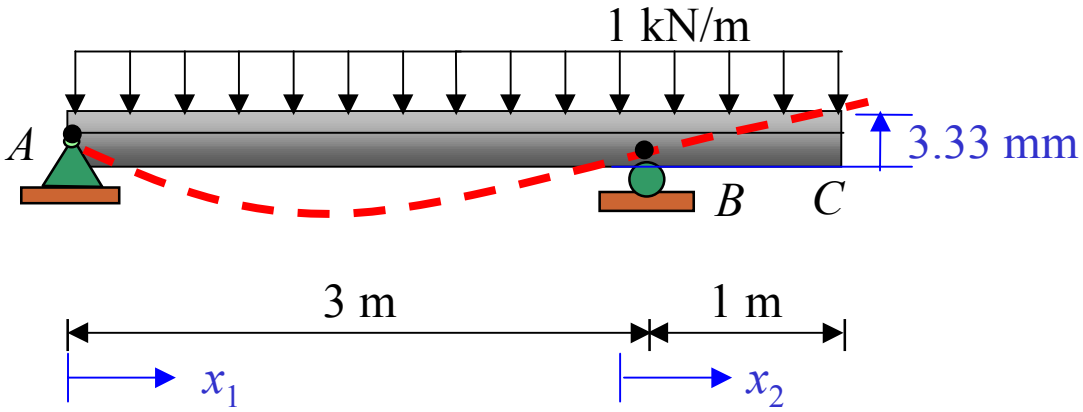
$$\left.\frac{d\upsilon_1}{dx_1}\right|_{x_1=3} = \left.\frac{d\upsilon_2}{dx_2}\right|_{x_2=0}$$

$$\frac{1}{EI}\left(\frac{2x_1^{\,2}}{3} - \frac{x_1^{\,3}}{6}\right) - \frac{7}{8EI} = \frac{1}{EI}\left(\frac{-x_2}{2} + \frac{x_2^{\,2}}{2} - \frac{x_2^{\,3}}{6}\right) + C_3$$

(3, 3, 0, 0, 0)

$$C_3 = \frac{5}{8EI}$$

$$\theta_B = \left(\frac{d\upsilon_2}{dx_2}\right)_{x_2=0} = C_3 = \frac{5}{8(10\times 15)} = 0.004167\ rad \Leftarrow$$



$$\upsilon_2 = \frac{1}{EI}\left(\frac{-x_2^{\,2}}{4} + \frac{x_2^{\,3}}{6} - \frac{x_2^{\,4}}{24}\right) + \frac{5}{8EI}x_2$$

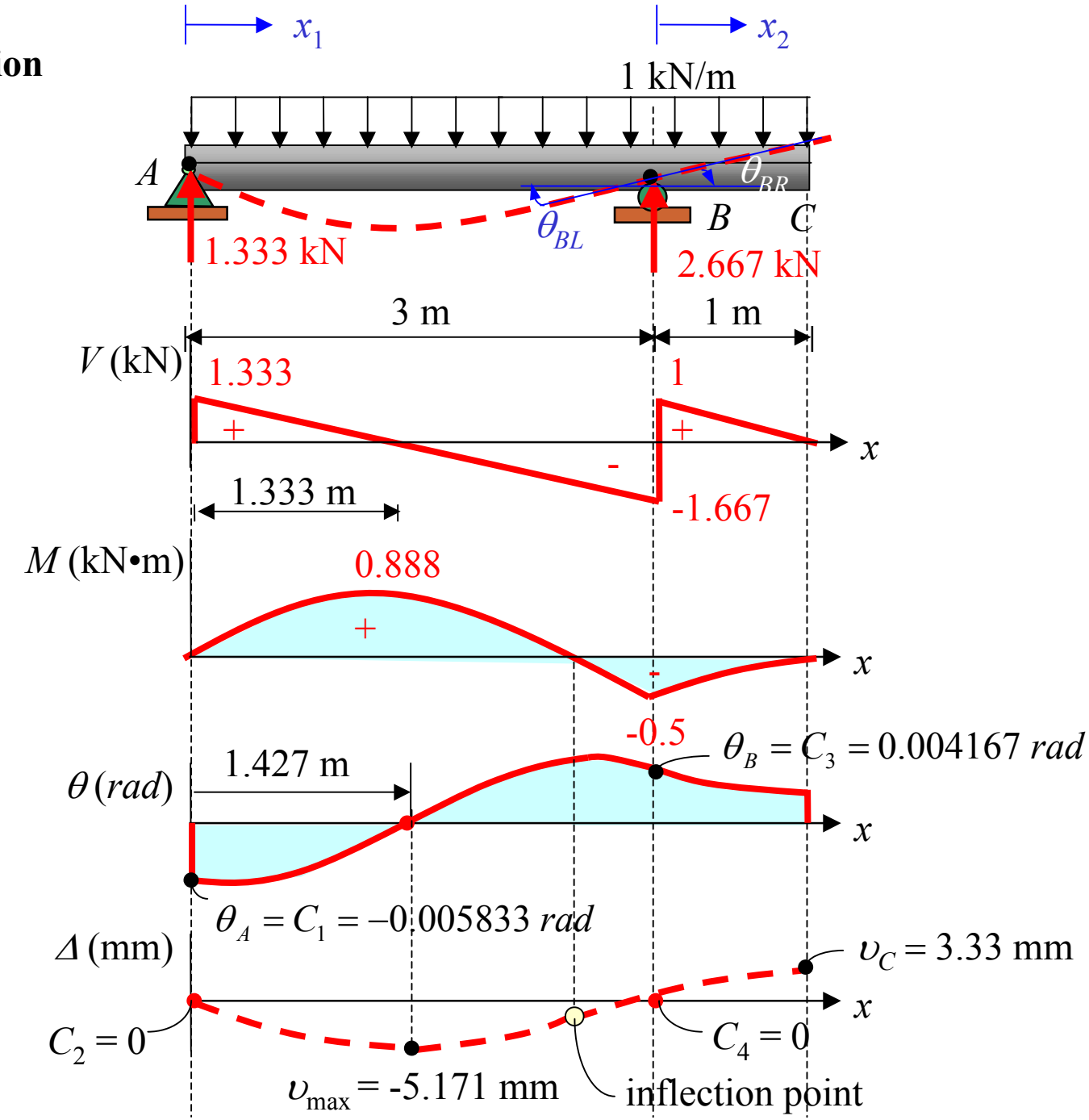
- *Displacement at C*

$$\upsilon_C = \upsilon_2\Big|_{x_2=1} = \frac{1}{EI}\left(\frac{-\overset{1}{x_2^{\,2}}}{4} + \frac{\overset{1}{x_2^{\,3}}}{6} - \frac{\overset{1}{x_2^{\,4}}}{24}\right) + \frac{5}{8EI}\overset{1}{x_2}$$

$$= -\frac{3}{24EI} + \frac{5}{8EI} = \frac{12}{24EI}$$

$$\upsilon_C = \frac{12}{24(10\times 15)} = 3.333\ mm\ , \quad \uparrow$$

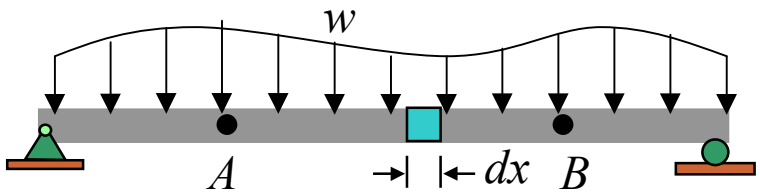
• **Conclusion**


x_1
x_2
1 kN/m
A
B
C
θ_{BR}
θ_{BL}
1.333 kN
2.667 kN
3 m
1 m
V (kN)
1.333
1
+
+
-
x
-1.667
1.333 m
M (kN•m)
0.888
+
-
x
-0.5
$\theta_B = C_3 = 0.004167\ rad$
1.427 m
$\theta\,(rad)$
x
$\theta_A = C_1 = -0.005833\ rad$
Δ (mm)
$\upsilon_C = 3.33$ mm
x
$C_2 = 0$
$C_4 = 0$
$\upsilon_{max} = -5.171$ mm
inflection point

Moment - Area Theorems

- **The angle between the tangents**

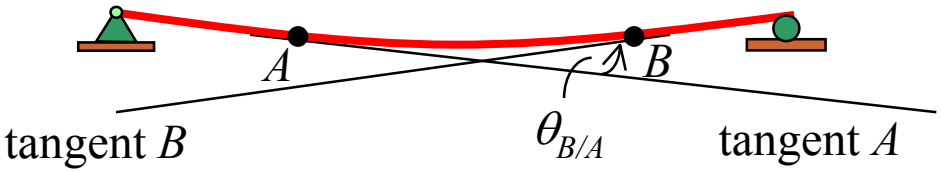
Theorem 1: $\theta_{B/A}$ *= The area under the M/EI diagram between B and A*



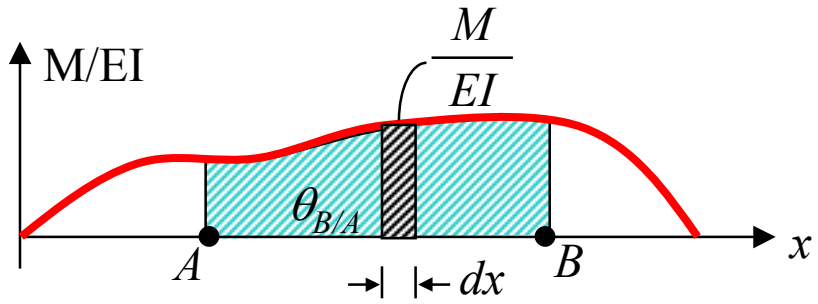
$$M = EI\frac{d^2\upsilon}{dx^2}$$

θ If *slope* is *small*

$$M = EI\frac{d}{dx}\left(\frac{d\upsilon}{dx}\right) = EI\frac{d\theta}{dx}$$



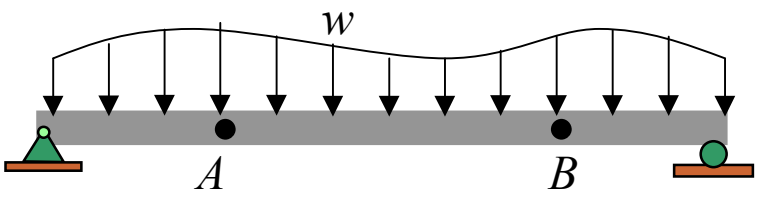
$$d\theta = \frac{M}{EI}dx$$

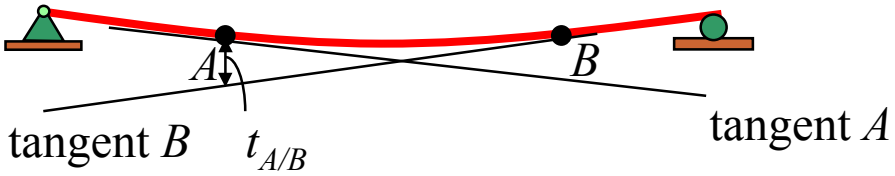


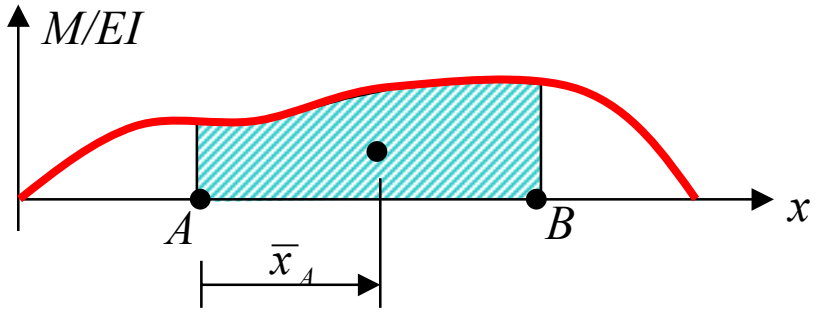
$$\theta_{B/A} = \int_A^B \frac{M}{EI}dx$$

- **The vertical deviation**

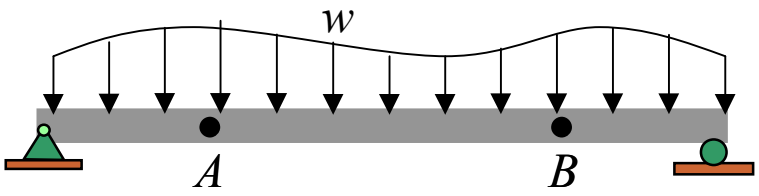
Theorem 2: *$t_{A/B}$ = The moment area under the M/EI diagram between B and A, this moment is computed about A.*

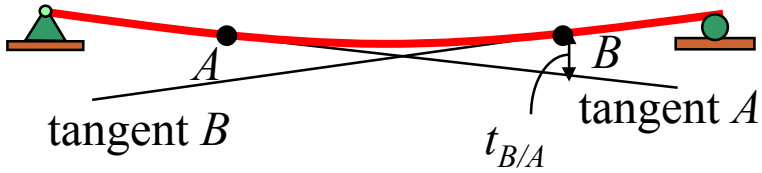


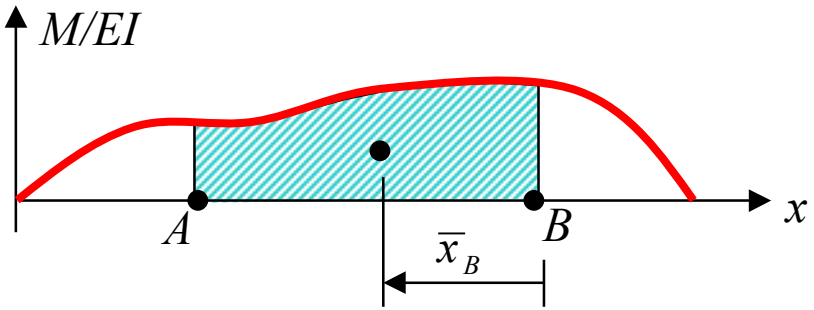




$$t_{A/B} = \bar{x}_A \int_A^B \frac{M}{EI} dx$$

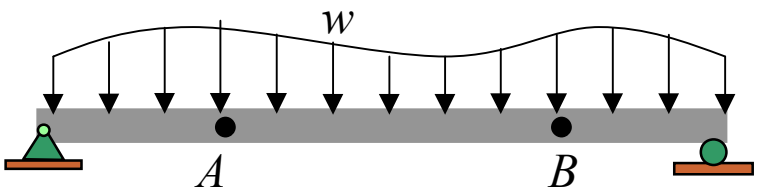


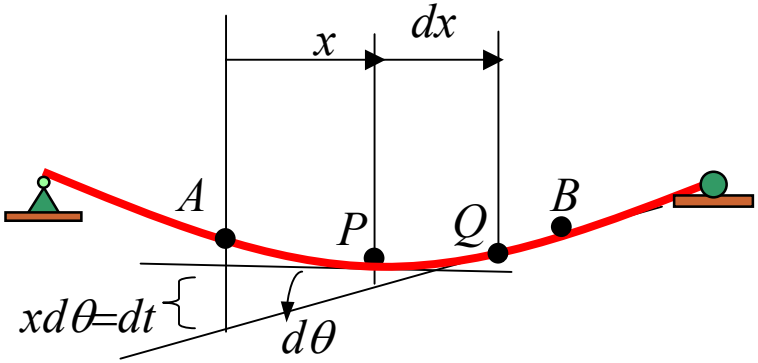


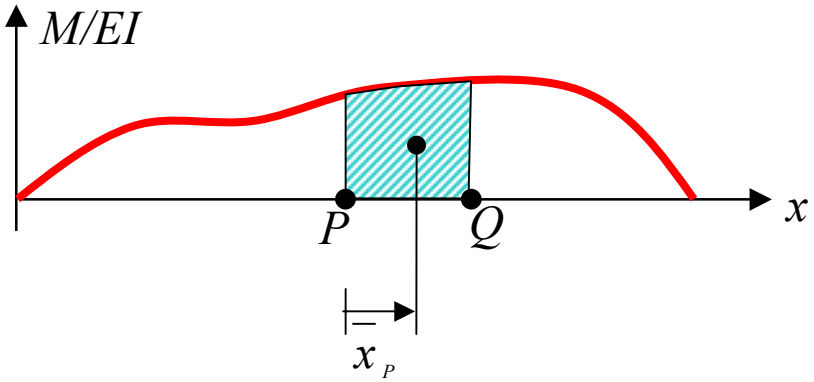


$$t_{B/A} = \bar{x}_B \int_A^B \frac{M}{EI} dx$$

General form







$$dt = xd\theta$$

$$dt = x\frac{M}{EI}dx$$

$$t = \int_x^{x+dx} x\frac{M}{EI}dx$$

$$t_{P/Q} = \int_P^Q x\frac{M}{EI}dx$$

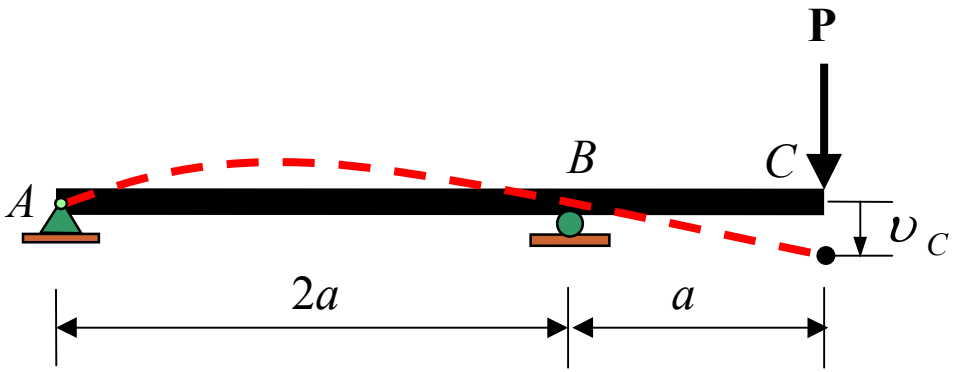
$$= \bar{x}_P \int \frac{M}{EI}dx$$

$$t_{A/x} = \int_A^x \frac{M}{EI}dx$$

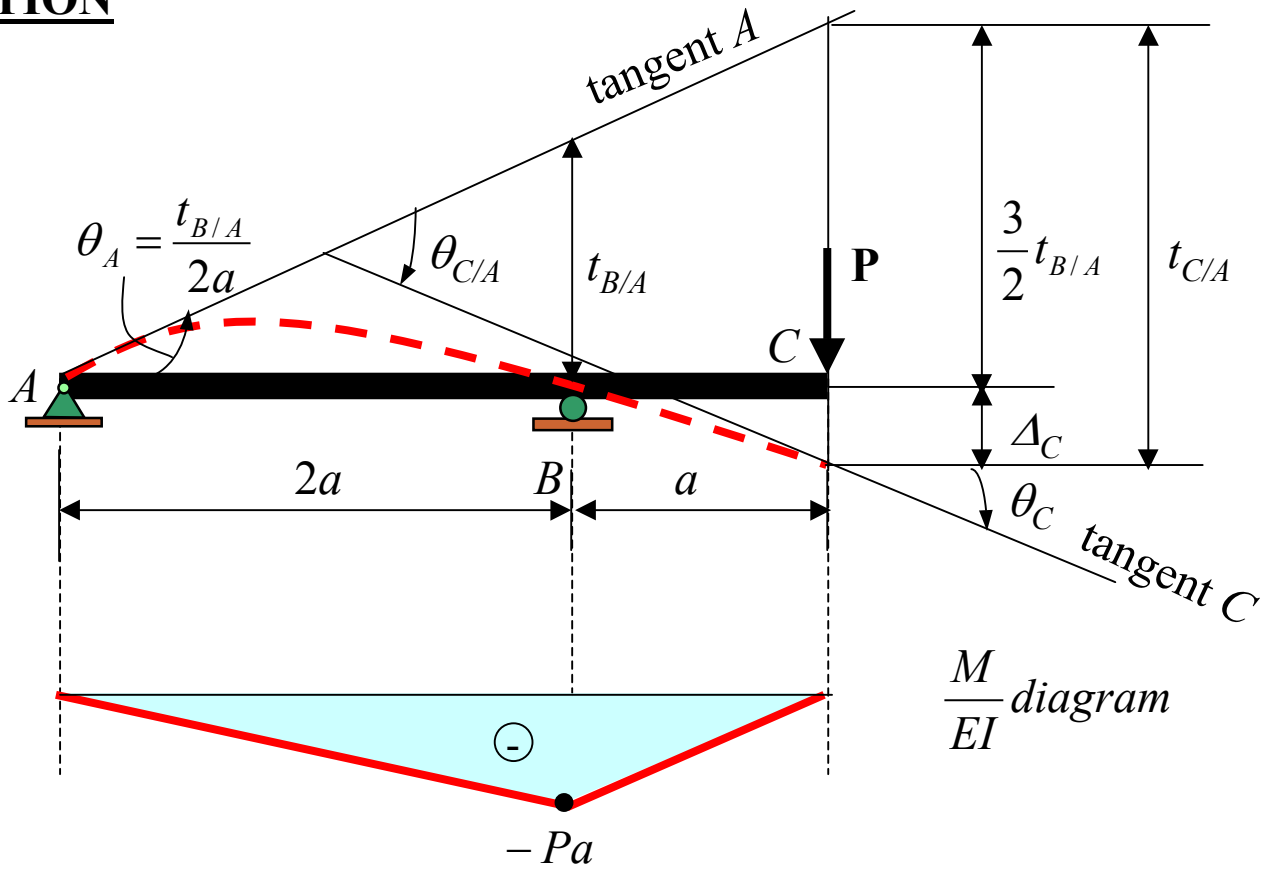
$$= \bar{x}_A \int \frac{M}{EI}dx$$

Example 8-5

The beam shown is subjected to a load **P** at its end. Determine the slope and displacement at *C*. *EI* is constant.



