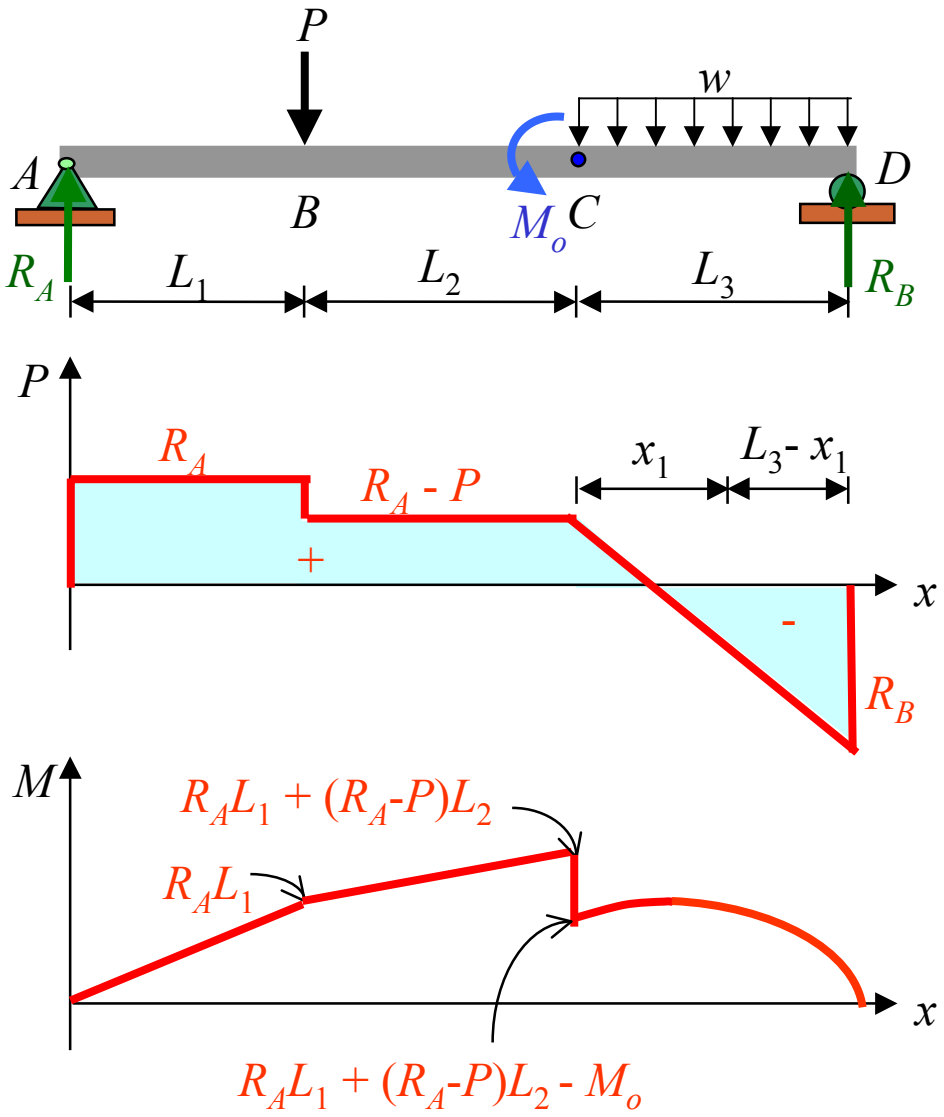
BENDING

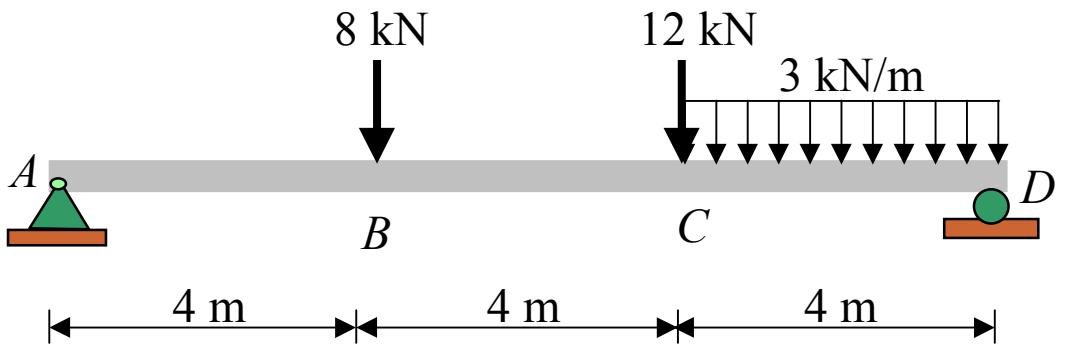
- Shear and moment diagrams
- Bending Deformation of a Straight Member
- The Flexure Formula
- Unsymmetric Bending
- Composite beams
- Reinforced Concrete Beams