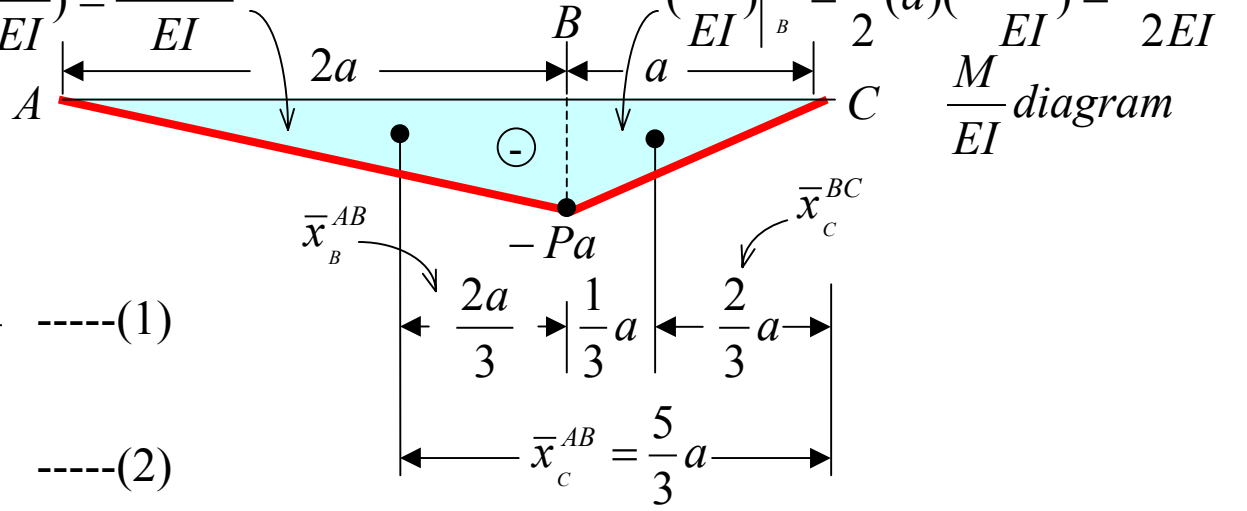
SOLUTION



$$\theta_C = \theta_{C/A} - \theta_A = \theta_{C/A} - \frac{t_{B/A}}{2a} \qquad \text{----------(1)}$$

$$\Delta_C = t_{C/A} - \frac{3}{2}t_{B/A} \qquad \text{----------(2)}$$

$$\left(\frac{M}{EI}\right)\Big|_A^B = \frac{1}{2}(2a)\left(-\frac{Pa}{EI}\right) = \frac{-Pa^2}{EI} \qquad \left(\frac{M}{EI}\right)\Big|_B^C = \frac{1}{2}(a)\left(-\frac{Pa}{EI}\right) = \frac{-Pa^2}{2EI}$$



$$\theta_C = \theta_{C/A} - \theta_A = \theta_{C/A} - \frac{t_{B/A}}{2a} \quad \text{-----(1)}$$

$$\Delta_C = t_{C/A} - \frac{3}{2}t_{B/A} \quad \text{-----(2)}$$

$$\theta_{C/A} = \left(\frac{M}{EI}\right)\Big|_A^C = \left(\frac{M}{EI}\right)\Big|_A^B + \left(\frac{M}{EI}\right)\Big|_B^C = \frac{-Pa^2}{EI} + \frac{-Pa^2}{2EI} = -\frac{3Pa^2}{2EI}$$

$$t_{B/A} = \left(\frac{M}{EI}\right)\Big|_A^B \bullet \bar{x}_B^{AB} = \left(-\frac{Pa^2}{EI}\right)\left(\frac{2a}{3}\right) = -\frac{2Pa^3}{3EI}$$

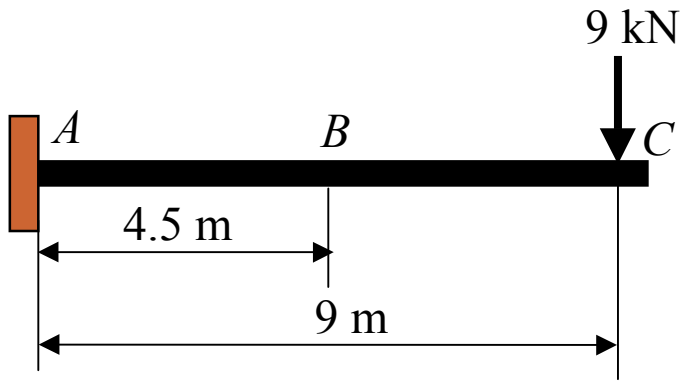
- $\theta_C = -\frac{3Pa^2}{2EI} - \frac{1}{2a}\left(-\frac{2Pa^3}{3EI}\right) = -\frac{7Pa^2}{6EI}$

$$t_{C/A} = \left(\frac{M}{EI}\right)\Big|_A^B \bullet \bar{x}_C^{AB} + \left(\frac{M}{EI}\right)\Big|_B^C \bullet \bar{x}_C^{BC} = \left(-\frac{Pa^2}{EI}\right)\left(\frac{5a}{3}\right) + \left(-\frac{Pa^2}{2EI}\right)\left(\frac{2a}{3}\right) = -\frac{2Pa^3}{EI}$$

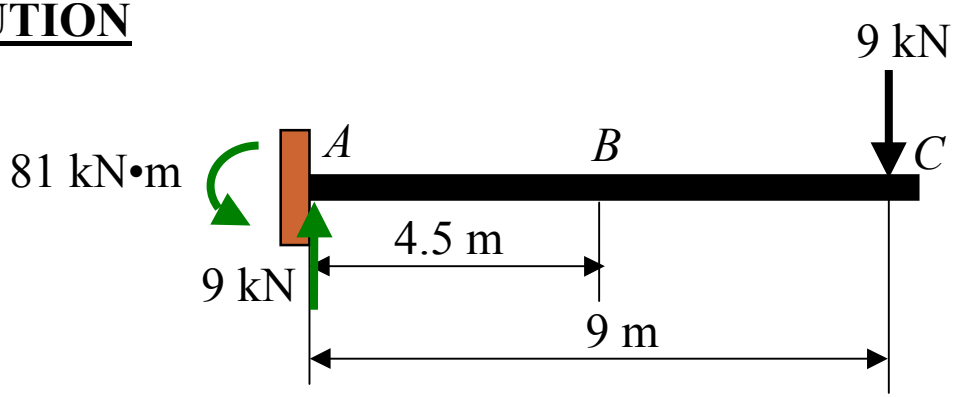
- $\Delta_C = -\frac{2Pa^3}{EI} - \frac{3}{2}\left(-\frac{2Pa^2}{3EI}\right) = -\frac{Pa^3}{EI}, \downarrow$

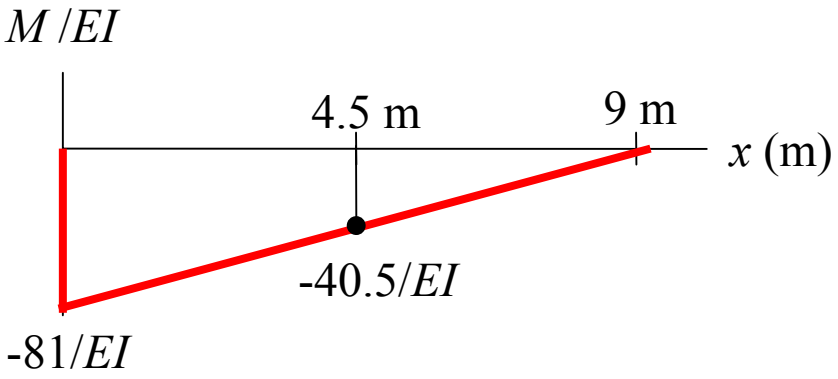
Example 8-6

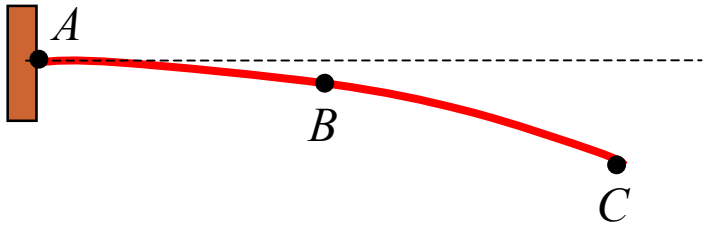
Determine the slope and deflection at points B and C of the beam shown in the figure. Take $E = 200$ GPa and $I = 250(10^6)$ mm^4

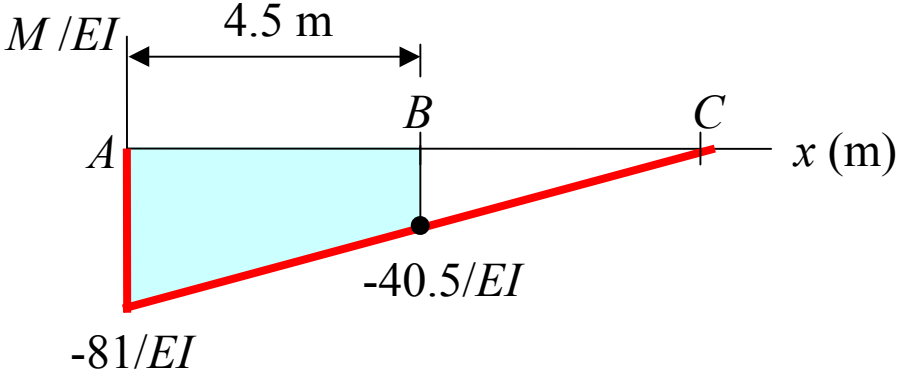


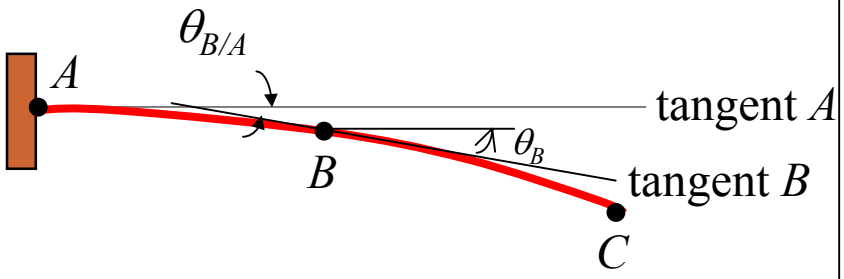
SOLUTION











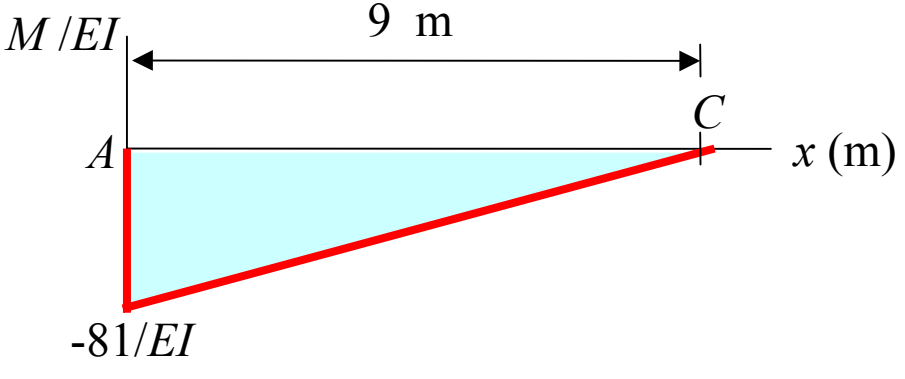
$\theta_B = \theta_{B/A}$ = the *area* under the *M/EI* diagram between point *A* and *B*

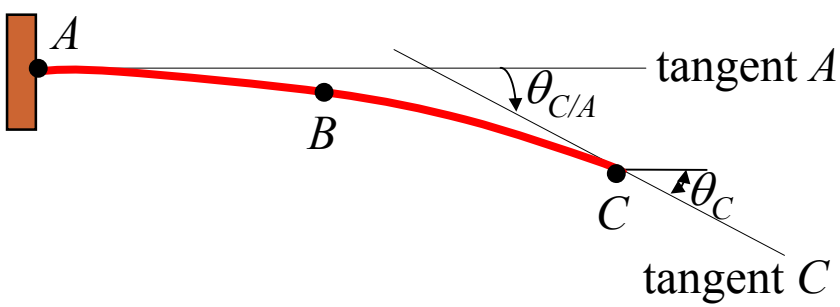
$$= -\frac{1}{2}\left(\frac{40.5}{EI} + \frac{81}{EI}\right)(4.5)$$

$$= -\frac{273.38}{EI}\, kN \bullet m^2$$

$$= -\frac{273.38\ kN \bullet m^2}{(200\times10^6\ \frac{kN}{m^2})(250\times10^{-6}\ m^4)}$$

θ_B = -0.00547 rad





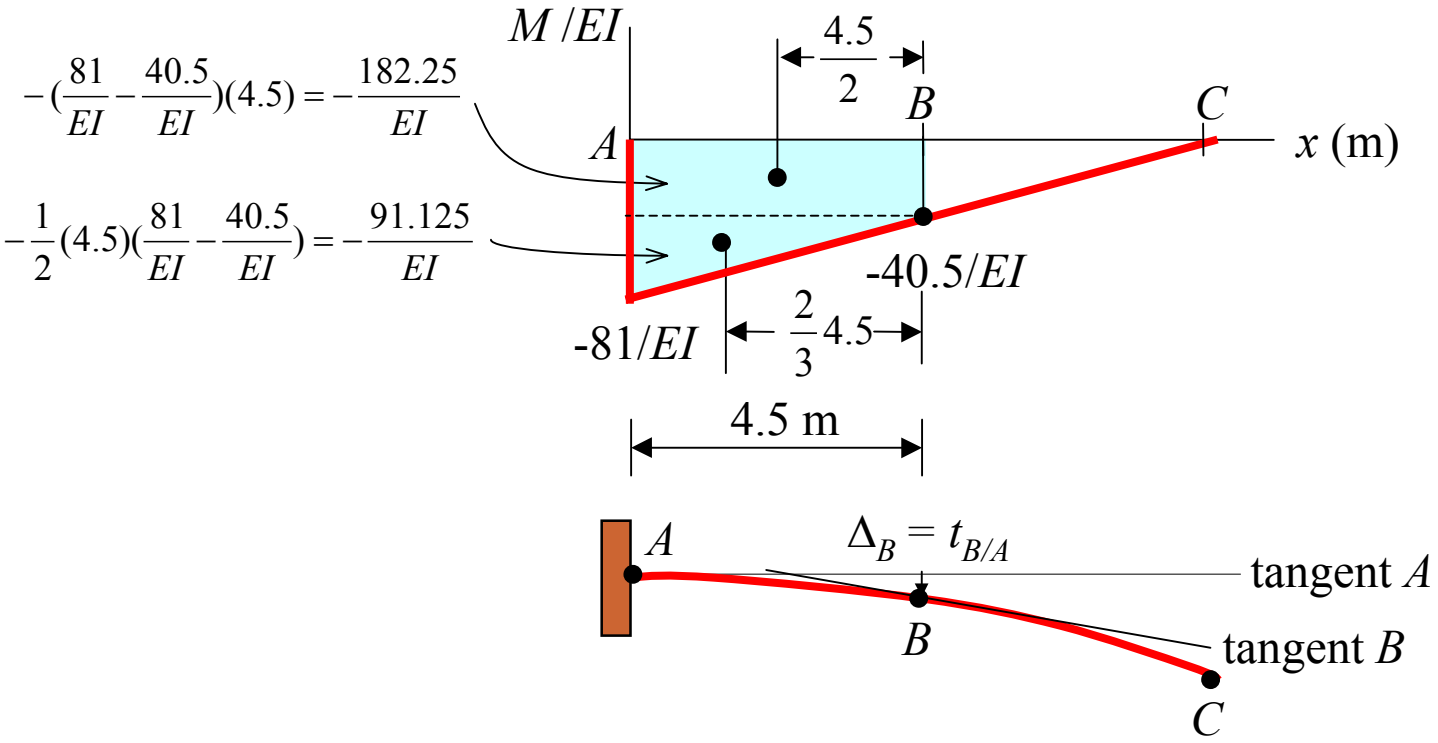
$\theta_C = \theta_{C/A}$ = the *area* under the *M/EI* diagram between point *A* and *C*

$$= \frac{1}{2}\left(-\frac{81}{EI}\right)(9)$$

$$= -\frac{364.50}{EI}\, kN \bullet m^2$$

$$= -\frac{364.5\ kN \bullet m^2}{(200\times10^6\ \frac{kN}{m^2})(250\times10^{-6}\ m^4)}$$

θ_C = -0.00729 rad

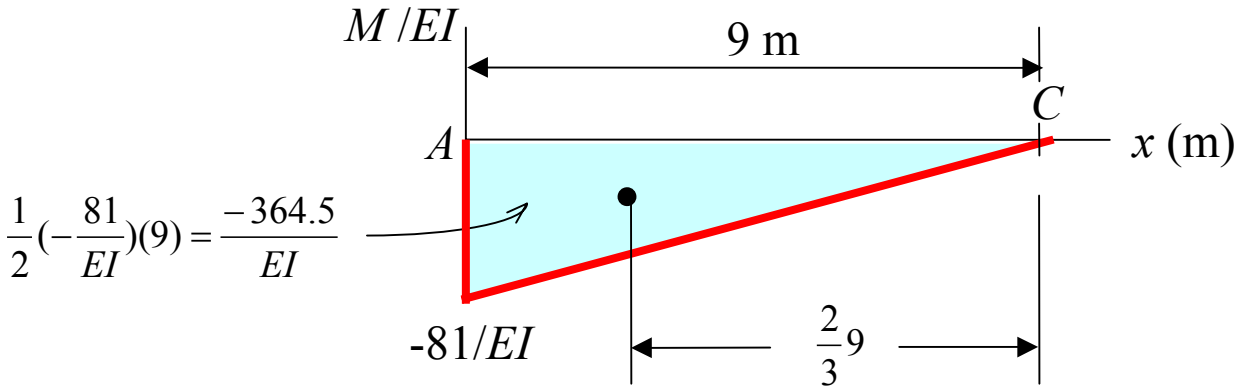


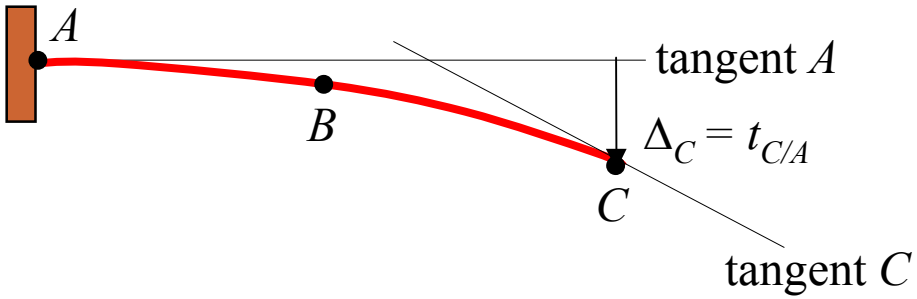
$\Delta_B = t_{B/A}$ = the *moment* of the *M/EI* diagram between *A* and *B* computed about point *B*.

$$= (-\frac{182.25}{EI}\,kN \bullet m^2)(\frac{4.5}{2}\,m) + (-\frac{91.125}{EI}\,kN \bullet m^2)(\frac{2}{3}4.5\,m)$$

$$= -\frac{683.44\,kN \bullet m^3}{EI} \qquad = -\frac{683.44\,kN \bullet m^3}{(200\times10^6\,\frac{kN}{m^2})(250\times10^{-6}\,m^4)}$$

Δ_B = -0.01367 m = -13.67 mm ↓





$\Delta_C = t_{C/A}$ = the *moment* of the *M/EI* diagram between *A* and *C* computed about point *C*.

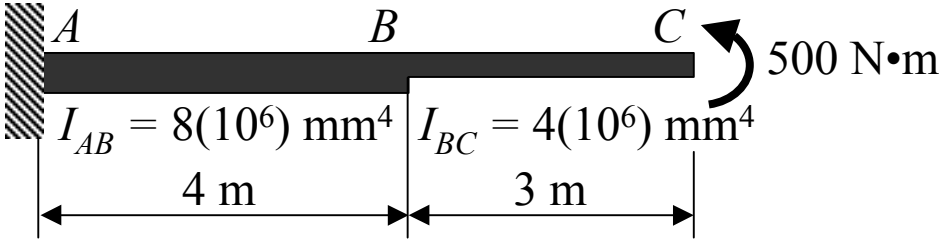
$$= (-\frac{364.5}{EI} kN \bullet m^2)(\frac{2}{3} 9\ m)$$

$$= -\frac{2187\ kN \bullet m^3}{EI} \qquad = -\frac{2187\ kN \bullet m^3}{(200 \times 10^6 \frac{kN}{m^2})(250 \times 10^{-6} m^4)}$$

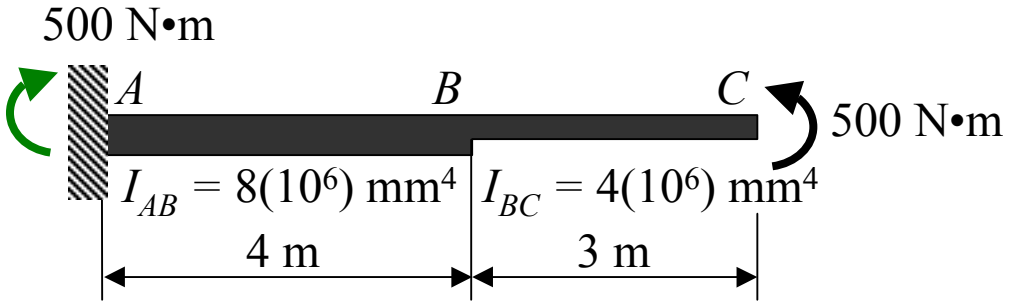
Δ_C = -0.04374 m = -43.74 mm ↓

Example 8-7

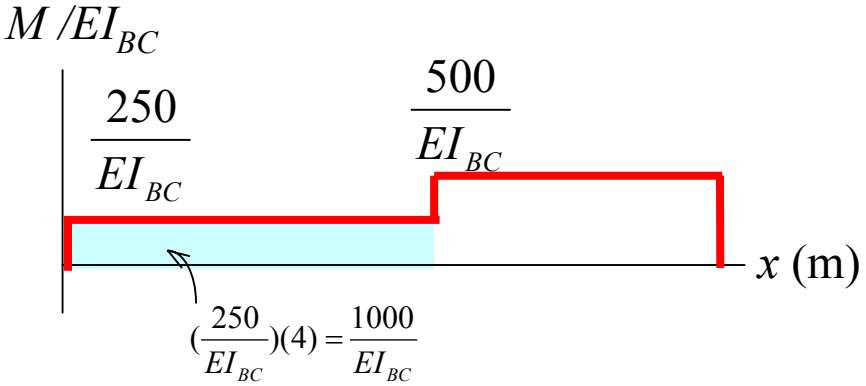
Determine the slope and deflection at points *B* and *C* of the beam shown in the figure. Values for the moment of inertia of each segment are indicated in the figure.
Take $E = 200$ GPa.

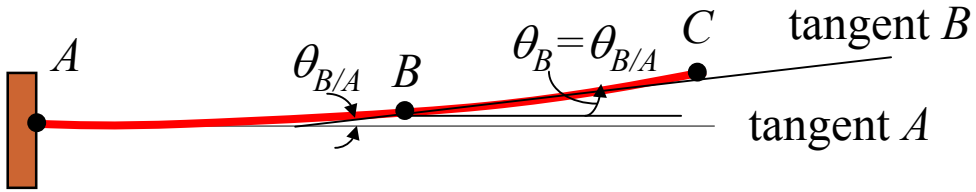


SOLUTION



$$I_{AB} = 2I_{BC}$$





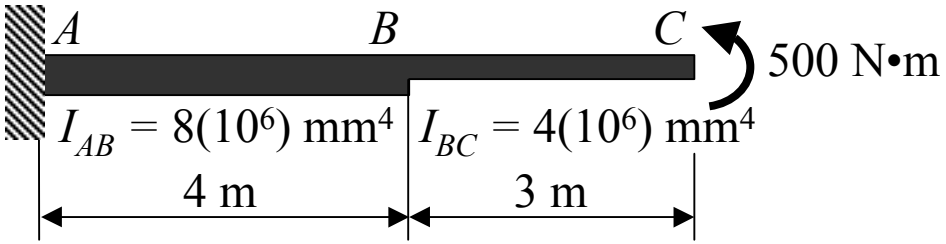
$\theta_B = \theta_{B/A}$ = the area under the M/EI diagram between A and B

$$= (\frac{250\ N \bullet m}{EI_{BC}})(4\ m)$$

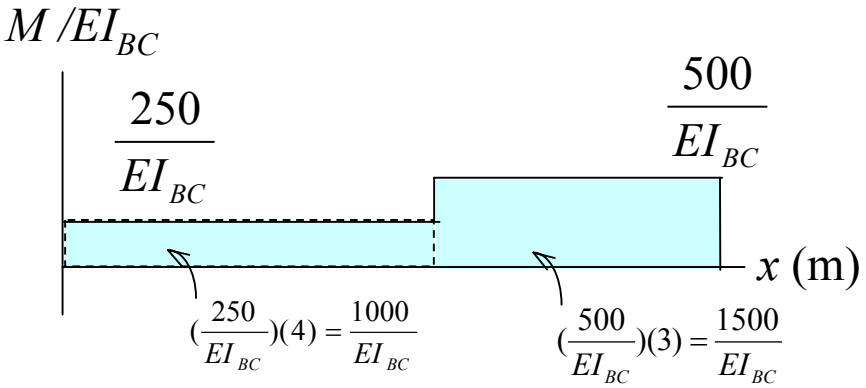
$$= \frac{100}{EI_{BC}}\ N \bullet m^2$$

$$= \frac{100\ N \bullet m^2}{(200 \times 10^9 \frac{N}{m^2})(4 \times 10^{-6} m^4)}$$

$$\theta_B = 0.00125 \text{ rad}$$



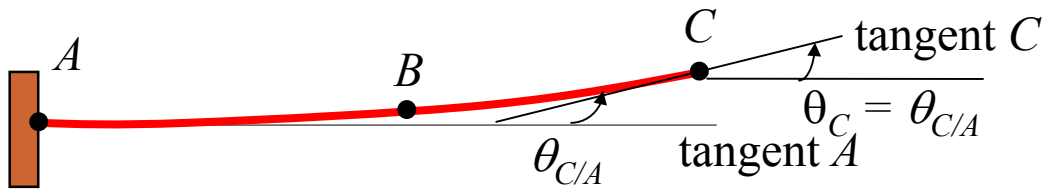
$$I_{AB} = 2I_{BC}$$



$\theta_C = \theta_{C/A}$ = the area under the *M/EI* diagram between *A* and *C*

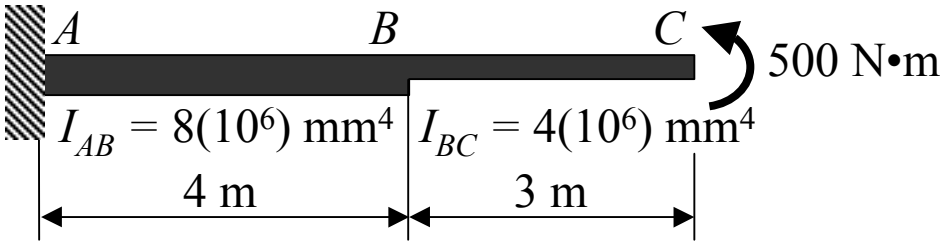
$$= \frac{100\ N \bullet m^2}{EI_{BC}} + \frac{1500\ N \bullet m^2}{EI_{BC}}$$

$$= \frac{2500\ N \bullet m^2}{EI_{BC}}$$

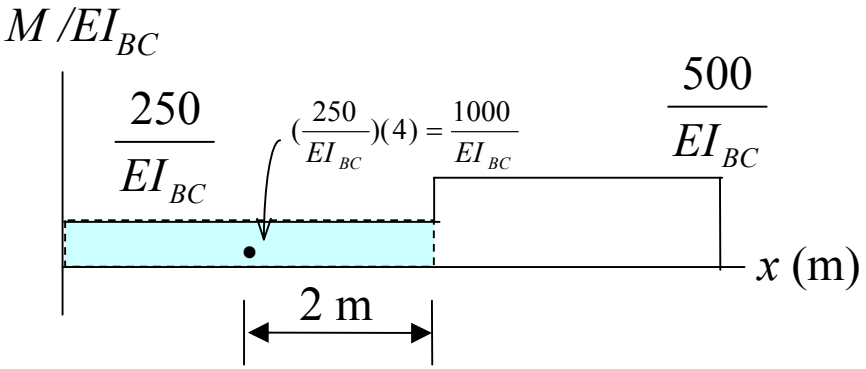


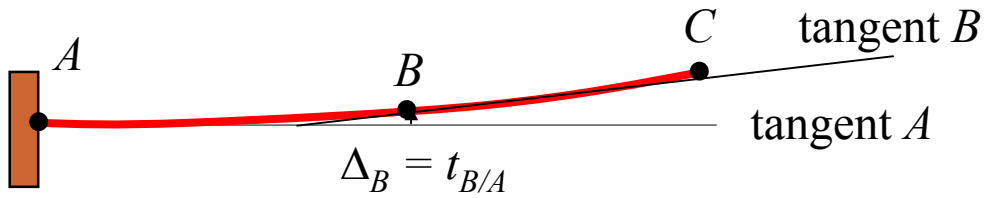
$$= \frac{2500\ N \bullet m^2}{(200 \times 10^9 \frac{N}{m^2})(4 \times 10^{-6} m^4)}$$

$\theta_C = 0.00313$ rad 



$$I_{AB} = 2I_{BC}$$





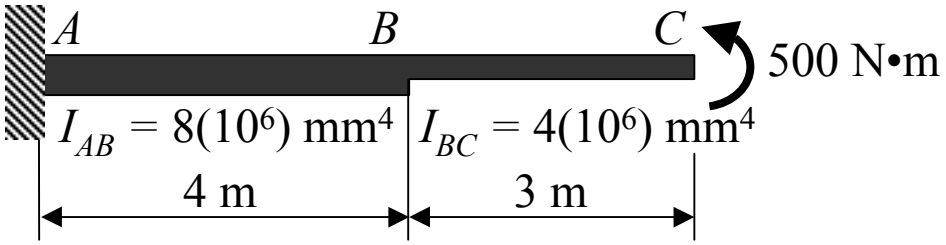
$\Delta_B = t_{B/A}$ = the moment of the M/EI diagram between A and B computed about point B.

$$= \frac{1000\ N \bullet m^2}{EI_{BC}}(2\ m)$$

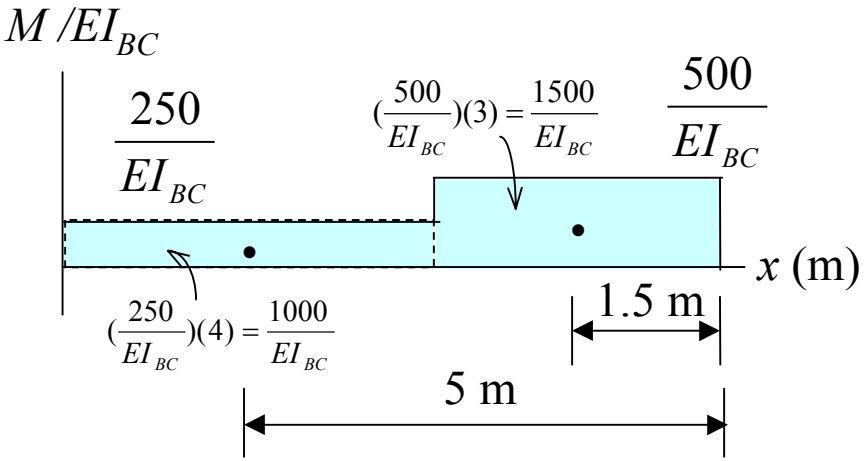
$$= \frac{2000}{EI_{BC}} N \bullet m^3$$

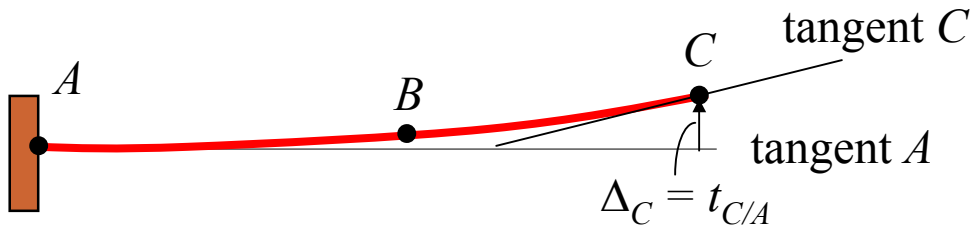
$$= \frac{2000\ N \bullet m^3}{(200\times10^9 \frac{N}{m^2})(4\times10^{-6} m^4)}$$

$$\Delta_B = 0.0025 \text{ m} = 2.5 \text{ mm} \uparrow$$



$$I_{AB} = 2I_{BC}$$





$\Delta_C = t_{C/A}$ = the moment of the M/EI diagram between A and C computed about point C.

$$= \frac{1000\ N \bullet m^2}{EI_{BC}}(5\ m) + \frac{1500\ N \bullet m^2}{EI_{BC}}(1.5\ m)$$

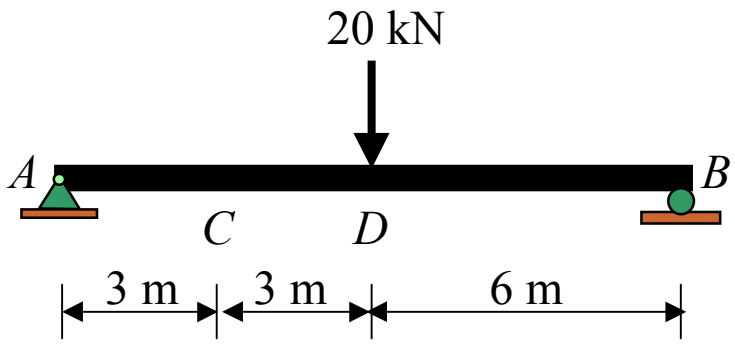
$$= \frac{7250}{EI_{BC}} N \bullet m^3$$

$$= \frac{7250\ N \bullet m^3}{(200 \times 10^9 \frac{N}{m^2})(4 \times 10^{-6} m^4)}$$

$\Delta_C = 0.00906$ m $= 9.06$ mm ↑

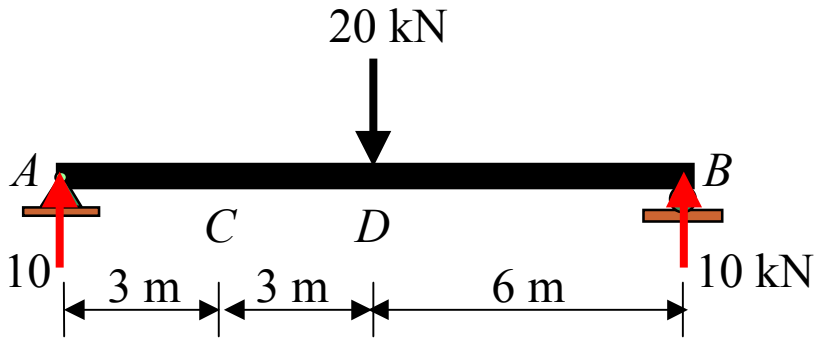
Example 8-8

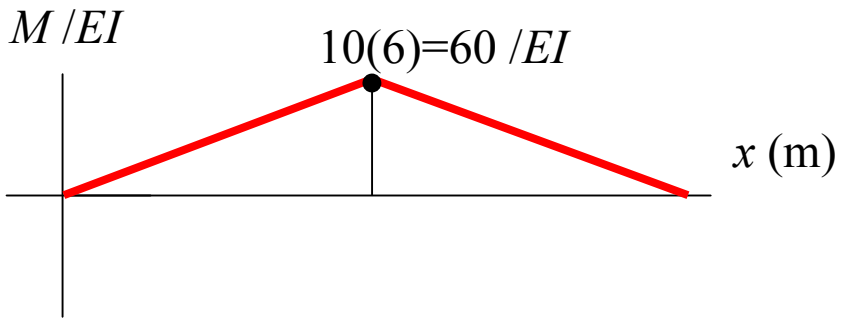
Determine the slope and deflection at points C of the beam shown in the figure. Take $E = 200$ GPa and $I = 60(10^6)$ mm^4

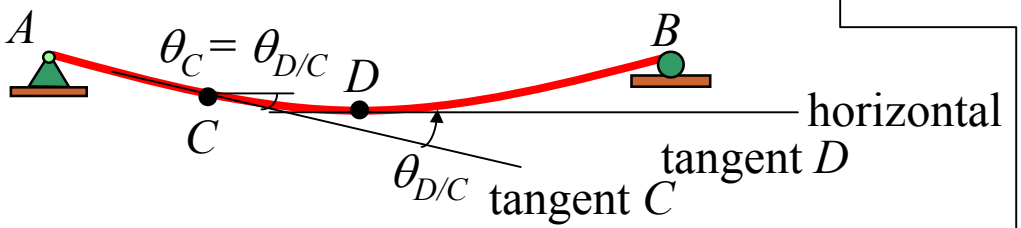


SOLUTION

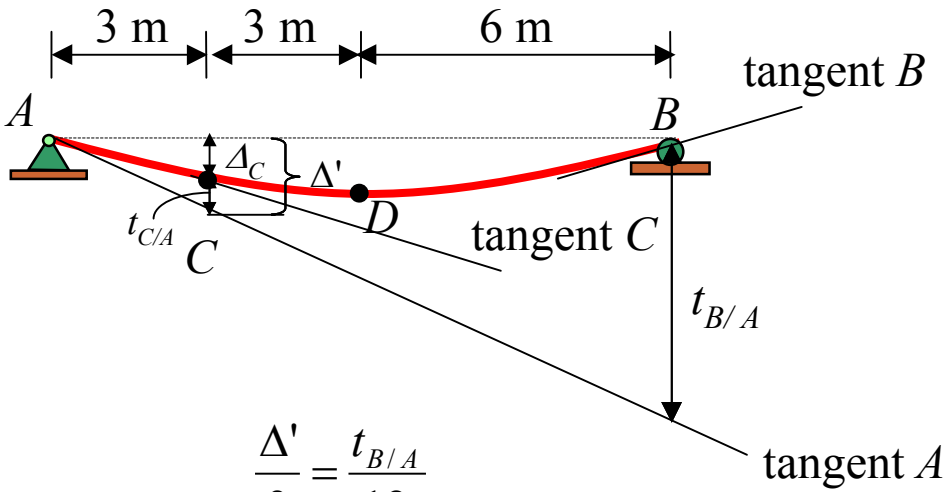
- **Elastic Curve**







$$\theta_C = \theta_{D/C} \qquad \text{--------(1)}$$



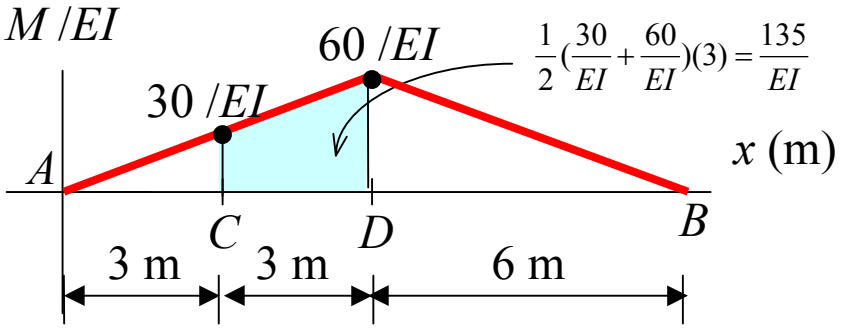
$$\frac{\Delta'}{3} = \frac{t_{B/A}}{12}$$

$$\Delta' = \frac{3}{12} t_{B/A} = \frac{1}{4} t_{B/A}$$

$$\Delta_C = \Delta' - t_{C/A}$$

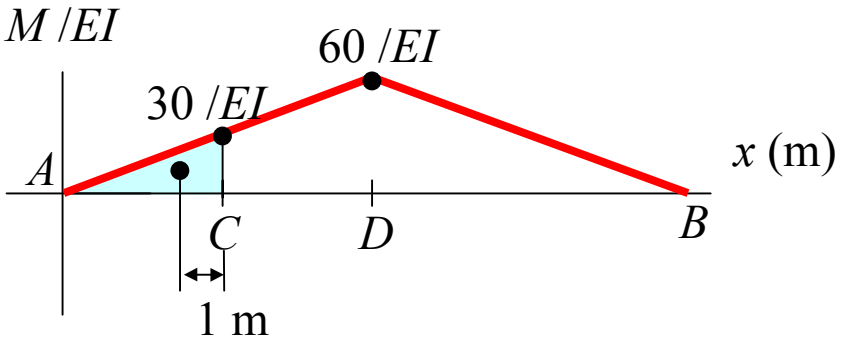
$$\Delta_C = \frac{1}{4} t_{B/A} - t_{C/A} \qquad \text{----(2)}$$

• Moment-Area Theorem



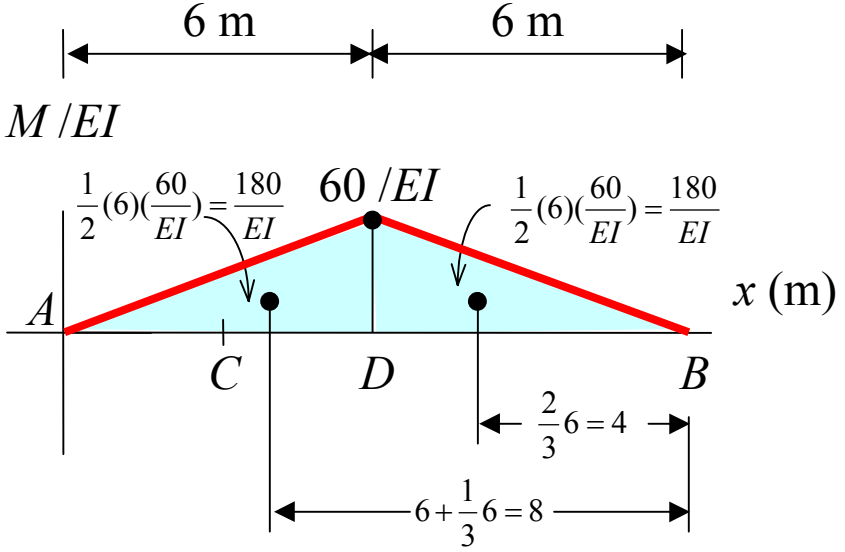
$\theta_{D/C}$ = the *area* under the *M/EI* diagram between *C* and *D*

$$\theta_{D/C} = \frac{135\ kN \bullet m^2}{EI}$$



$t_{C/A}$ = the *moment* of the *M/EI* diagram between *A* and *C* computed about point *C*.

$$t_{C/A} = [\frac{1}{2}(3\ m)(\frac{30\ kN \bullet m}{EI})](1\ m) = \frac{45}{EI}\ kN \bullet m^3$$



$t_{B/A}$ = the *moment* of the *M/EI* diagram between *A* and *B* computed about point *B*.

$$t_{B/A} = (\frac{180\ kN \bullet m^2}{EI})](4\ m) + (\frac{180\ kN \bullet m^2}{EI})](8\ m)$$

$$t_{B/A} = \frac{2160}{EI}\ kN \bullet m^3$$

Substitute $\theta_{D/C}$ in (1):

$$\theta_C = \theta_{D/C} = \frac{135\ kN \bullet m^2}{EI}$$

$$= \frac{135\ kN \bullet m^2}{(200\times10^6\ \frac{kN}{m^2})(60\times10^{-6}\ m^4)}$$

$\theta_C = 0.01125$ rad

Substitute $t_{B/A}$ and $t_{C/A}$ in (2) :

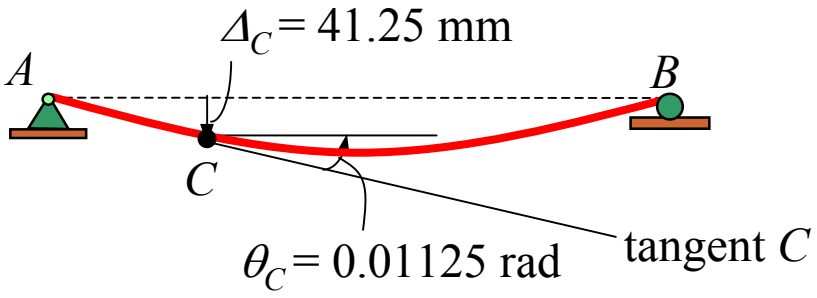
$$\Delta_C = \frac{1}{4}t_{B/A} - t_{C/A}$$

$$= \frac{1}{4}(\frac{2160}{EI}) - \frac{45}{EI}$$

$$= \frac{495}{EI}$$

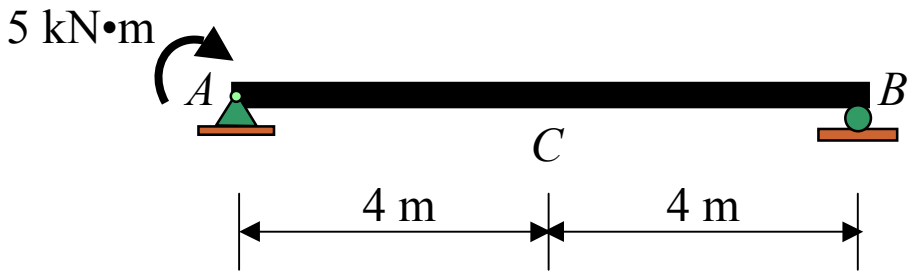
$$= \frac{495}{(200)(60)}$$

$\Delta_C = 0.04125$ m = 41.25 mm, ↓

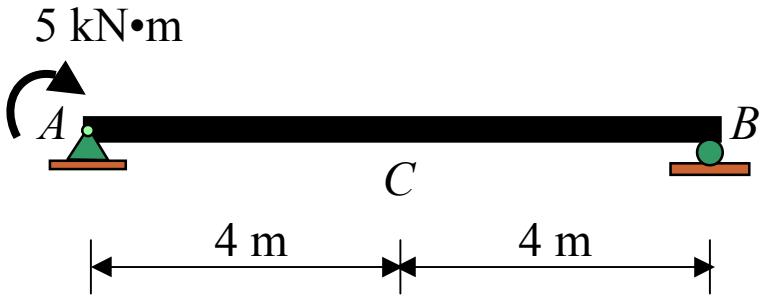


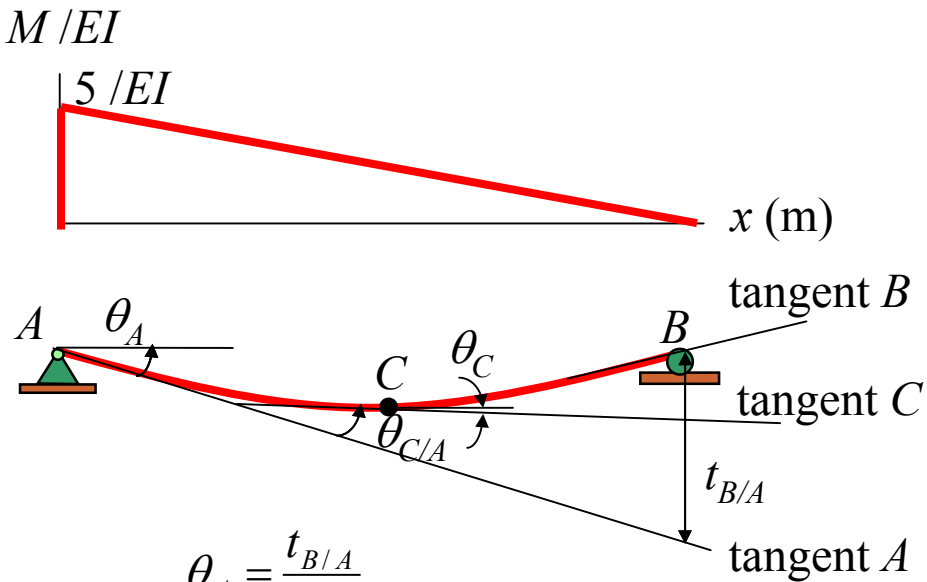
Example 8-9

Determine the slope and deflection at points C of the beam shown in the figure. Take $E = 200$ GPa and $I = 6(10^6)$ mm^4



SOLUTION



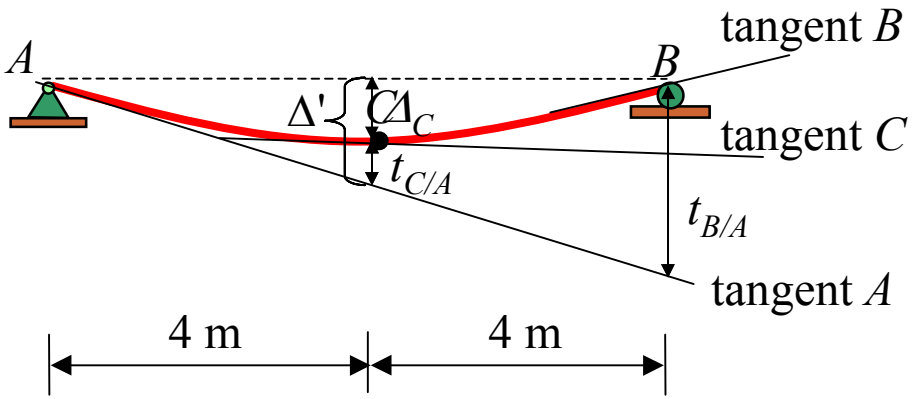


$$\theta_A = \frac{t_{B/A}}{8}$$

$$\theta_C = \theta_A - \theta_{C/A}$$

$$So; \quad \theta_C = \frac{t_{B/A}}{8} - \theta_{C/A} \qquad (1)$$

- **Elastic Curve**



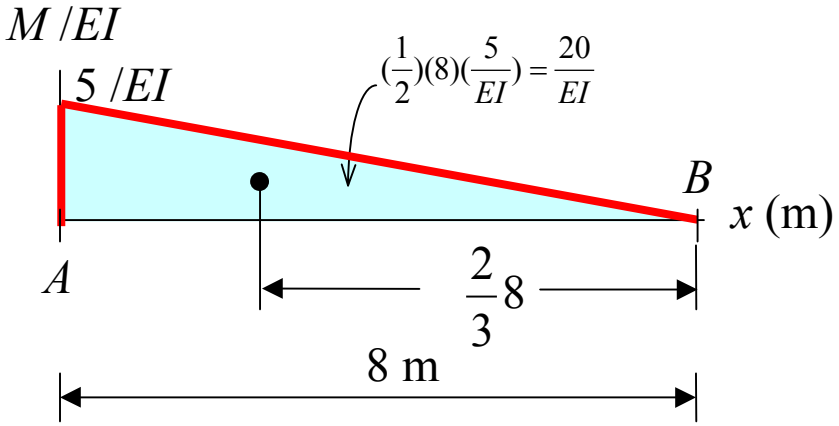
$$\frac{\Delta'}{4} = \frac{t_{B/A}}{8}$$

$$\Delta' = \frac{1}{2} t_{B/A}$$

$$\Delta_C = \Delta' - t_{C/A}$$

$$So; \quad \Delta_C = \frac{1}{2} t_{B/A} - t_{C/A} \qquad -----(2)$$

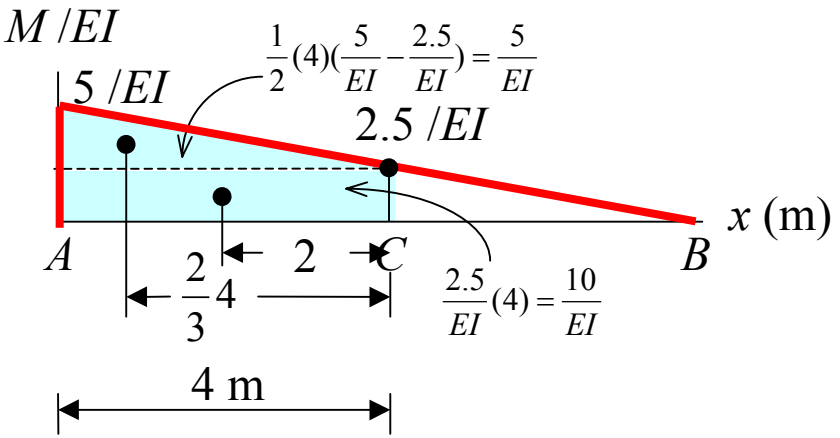
• Moment-Area Theorem



$t_{B/A}$ = the *moment* of the *M/EI* diagram between *A* and *B* computed about point *B*.

$$t_{B/A} = \frac{20\ kN \bullet m^2}{EI}(\frac{2}{3} \times 8\ m)$$

$$= \frac{106.667\ kN \bullet m^3}{EI}$$



$\theta_{C/A}$ = the *area* under the *M/EI* diagram between *A* and *C*

$$\theta_{C/A} = \frac{10}{EI} + \frac{5}{EI} = \frac{15\ kN \bullet m^2}{EI}$$

$t_{C/A}$ = the *moment* of the *M/EI* diagram between *A* and *C* computed about point *C*.

$$t_{B/A} = \frac{10\ kN \bullet m^2}{EI}(2\ m) + \frac{5\ kN \bullet m^2}{EI}(\frac{2}{3}4\ m)$$

$$= \frac{33.333\ kN \bullet m^3}{EI}$$

Substitute $t_{B/A}$ and $\theta_{C/A}$ in (1) :

$$\theta_C = \frac{t_{B/A}}{8} - \theta_{C/A}$$

$$\theta_C = \frac{1}{8}\left(\frac{106.667}{EI}\right) - \left(\frac{15}{EI}\right)$$

$$= -\frac{1.667\ kN \bullet m^2}{EI}$$

$$= -\frac{1.667\ kN \bullet m^2}{(200\times10^6\ \frac{kN}{m^2})(6\times10^{-6}\ m^4)}$$

= -0.0014 rad

Substitute $t_{B/A}$ and $t_{C/A}$ in (2) :

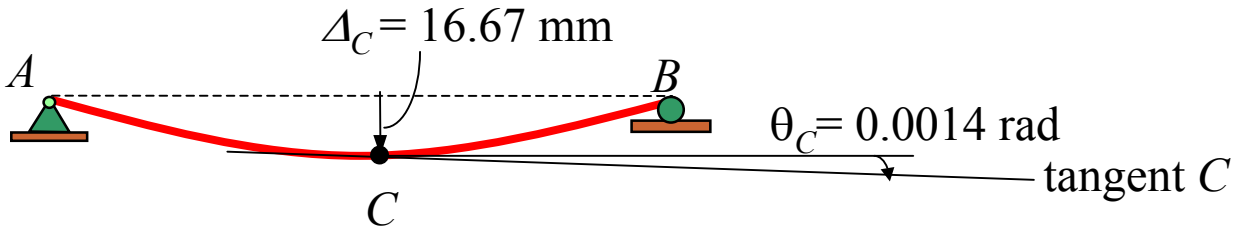
$$\Delta_C = \frac{1}{2}t_{B/A} - t_{C/A}$$

$$= \frac{1}{2}\left(\frac{106.667\ kN \bullet m^3}{EI}\right) - \frac{33.333\ kN \bullet m^3}{EI}$$

$$= \frac{20\ kN \bullet m^3}{EI}$$

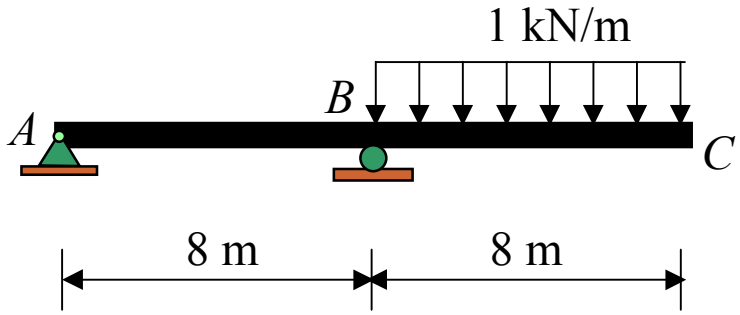
$$= \frac{20\ kN \bullet m^3}{(200\times10^6\ \frac{kN}{m^2})(6\times10^{-6}\ m^4)}$$

Δ_C = 0.01667 m = 16.67 mm, ↓



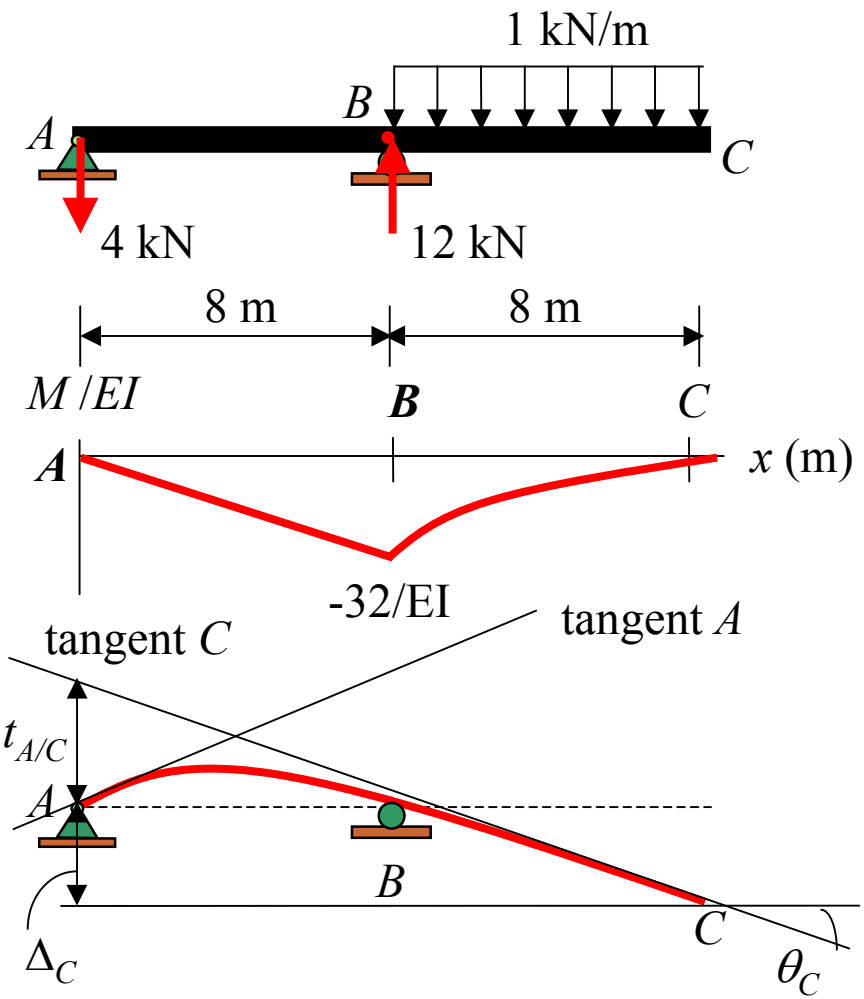
Example 8-10

Determine the slope and deflection at points C of the beam shown in the figure. Take $E = 200$ GPa and $I = 250(10^6)$ mm^4

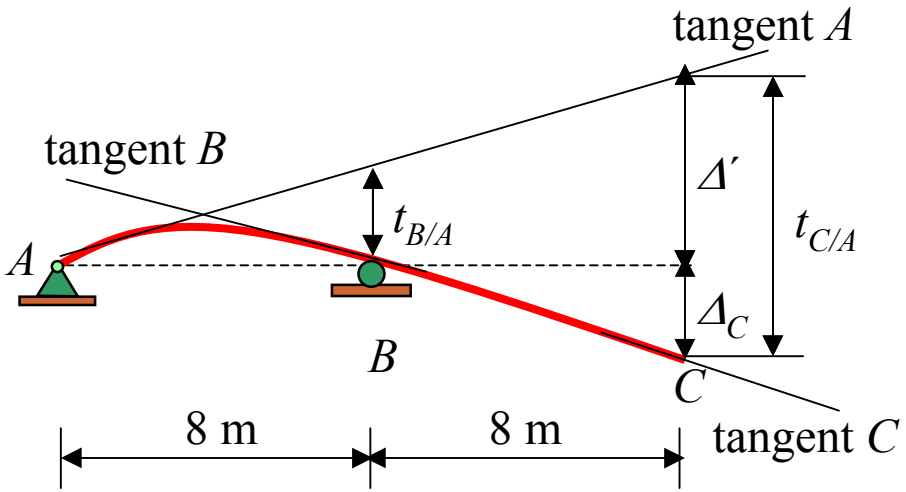


SOLUTION

• **Elastic Curve**



$$\theta_C = \frac{|t_{A/C}| + |\Delta_C|}{16} \qquad ---(1)$$



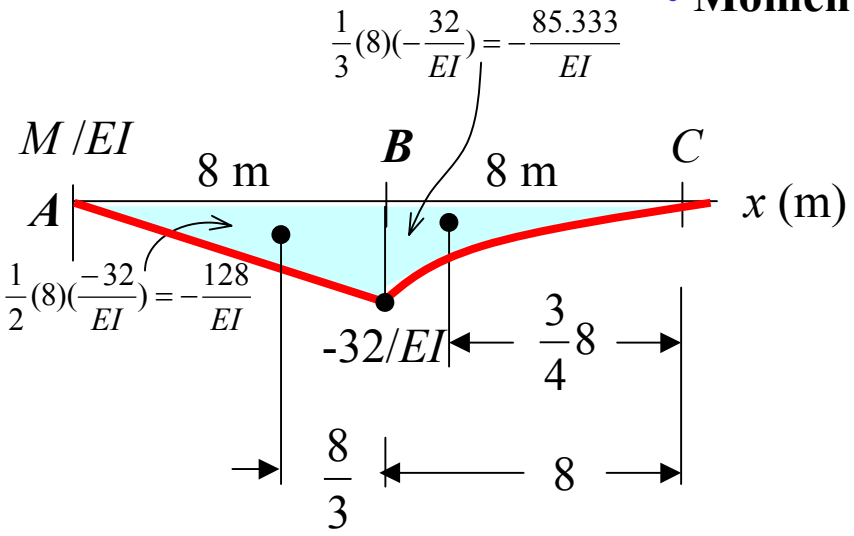
$$\frac{\Delta'}{16} = \frac{t_{B/A}}{8}$$

$$\Delta' = 2t_{B/A}$$

$$\Delta_C = |t_{C/A}| - |\Delta'|$$

$$\Delta_C = |t_{C/A}| - 2|t_{B/A}| \qquad ---(2)$$

• Moment-Area Theorem



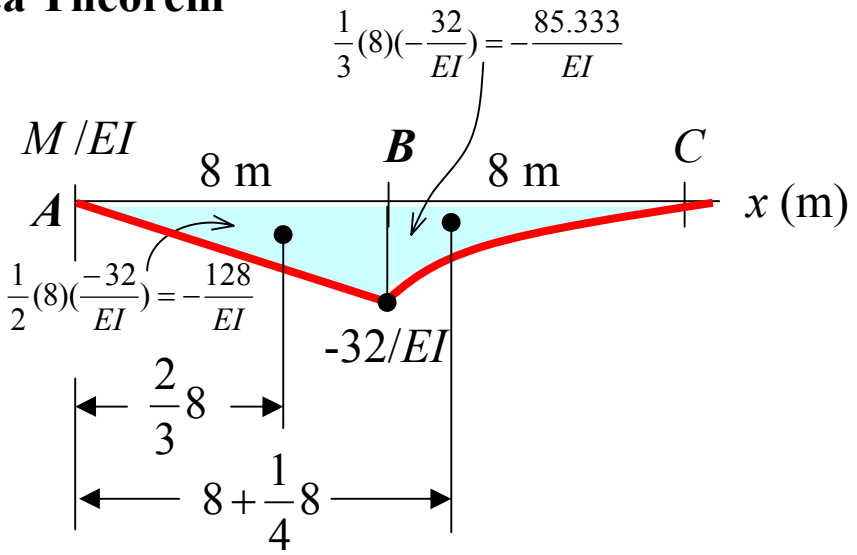
$t_{C/A}$ = the *moment* of the *M/EI* diagram between *A* and *C* computed about point *C*.

$$t_{C/A} = (\frac{-85.333}{EI})(\frac{3}{4}8) + (\frac{-128}{EI})(8 + \frac{1}{3}8)$$

$$= \frac{-1877.37\ kN \bullet m^3}{EI}$$

$$t_{B/A} = (\frac{-128}{EI})(\frac{1}{3}8)$$

$$= \frac{-341.33\ kN \bullet m^3}{EI}$$

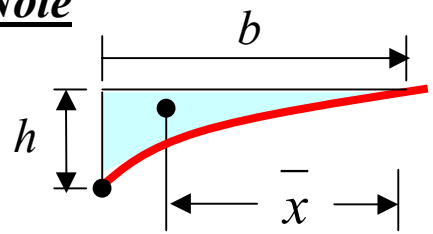


$t_{A/C}$ = the *moment* of the *M/EI* diagram between *A* and *C* computed about point *A*.

$$t_{A/C} = (\frac{-128}{EI})(\frac{2}{3}8) + (\frac{-85.33}{EI})(8 + \frac{1}{4}8)$$

$$= \frac{-1536\ kN \bullet m^3}{EI}$$

Note



$$A = \frac{1}{3}bh$$

$$\bar{x} = \frac{3}{4}b$$

Substitute $t_{C/A}$ and $\theta_{B/A}$ in (2) :

$$\Delta_C = |t_{C/A}| - |2t_{B/A}|$$

$$= \frac{1877.37}{EI} - 2\left(\frac{341.33}{EI}\right)$$

$$= \frac{1194.71\ kN \bullet m^3}{EI}$$

$$= \frac{1194.71\ kN \bullet m^3}{(200\times10^6\ \frac{kN}{m^2})(250\times10^{-6}\ m^4)}$$

Δ_C = 0.02389 m = 23.89 mm, ↓

Substitute $t_{A/C}$ and Δ_C in (1) :

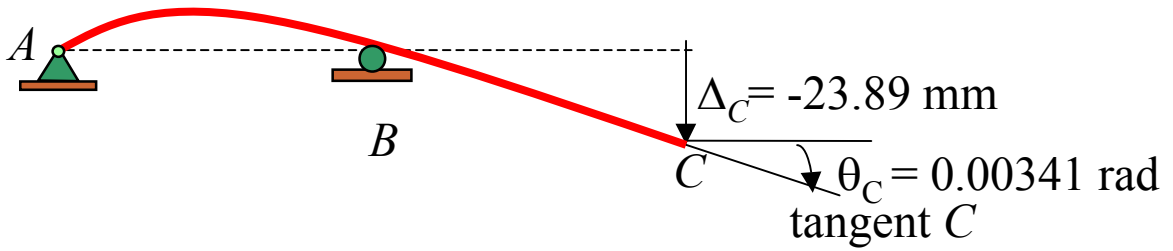
$$\theta_C = \frac{1}{16\ m}(|t_{A/C}| + |\Delta_C|)$$

$$\theta_C = \frac{1}{16\ m}\left(\frac{1536\ kN \bullet m^3}{EI} + \frac{1194.71\ kN \bullet m^3}{EI}\right)$$

$$= \frac{170.67\ kN \bullet m^2}{EI}$$

$$= \frac{170.67\ kN \bullet m^2}{(200\times10^6\ \frac{kN}{m^2})(250\times10^{-6}\ m^4)}$$

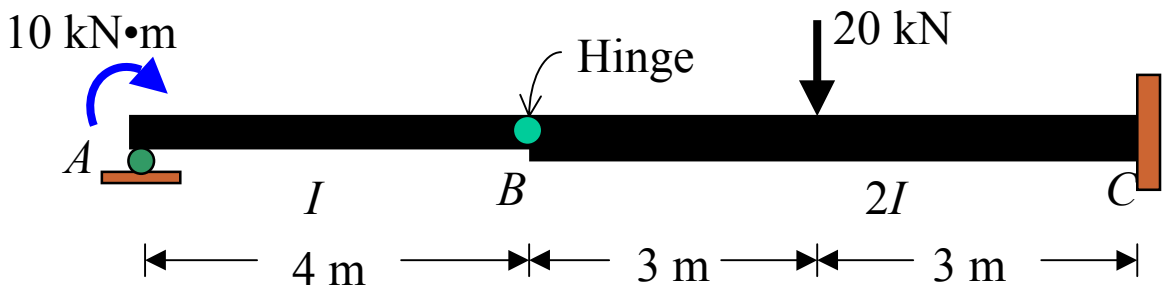
θ_C = 0.00341 rad

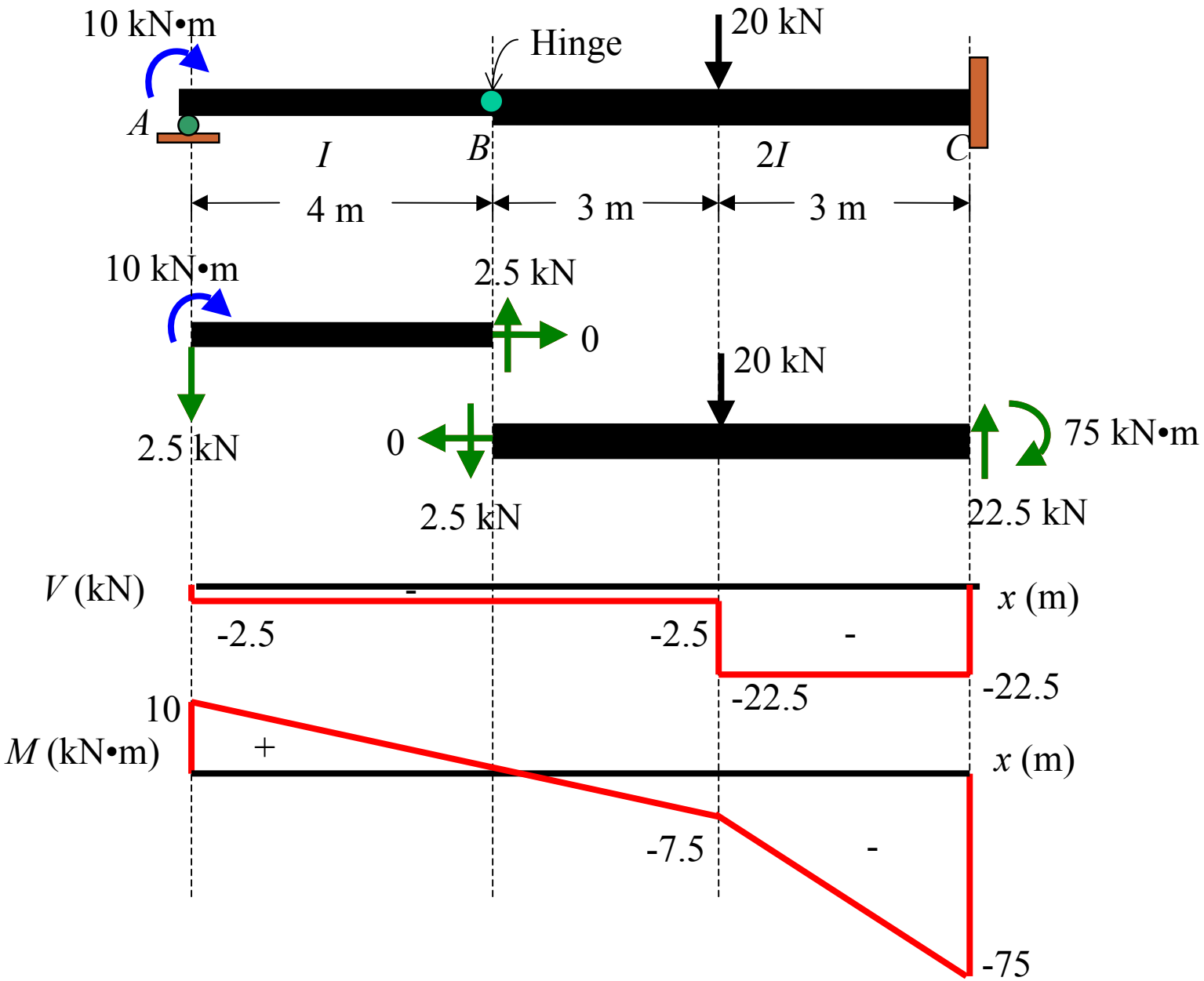


Example 8-11

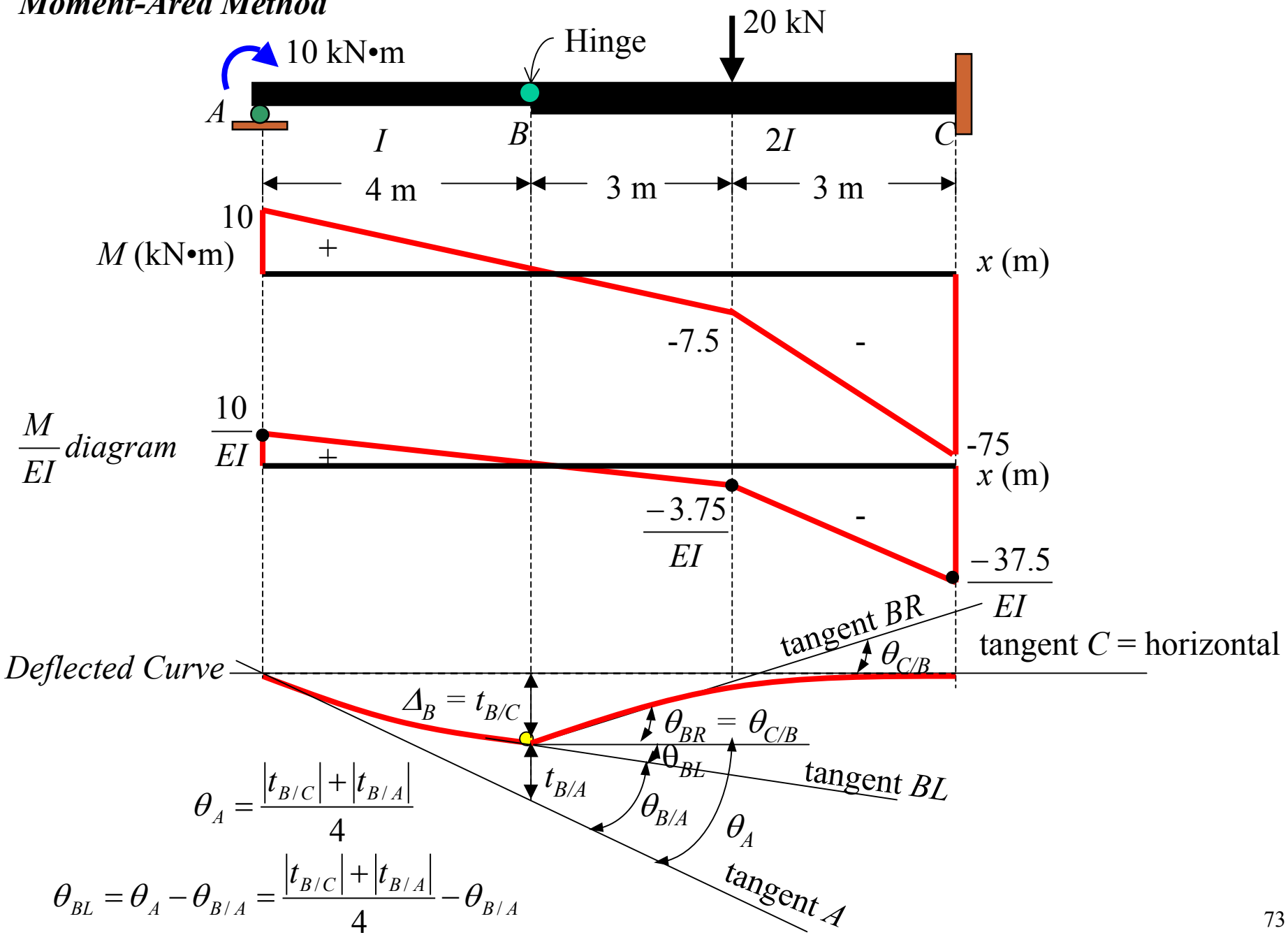
(a) Determine the deflection and slopes at point *B*.
(b) Draw shear, bending moment diagrams and deflected curve.

$E = 200$ GPa
$I = 200(10^6)$ mm^4



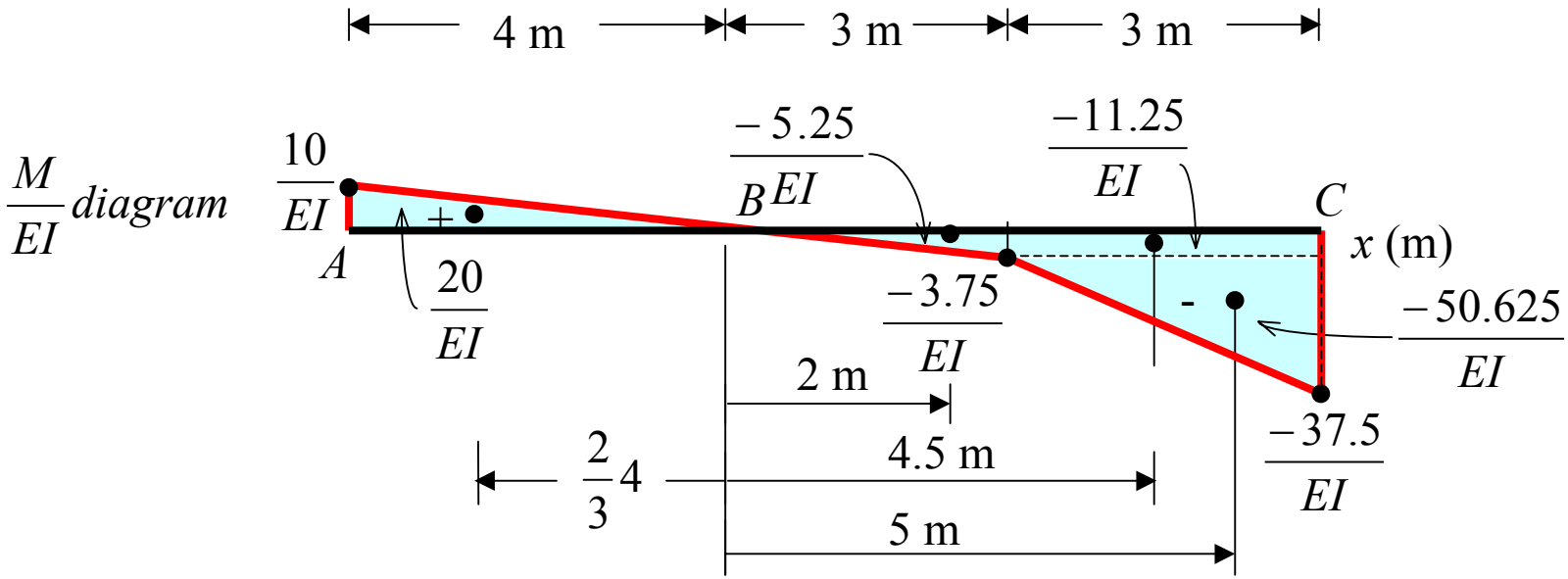

10 kN•m
Hinge
20 kN
A
B
C
I
2I
4 m
3 m
3 m
10 kN•m
2.5 kN
0
20 kN
2.5 kN
0
75 kN•m
2.5 kN
22.5 kN
V (kN)
x (m)
-2.5
-2.5
-22.5
-22.5
10
M (kN•m)
x (m)
-7.5
-75

Moment-Area Method



$$\theta_A = \frac{|t_{B/C}| + |t_{B/A}|}{4}$$

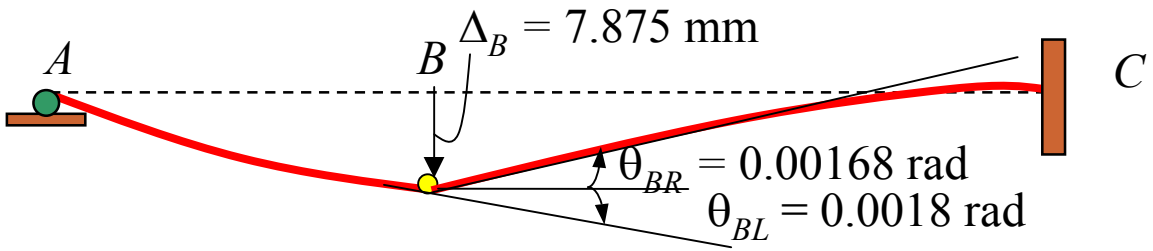
$$\theta_{BL} = \theta_A - \theta_{B/A} = \frac{|t_{B/C}| + |t_{B/A}|}{4} - \theta_{B/A}$$



$$\theta_{BR} = |\theta_{B/C}| = \left|\frac{-5.625}{EI} + \frac{-11.25}{EI} + \frac{-50.265}{EI}\right| = \frac{67.5}{EI} = \frac{67.5}{(200)(200)} = 0.00168 \text{ rad}$$

$$\Delta_B = |t_{B/C}| = \left|\frac{-5.625}{EI}(2) + \frac{-11.25}{EI}(4.5) + \frac{-50.265}{EI}(5)\right| = \frac{315}{EI} = \frac{315}{(200)(200)} = 7.875 \text{ mm}$$

$$\theta_{BL} = \frac{|t_{B/C}| + |t_{B/A}|}{4} - \theta_{B/A} = \left(\frac{\frac{315}{EI} + \left(\frac{20}{EI}\right)\left(\frac{2}{3}4\right)}{4}\right) - \frac{20}{EI} = \frac{72.08}{EI} = \frac{72.08}{(200)(200)} = 0.0018 \text{ rad}$$



Conjugate-Beam Method

$$\frac{dV}{dx} = -w \qquad \frac{d^2M}{dx^2} = -w$$

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad \frac{d^2\upsilon}{dx^2} = \frac{M}{EI}$$

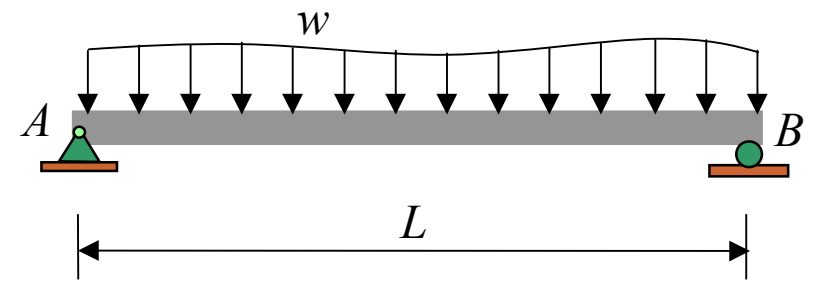
Or integrating,

$$V = -\int w dx \qquad M = \int [-\int w dx] dx$$

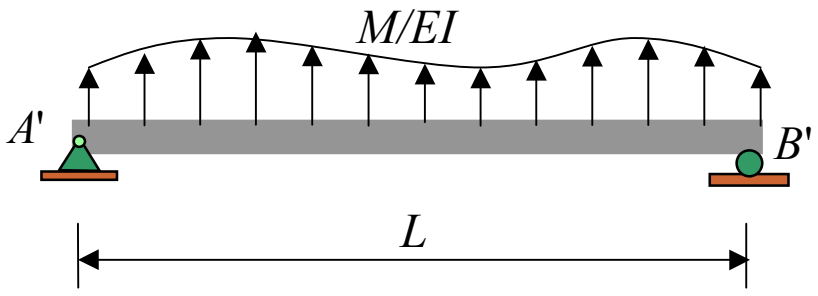
$$\theta = \int (\frac{M}{EI}) dx \qquad \upsilon = \int [\int (\frac{M}{EI}) dx] dx$$

Theorem 1: The slope at a point in the real beam is numerically equal to the shear at the corresponding point in the conjugate beam.

Theorem 2: The displacement of a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam.

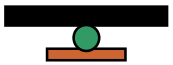
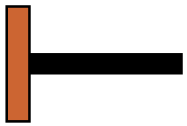


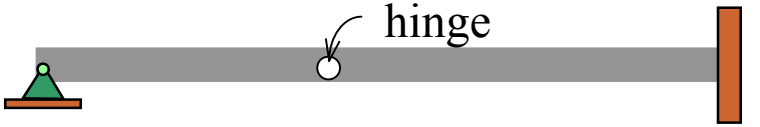
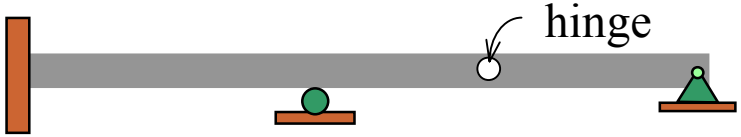
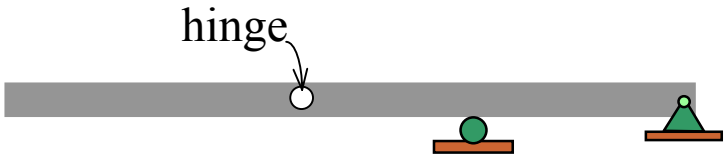
real beam



conjugate beam

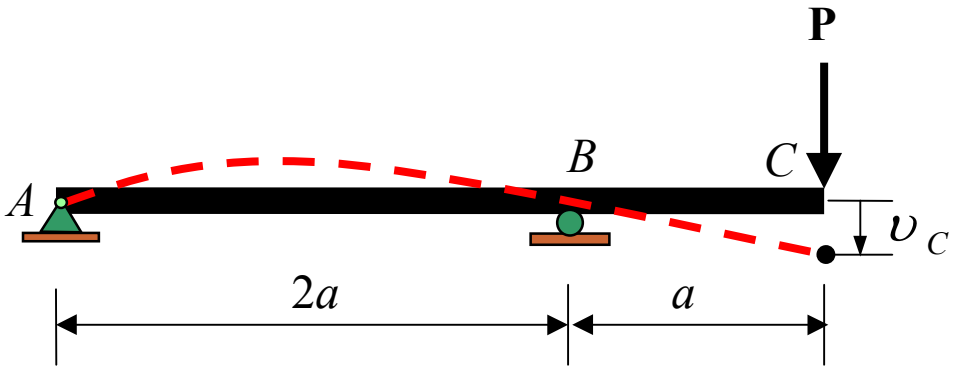
- **Conjugate - Beam Supports**

Real Beam	Conjugate Beam
1) θ $\Delta = 0$  external pin	V $M = 0$  pin
2) θ $\Delta = 0$  external roller	V $M = 0$  roller
3) θ $\Delta = 0$  internal pin	V $M = 0$  hinge
4) θ $\Delta = 0$  internal roller	V $M = 0$  hinge
5) θ Δ  hinge	V M  internal roller
6) $\theta = 0$ $\Delta = 0$  fixed	$V = 0$ $M = 0$  free
7) θ Δ  free	V M  fixed

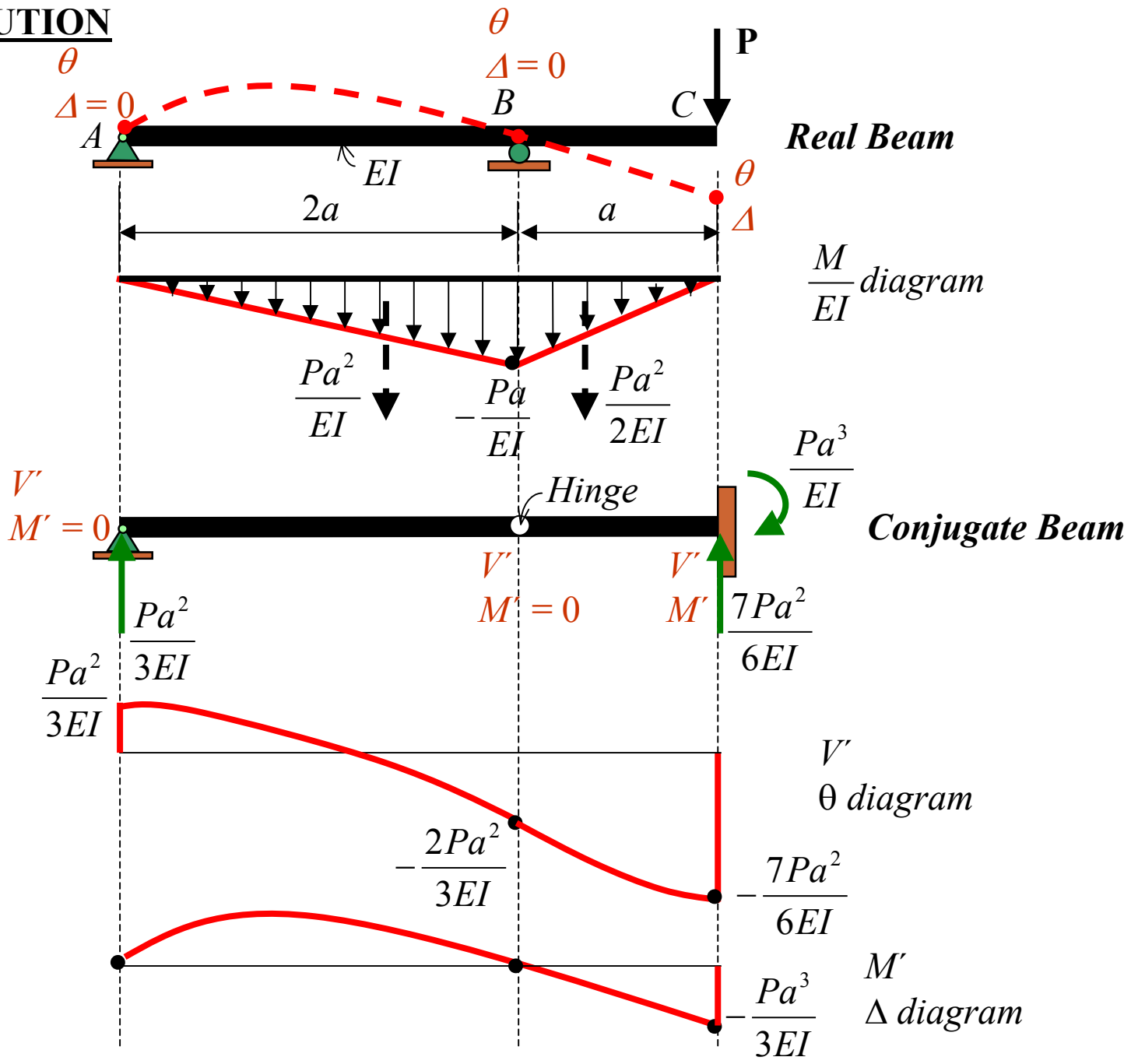
Real Beam	Conjugate Beam
	
	
	
	
	

Example 8-10

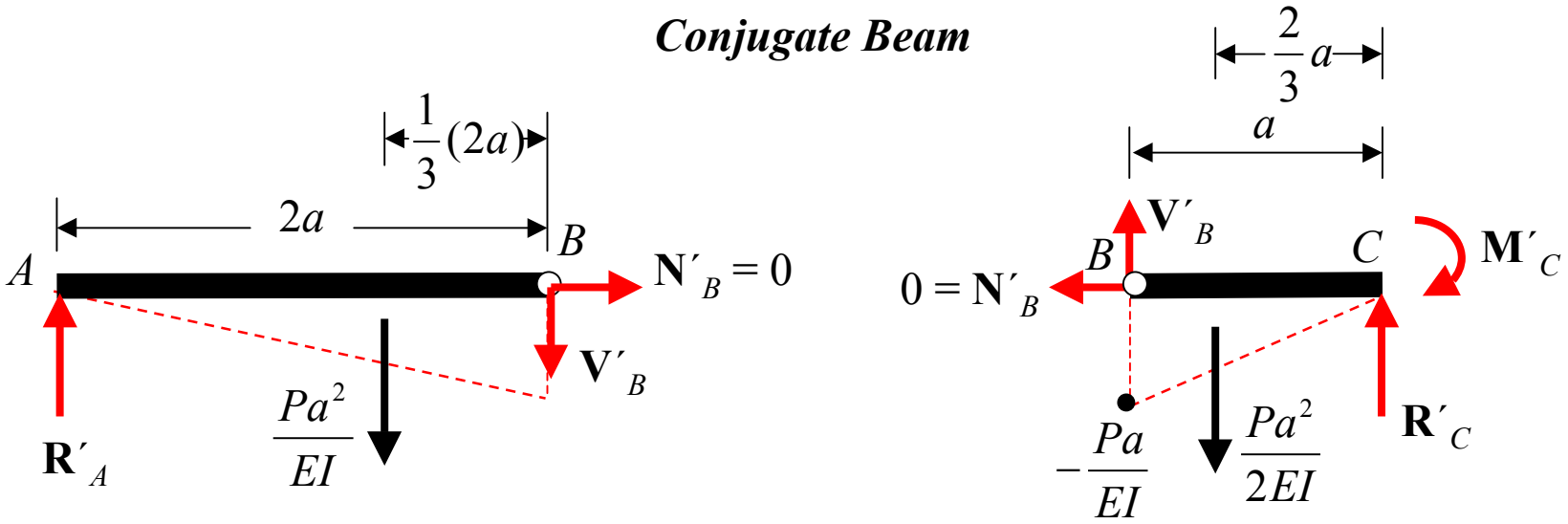
The beam shown is subjected to a load **P** at its end. Determine the slope and displacement at *C*. *EI* is constant.



SOLUTION


θ
Δ = 0
A
θ
Δ = 0
B
C
P
Real Beam
EI
2a
a
θ
Δ
M/EI diagram
Pa²/EI
−Pa/EI
Pa²/2EI
V′
M′ = 0
Hinge
Pa³/EI
Conjugate Beam
V′
M′ = 0
V′
M′
Pa²/3EI
7Pa²/6EI
Pa²/3EI
V′
θ diagram
−2Pa²/3EI
−7Pa²/6EI
M′
Δ diagram
−Pa³/3EI

Conjugate Beam



Member AB:

$+\circlearrowright \Sigma M_B = 0: \quad \frac{Pa^2}{EI}(\frac{1}{3}2a) - R'_A(2a) = 0$

$$R'_A = \frac{Pa^2}{3EI}$$

$+\uparrow \Sigma F_y = 0: \quad \frac{Pa^2}{3EI} - \frac{Pa^2}{EI} - V'_B = 0$

$$V'_B = -\frac{2Pa^2}{3EI}$$

Member BC:

$+\uparrow \Sigma F_y = 0: \quad -\frac{2Pa^2}{3EI} - \frac{Pa^2}{2EI} + R'_C = 0$

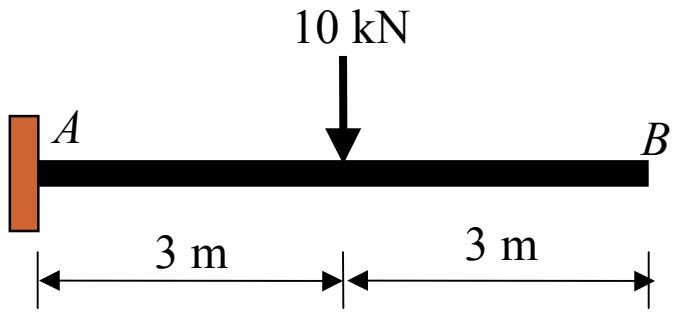
$$R'_C = \frac{7Pa^2}{6EI}$$

$+\circlearrowright \Sigma M_C = 0: \quad -M'_C + \frac{Pa^2}{2EI}(\frac{2}{3}a) + \frac{2Pa^2}{3EI}(a) = 0$

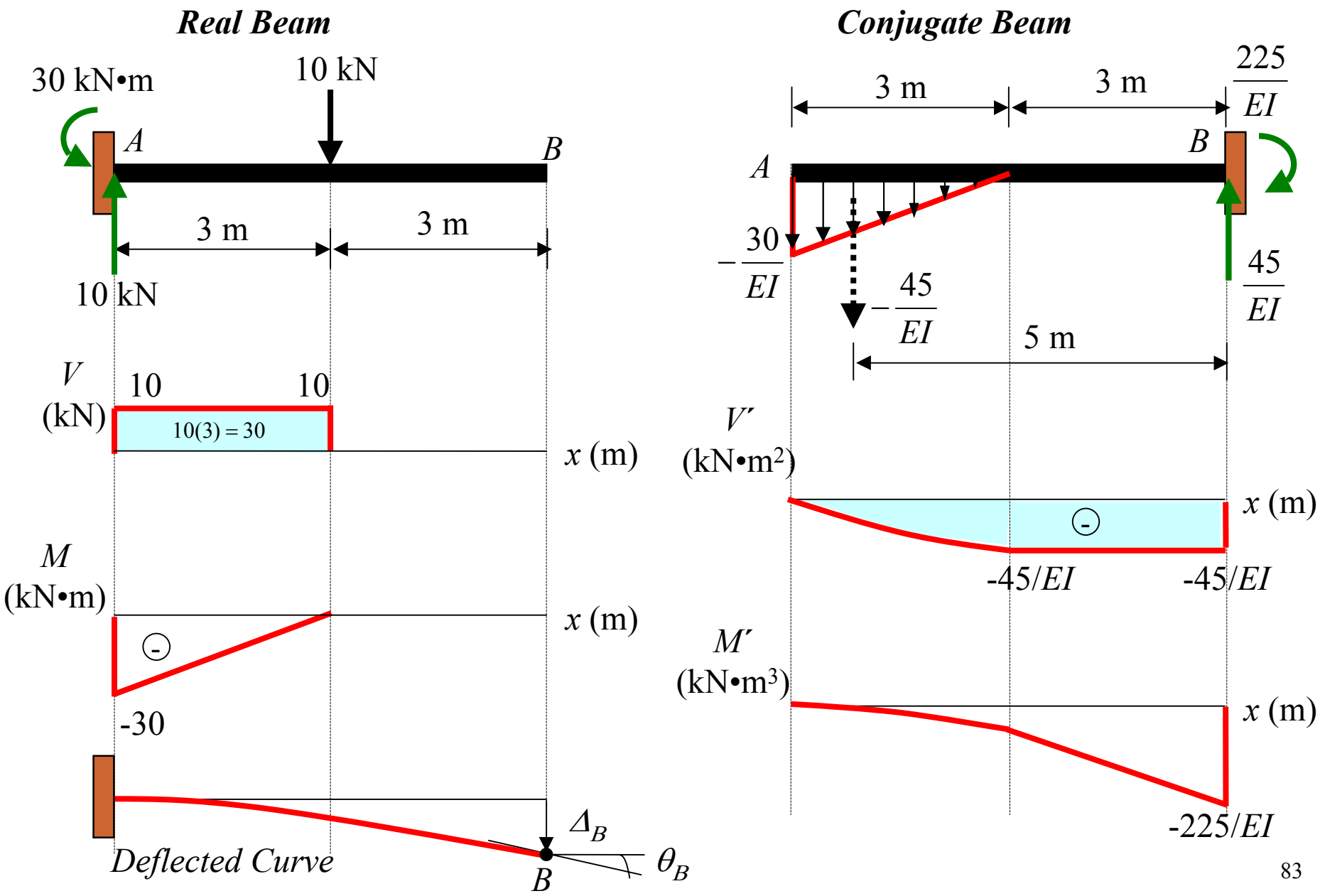
$$M'_C = \frac{Pa^3}{EI}$$

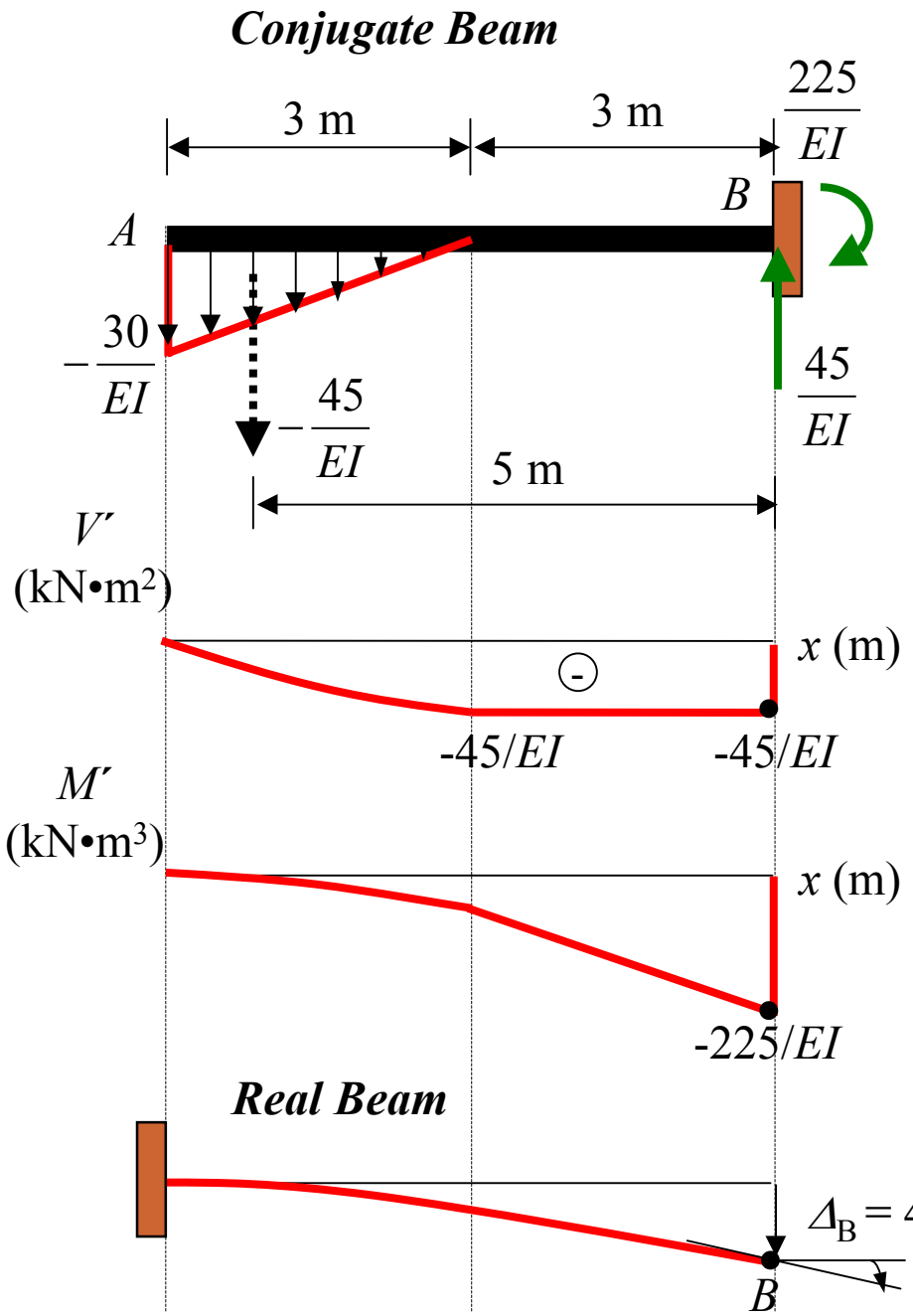
Example 8-11

Use the conjugate beam method for
Determine the slope and deflection at points B of the beam shown in the figure.
Take $E = 200$ GPa and $I = 250(10^6)$ mm^4.



SOLUTION


Real Beam
30 kN•m
10 kN
A
B
3 m
3 m
10 kN
V
(kN)
10
10
10(3) = 30
x (m)
M
(kN•m)
-30
x (m)
Deflected Curve
Δ_B
θ_B
B
Conjugate Beam
3 m
3 m
225/EI
A
B
30/EI
45/EI
45/EI
5 m
V′
(kN•m²)
-45/EI
-45/EI
x (m)
M′
(kN•m³)
-225/EI
x (m)



$$\theta_B = V'_B = -\frac{45}{EI}$$

$$= -\frac{45kN \bullet m^2}{(200\times10^6 \frac{kN}{m^2})(250\times10^{-6} m^4)}$$

θ_B = -0.0009 rad

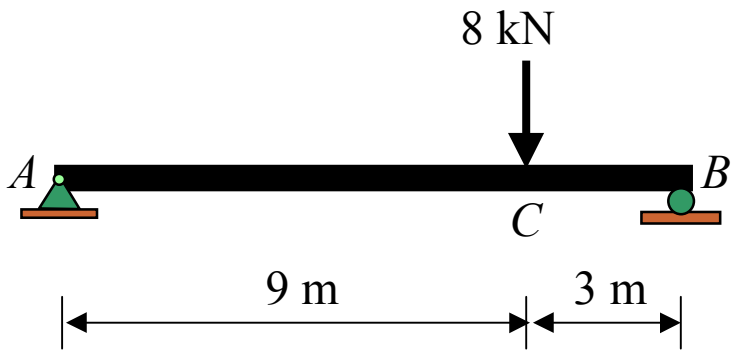
$$\Delta_B = M'_B = -\frac{225}{EI}$$

$$= -\frac{225kN \bullet m^3}{(200\times10^6 \frac{kN}{m^2})(250\times10^{-6} m^4)}$$

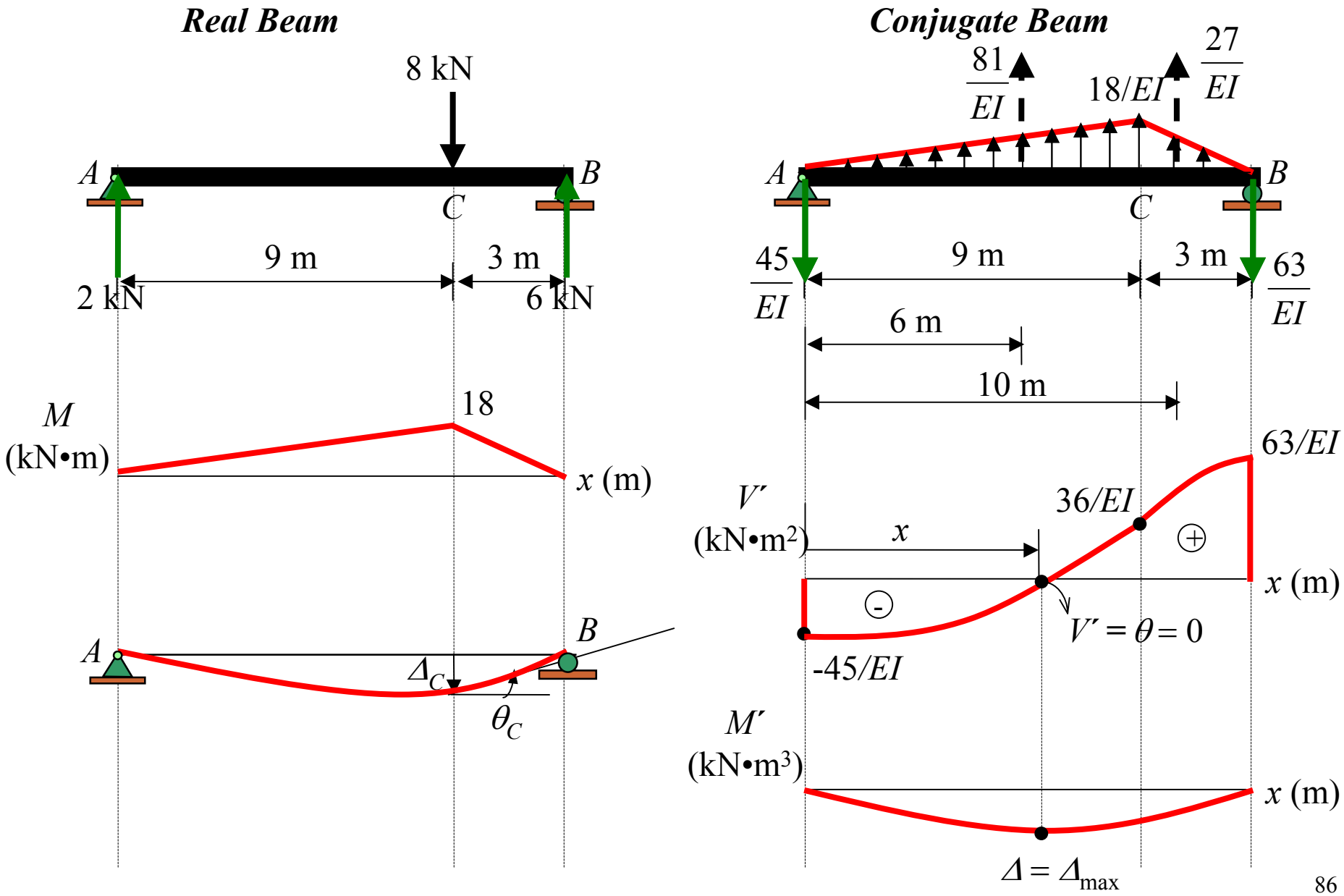
Δ_B = -0.0045 m = -4.5 mm

Example 8-12

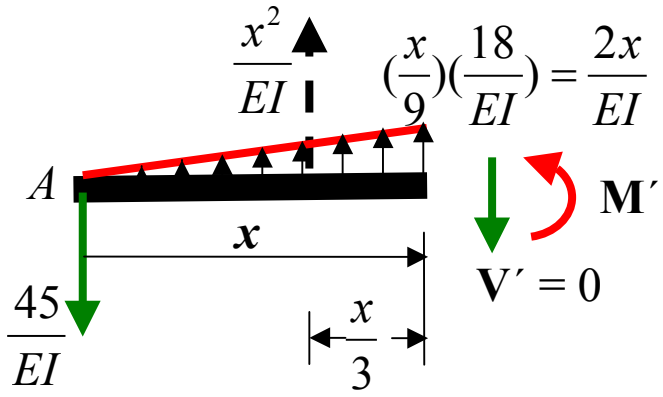
Use the conjugate beam method for
Determine the maximum deflection , the slope and deflection at points C of the beam shown in the figure. Take $E = 200$ GPa and $I = 60(10^6)$ mm^4.



SOLUTION


Real Beam
8 kN
A
B
C
9 m
3 m
2 kN
6 kN
M
(kN•m)
18
x (m)
A
B
Δ_C
θ_C
Conjugate Beam
81/EI
18/EI
27/EI
A
B
C
45/EI
9 m
3 m
63/EI
6 m
10 m
63/EI
V′
(kN•m²)
36/EI
x
x (m)
V′ = θ = 0
-45/EI
M′
(kN•m³)
x (m)
Δ = Δ_max

- ***maximum deflection***



$$+\uparrow \Sigma F_y = 0: \qquad -\frac{45}{EI} + \frac{x^2}{EI} = 0$$

$$x = 6.71 \text{ m}$$

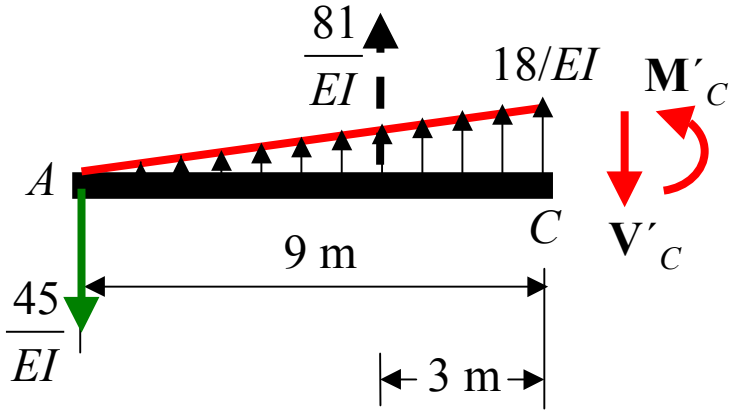
$$+\circlearrowleft \Sigma M_{x=6.71} = 0:$$

$$M' - \frac{(6.71)^2}{EI}(\frac{6.71}{3}) + \frac{45}{EI}(6.71) = 0$$

$$M' = -\frac{201.246}{EI}$$

$$\Delta_{\max} = M' = -\frac{201.246}{EI}$$

- ***slope and deflection at point C***



$$+\uparrow \Sigma F_y = 0: \qquad -\frac{45}{EI} + \frac{81}{EI} - V'_C = 0$$

$$V'_C = \frac{36}{EI}$$

Note : *V'_C is the same value from previously obtained V' diagram.*

$$+\circlearrowleft \Sigma M_C = 0:$$

$$M'_C - \frac{81}{EI}(3) + \frac{45}{EI}(9) = 0$$

$$M'_C = -\frac{162}{EI}$$

$$\Delta_{max} = M' = -\frac{201.25}{EI} = -\frac{201.25\ kN \bullet m^3}{(200\times10^6\ \frac{kN}{m^2})(60\times10^{-6}\ m^4)}$$

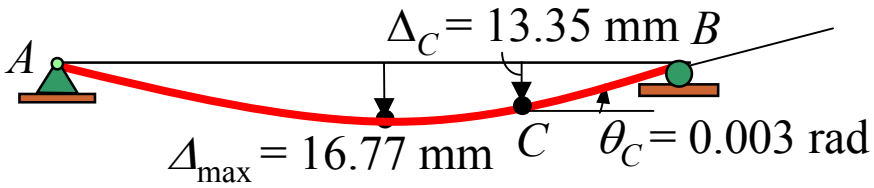
Δ_{max} = -0.01677 m = -16.77 mm ↓

$$\theta_C = V'_C = \frac{81}{EI} = \frac{81\ kN \bullet m^2}{(200\times10^6\ \frac{kN}{m^2})(60\times10^{-6}\ m^4)}$$

θ_C = 0.003 rad 

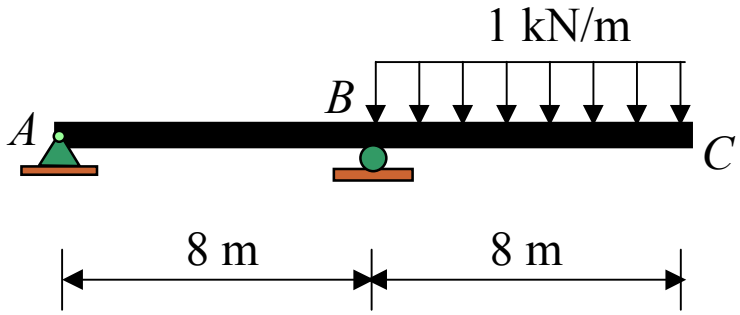
$$\Delta_C = M'_C = -\frac{162}{EI} = -\frac{162\ kN \bullet m^3}{(200\times10^6\ \frac{kN}{m^2})(60\times10^{-6}\ m^4)}$$

Δ_C = -0.0135 m = -13.35 mm ↓

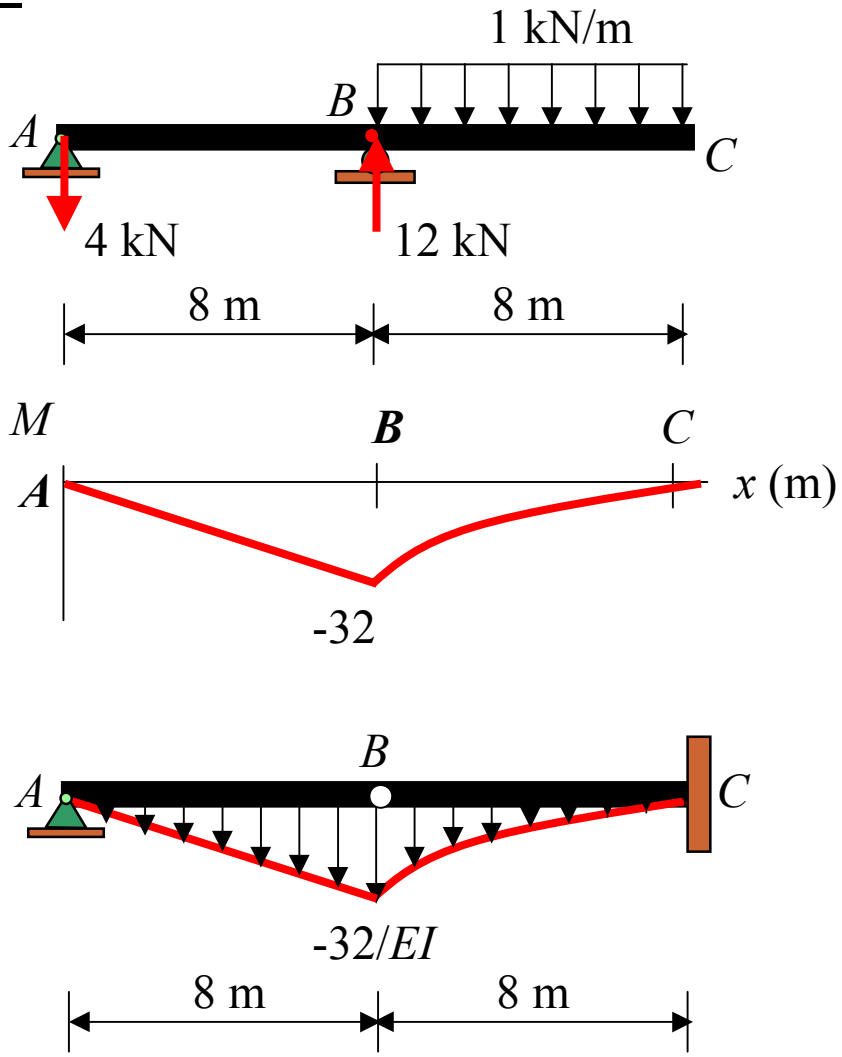


Example 8-13

Determine the slope and deflection at points C of the beam shown in the figure. Take $E = 200$ GPa and $I = 250(10^6)$ mm^4

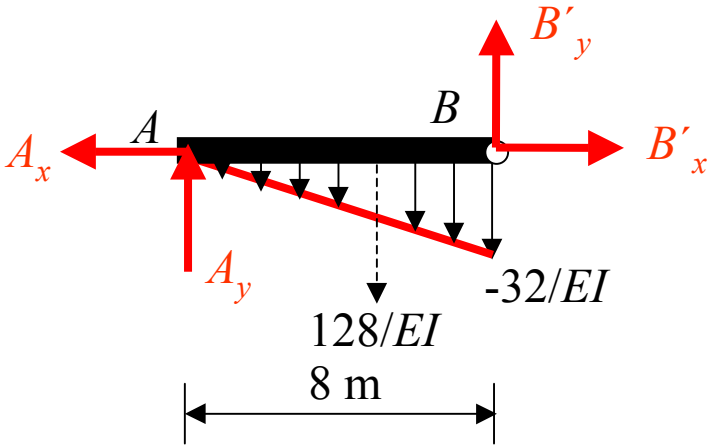


SOLUTION



Real beam

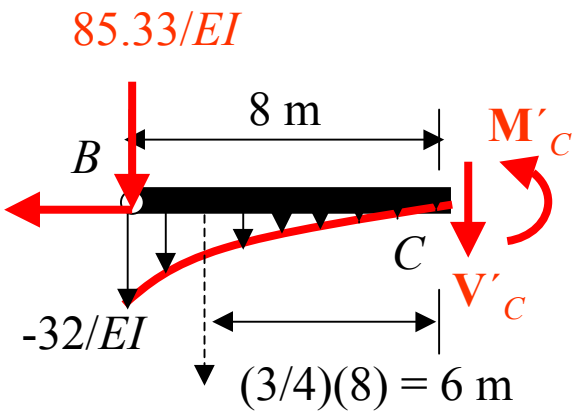
Conjugate beam



$+\circlearrowleft \Sigma M_A = 0$:

$$8B'_y - \frac{128}{EI}\left(\frac{2}{3}8\right) = 0$$

$$B'_y = \frac{85.33}{EI}$$



$$\frac{1}{3}(8)\left(\frac{32}{EI}\right) = \frac{85.33}{EI}$$

$+\uparrow \Sigma F_y = 0$:

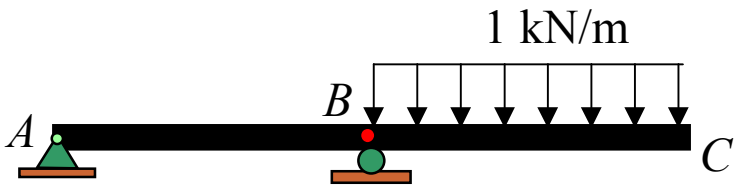
$$V'_C = \theta_C = -\frac{170.67}{EI}$$

$$= \frac{170.67}{(200)(250)} = -0.00341\ rad$$

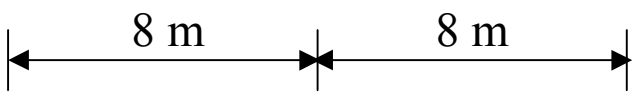
$+\circlearrowleft \Sigma M_C = 0$:

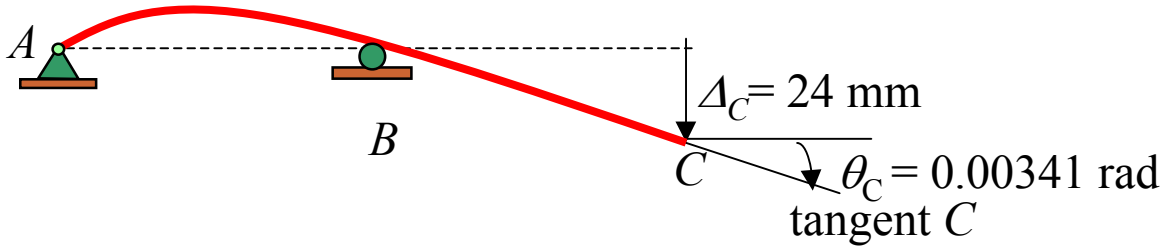
$$+M'_C + \frac{85.33}{EI}(8) + \frac{85.33}{EI}(6) = 0$$

$$M'_C = \Delta_C = -\frac{1194.62}{EI} = -\frac{1194.62}{(200)(250)} = -0.024\ m$$



Real beam

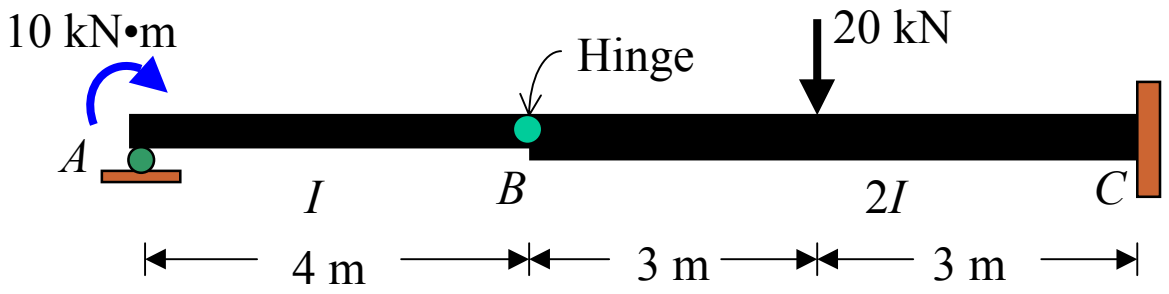




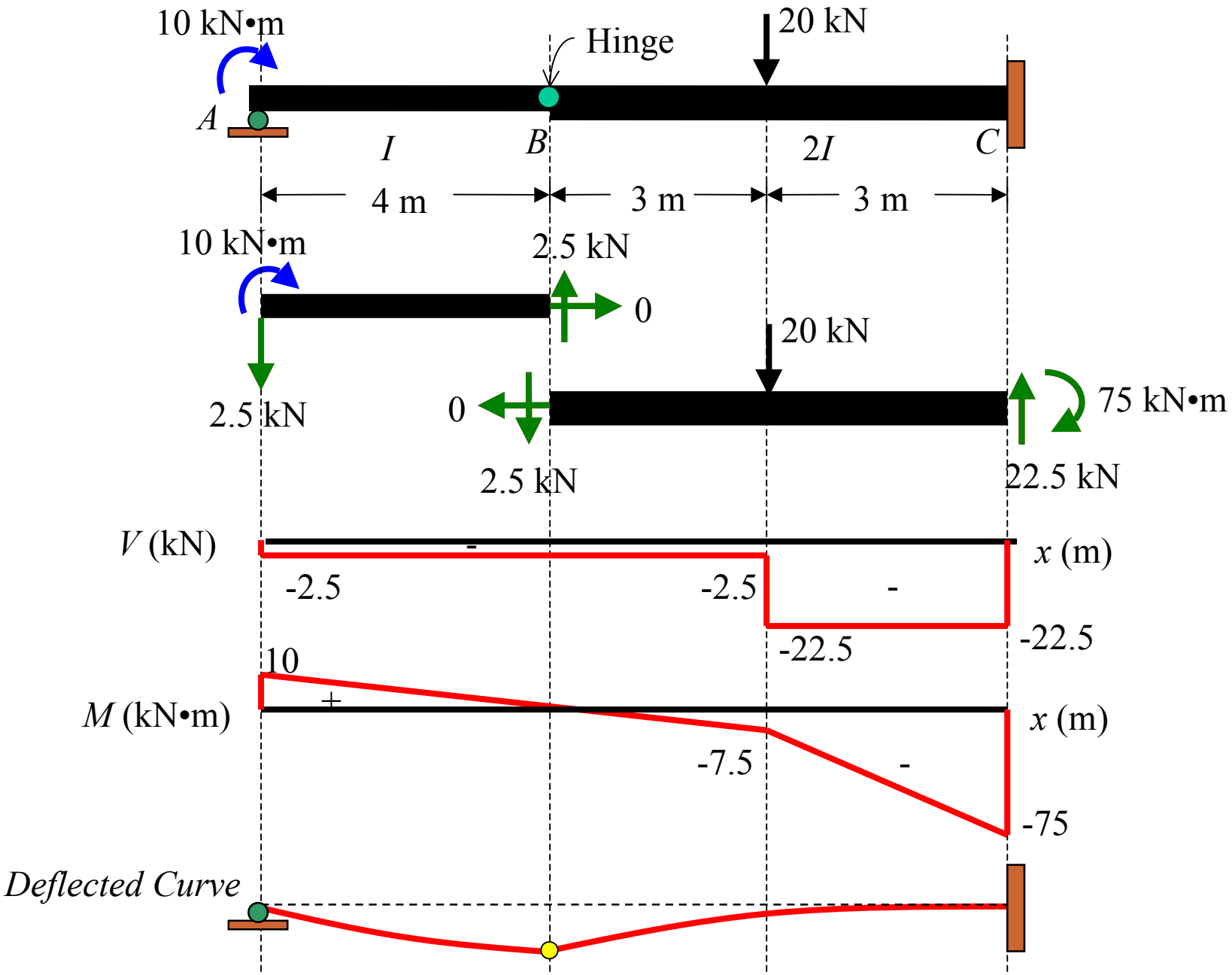
Example 8-14

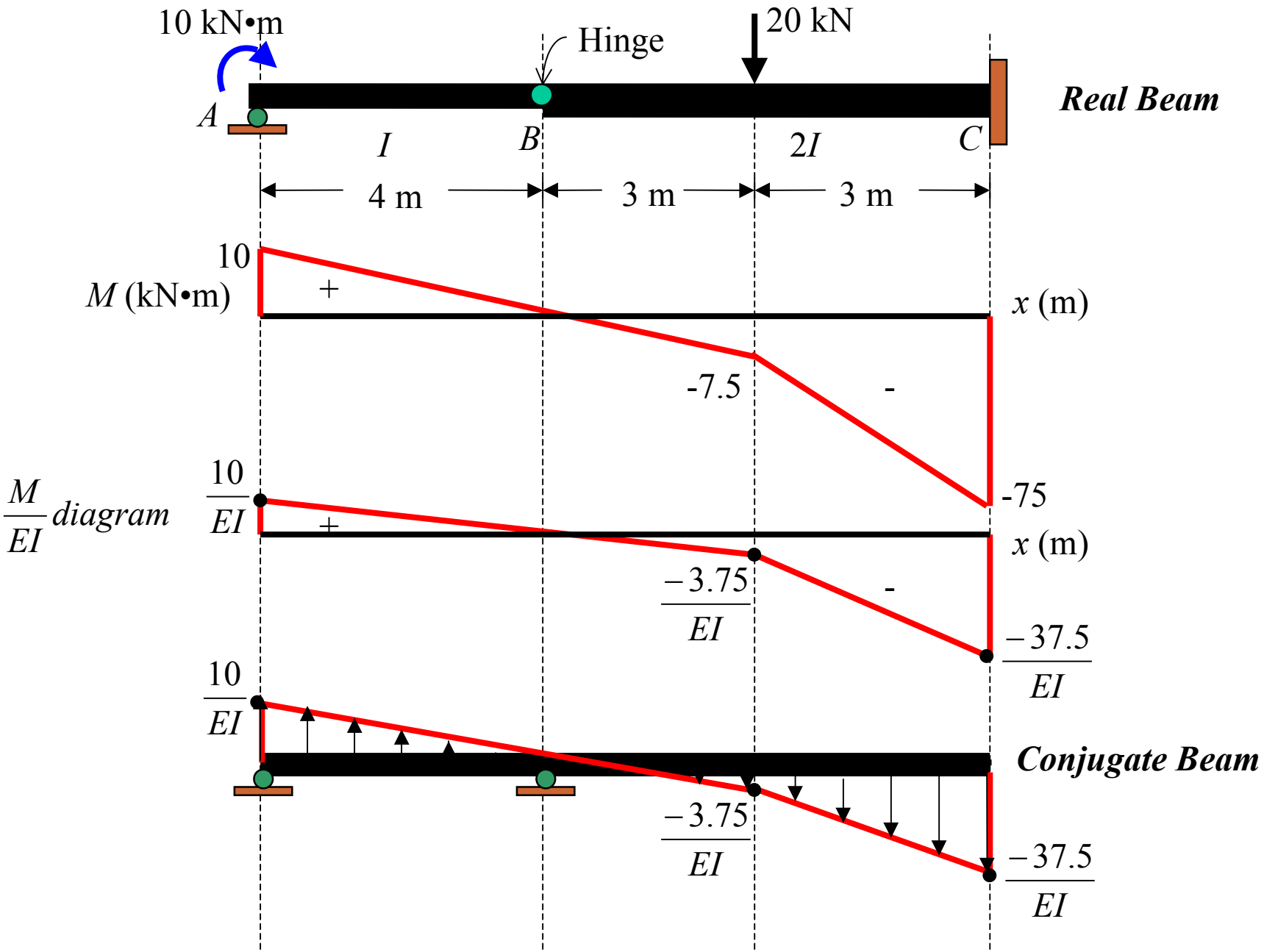
(a) Determine the deflection and slopes at point *B*.
(b) Draw shear, bending moment diagrams and deflected curve.

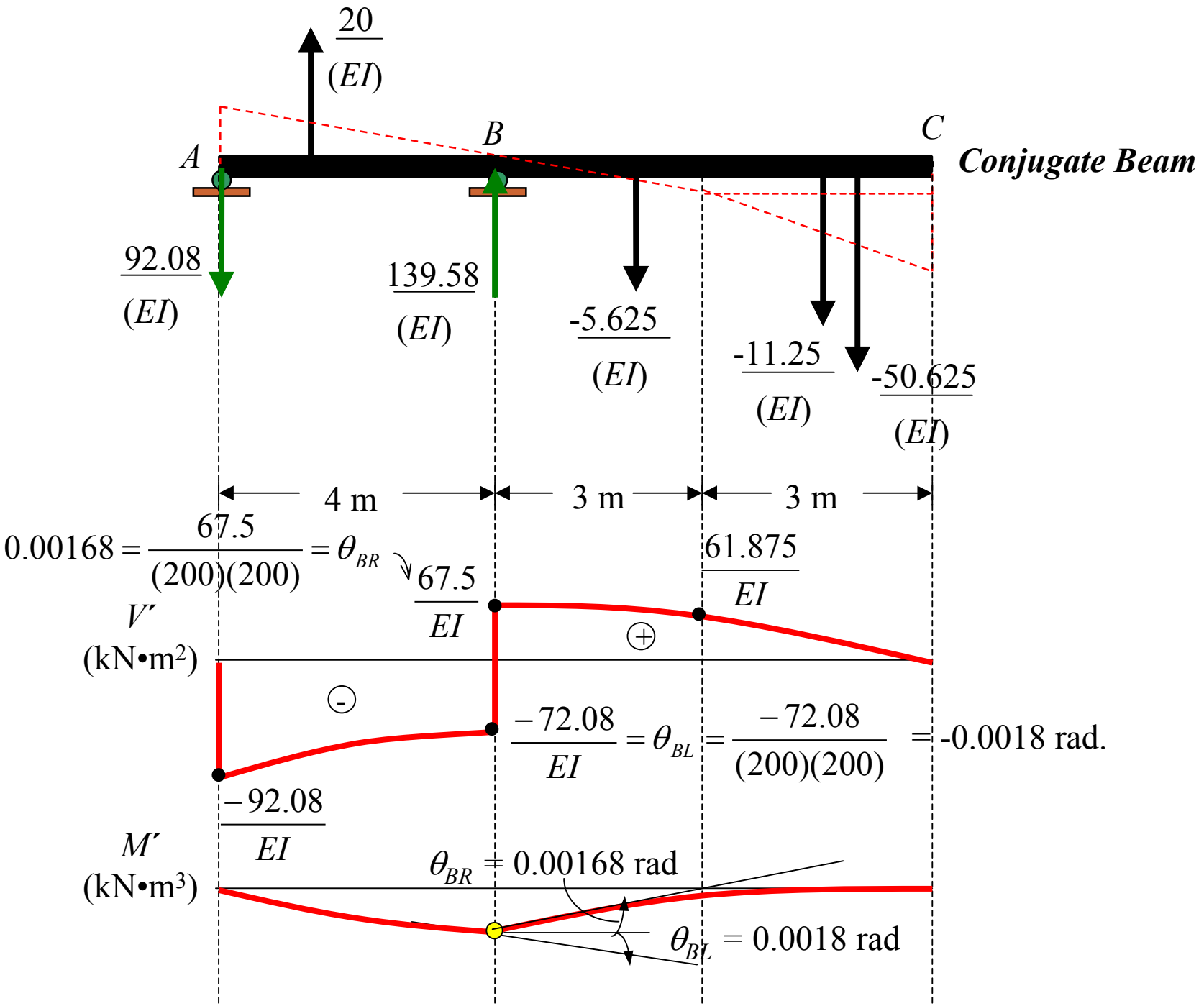
$E = 200$ GPa
$I = 200(10^6)$ mm^4



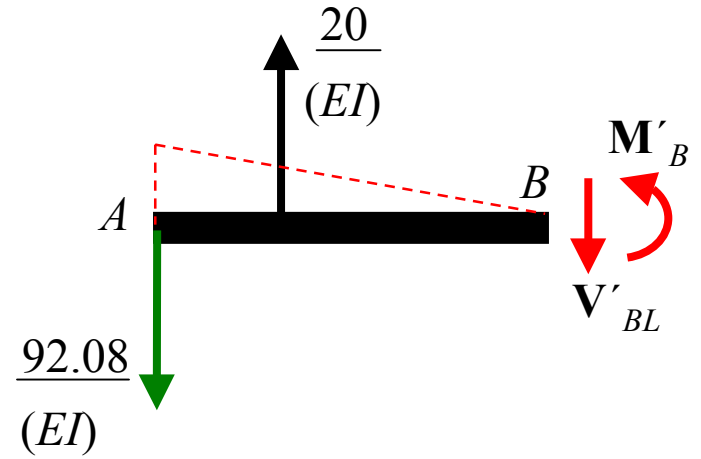
SOLUTION


10 kN•m
Hinge
20 kN
A
B
C
I
2I
4 m
3 m
3 m
10 kN•m
2.5 kN
0
20 kN
2.5 kN
0
75 kN•m
2.5 kN
22.5 kN
V (kN)
x (m)
-2.5
-2.5
-22.5
-22.5
10
M (kN•m)
x (m)
-7.5
-75
Deflected Curve


10 kN•m
Hinge
20 kN
A
B
C
I
2I
4 m
3 m
3 m
Real Beam
10
M (kN•m)
+
x (m)
-7.5
-
-75
M/EI diagram
10/EI
+
x (m)
−3.75/EI
-
−37.5/EI
10/EI
Conjugate Beam
−3.75/EI
−37.5/EI


20/(EI)
A
B
C
Conjugate Beam
92.08/(EI)
139.58/(EI)
-5.625/(EI)
-11.25/(EI)
-50.625/(EI)
4 m
3 m
3 m
0.00168 = 67.5/(200)(200) = θBR
67.5/EI
61.875/EI
V′ (kN•m²)
⊕
⊖
−72.08/EI = θBL = −72.08/(200)(200) = -0.0018 rad.
−92.08/EI
M′ (kN•m³)
θBR = 0.00168 rad
θBL = 0.0018 rad

- ***Slope before support B***

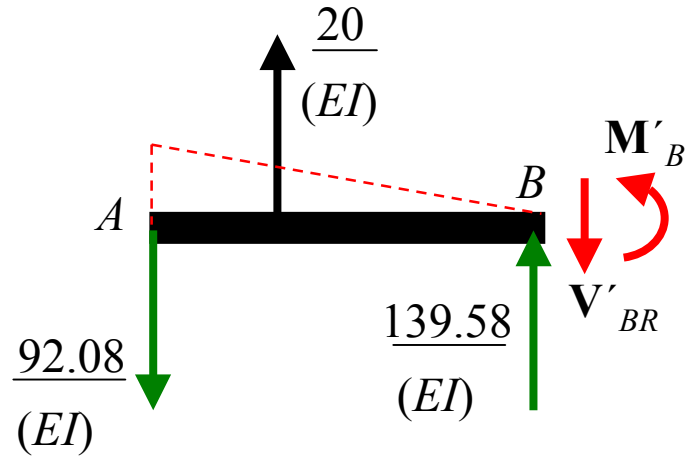


$$+\uparrow \Sigma F_y = 0: \qquad -\frac{92.08}{EI} + \frac{20}{EI} - V'_{BL} = 0$$

$$V'_{BL} = -\frac{72.08}{EI}$$

$$\theta_{BL} = V'_{BL} = -\frac{72.08}{(200)(200)} = -0.0018\ rad$$

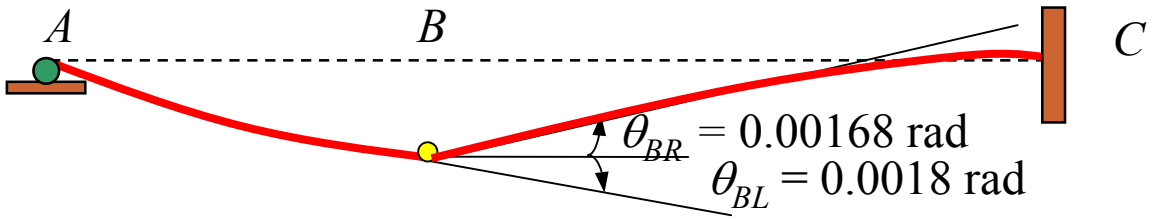
- ***Slope after support B***

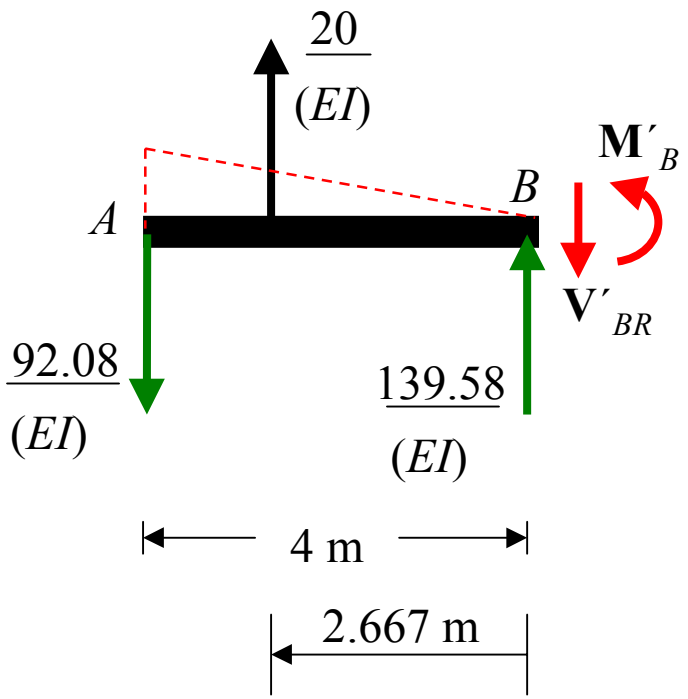


$$+\uparrow \Sigma F_y = 0: -\frac{92.08}{EI} + \frac{20}{EI} + \frac{139.58}{EI} - V'_{BR} = 0$$

$$V'_{BR} = \frac{67.5}{EI}$$

$$\theta_{BR} = V'_{BR} = \frac{67.5}{(200)(200)} = 0.00168\ rad$$





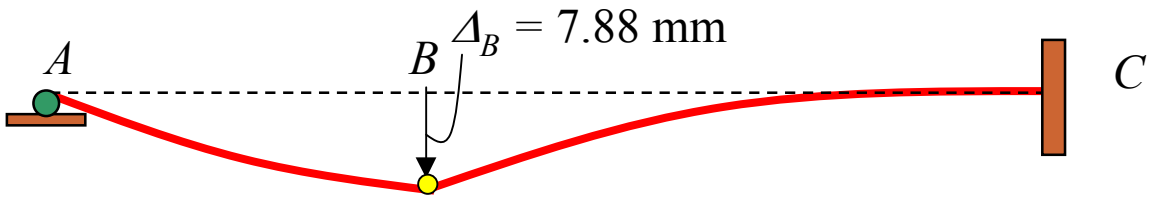
- ***Deflection at point B***

$$+\circlearrowleft \Sigma M_B = 0:$$

$$M'_B - \frac{20}{EI}(2.667) + \frac{92.08}{EI}(4) = 0$$

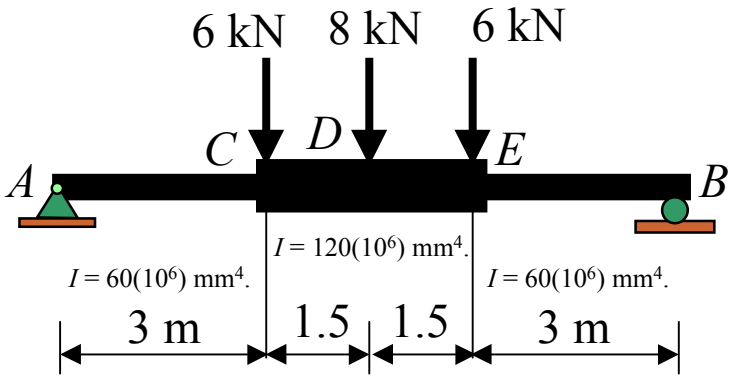
$$M'_B = -\frac{315}{EI}$$

$$M'_B = -\frac{315}{EI} = -\frac{315}{(200)(200)} = -7.88\ mm$$



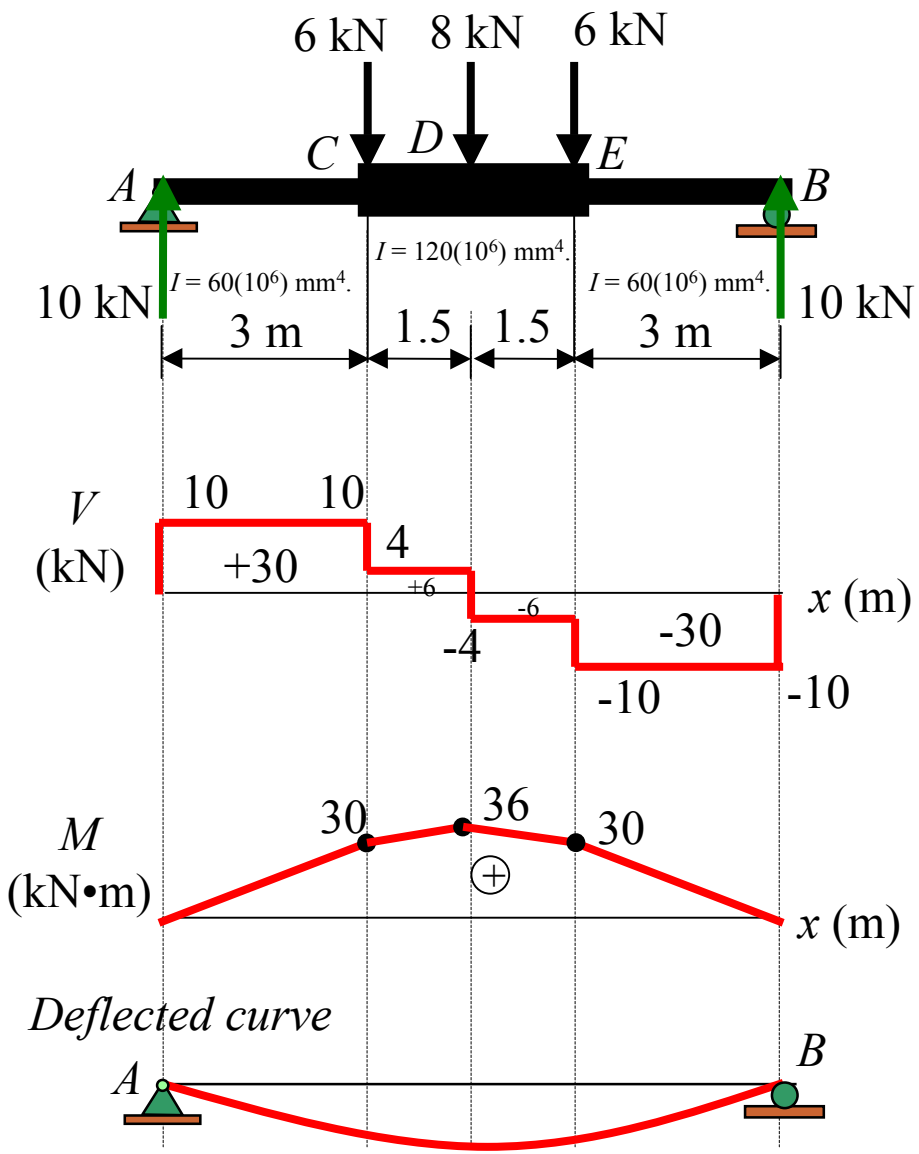
Example 8-15

The girder in the figure below is made from a continuous beam and reinforced at its center with cover plates where its moment of inertia is larger. The 3 -m end segments have a moment of inertia of $I = 60(10^6)\ \text{mm}^4$, and the center portion has a moment of inertia of $I' = 120(10^6)\ \text{mm}^4$. Determine the slope and deflection at point C, the maximum deflection. Take $E = 200$ GPa.

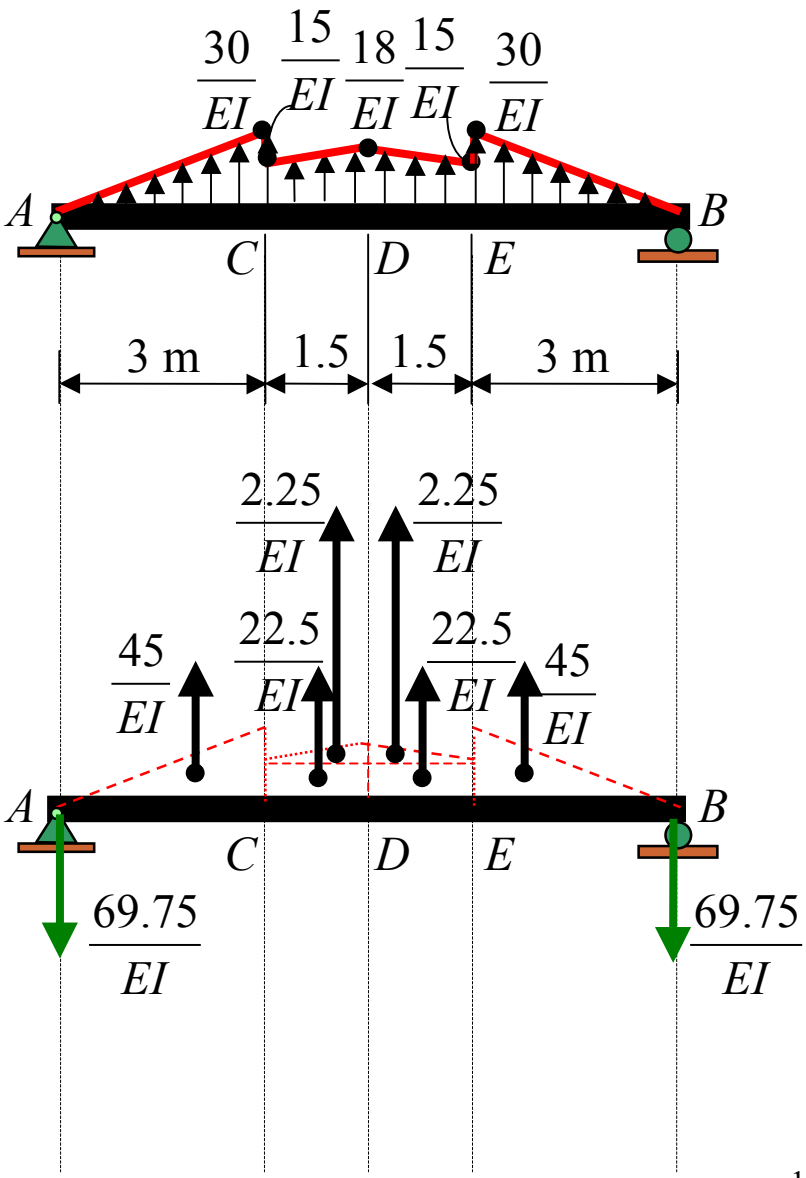


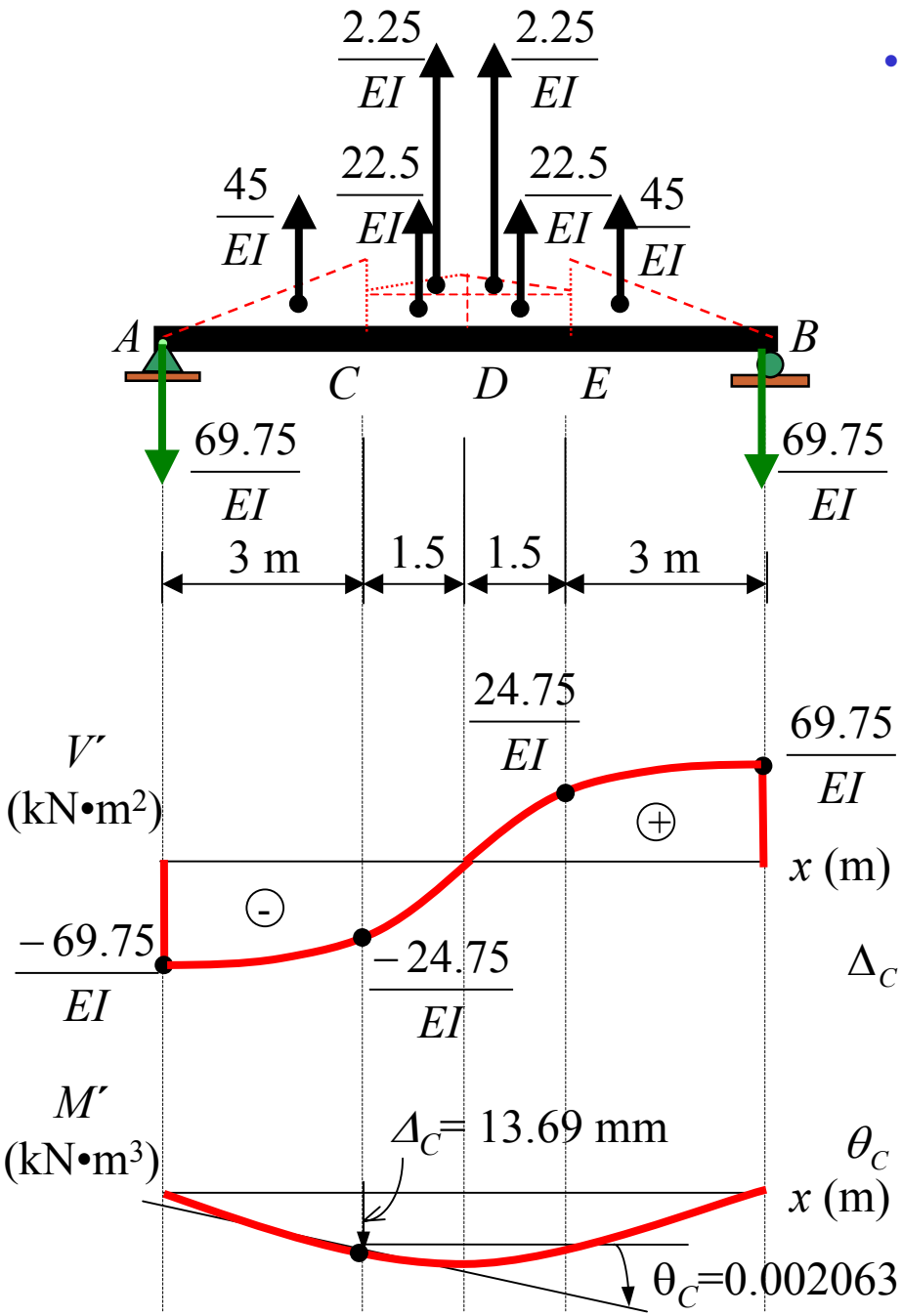
SOLUTION

Real Beam

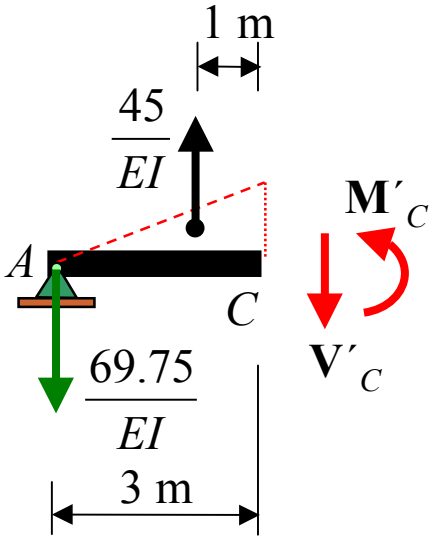


Conjugate Beam





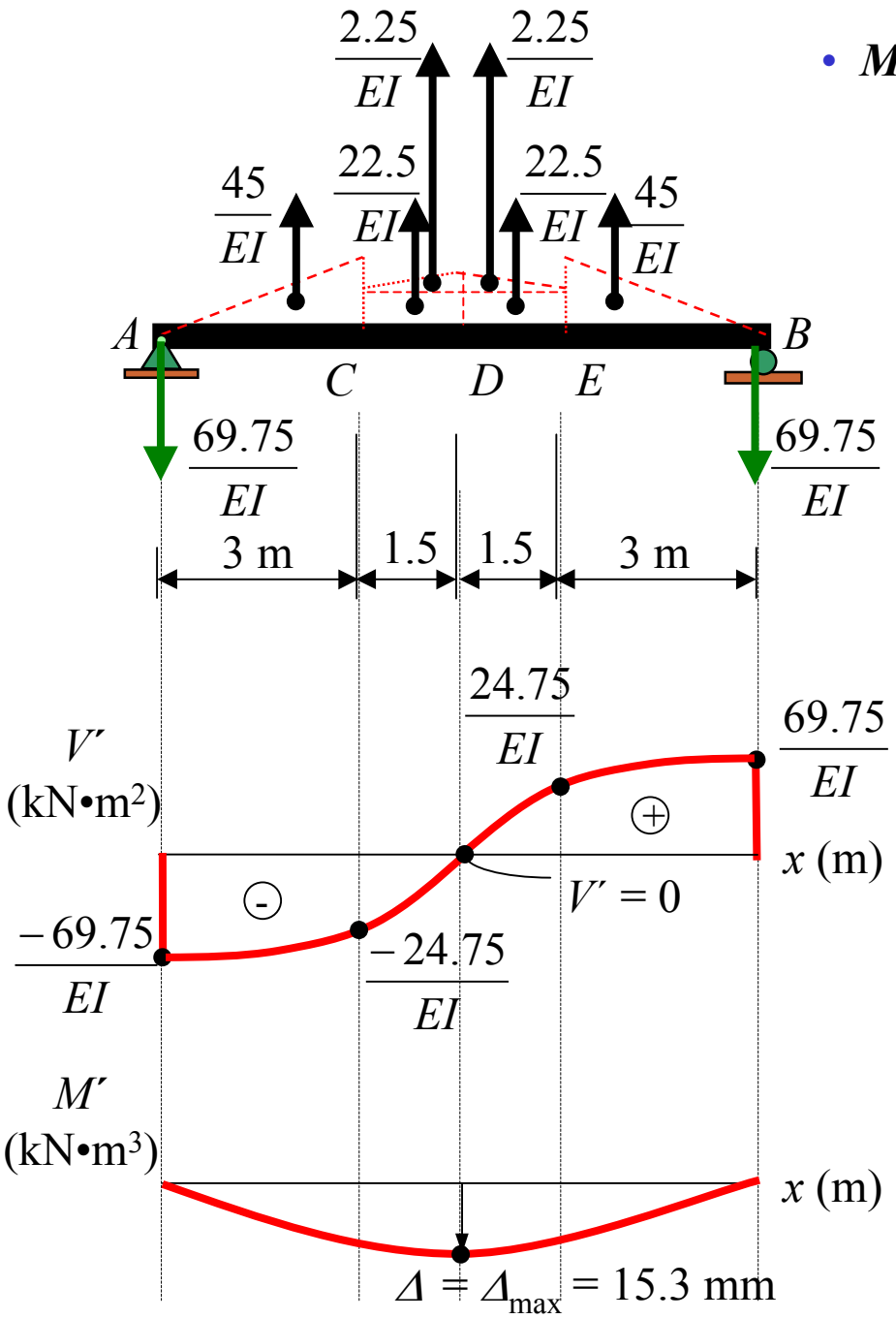
- ***Slope and deflection at point C***



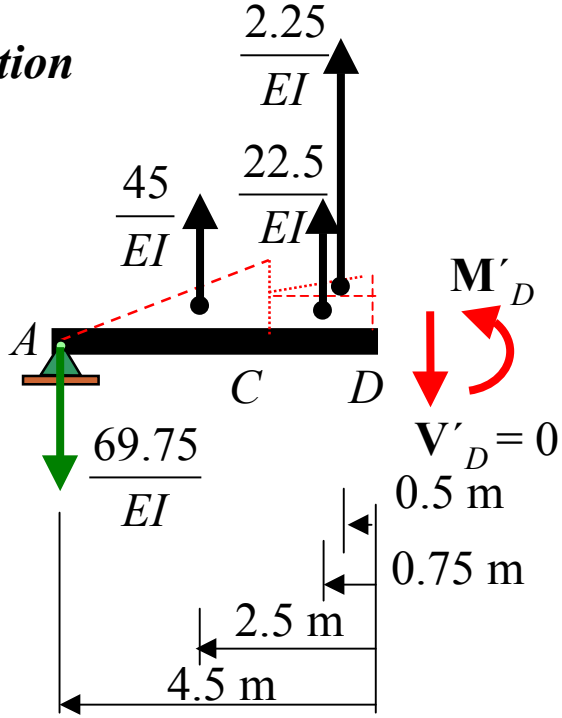
$$+\circlearrowleft \Sigma M_C = 0: \quad M'_C - \frac{45}{EI}(1) + \frac{69.75}{EI}(3) = 0$$

$$\Delta_C = M'_C = -\frac{164.25}{EI} = -\frac{164.25}{(200)(60)} = \text{-13.69 mm}$$

$$\theta_C = V'_C = \frac{-24.75}{EI} = \frac{-24.75}{(200)(200)} = \text{-0.002063 rad}$$



- ***Maximum deflection***



$+\circlearrowleft \Sigma M_D = 0$:

$$M'_D - \frac{2.25}{EI}(0.5) - \frac{22.5}{EI}(0.75)$$

$$-\frac{45}{EI}(2.5) + \frac{69.75}{EI}(4.5) = 0$$

$$M'_D = -\frac{183.375}{EI}$$

$$\Delta_{max} = \Delta_D = M'_D = -\frac{183.375}{(200)(60)} = \text{-15.3 mm}$$