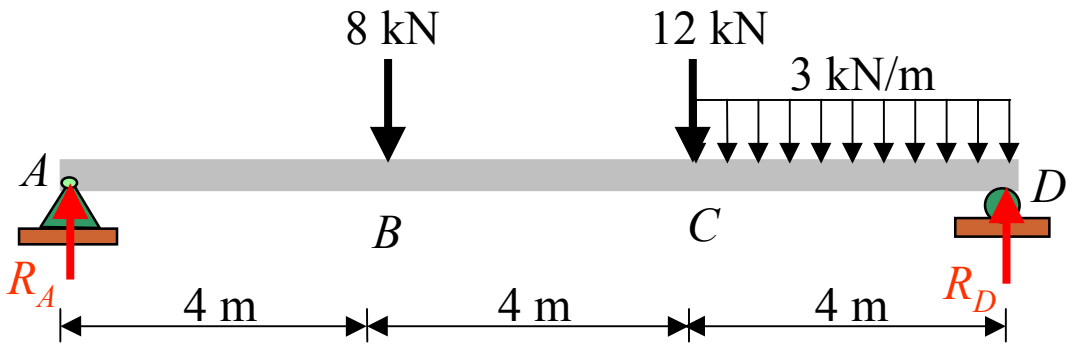
Shear and bending moment diagram



Example 1

Draw the shear and moment diagram for the beam shown.





$+\circlearrowleft \Sigma M_A = 0$:

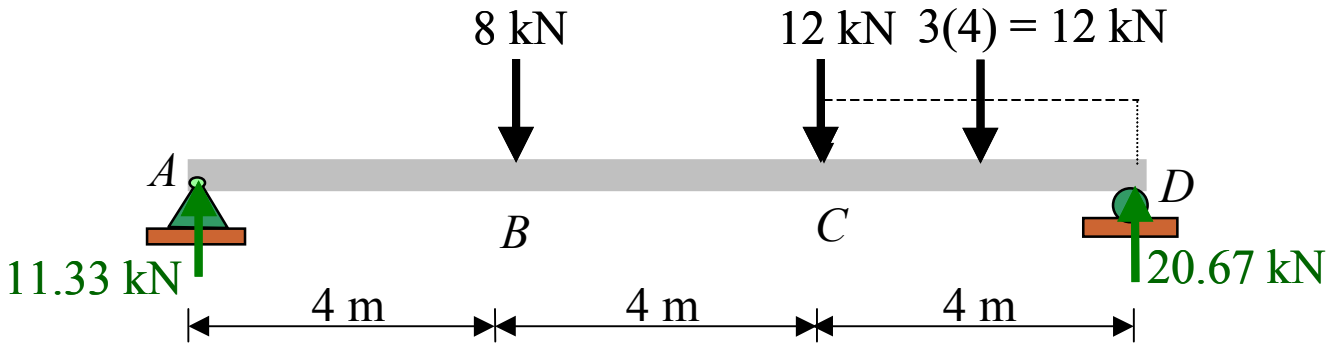
$$-8(4) -12(8) - (3x4)(10) + R_D(12) = 0$$

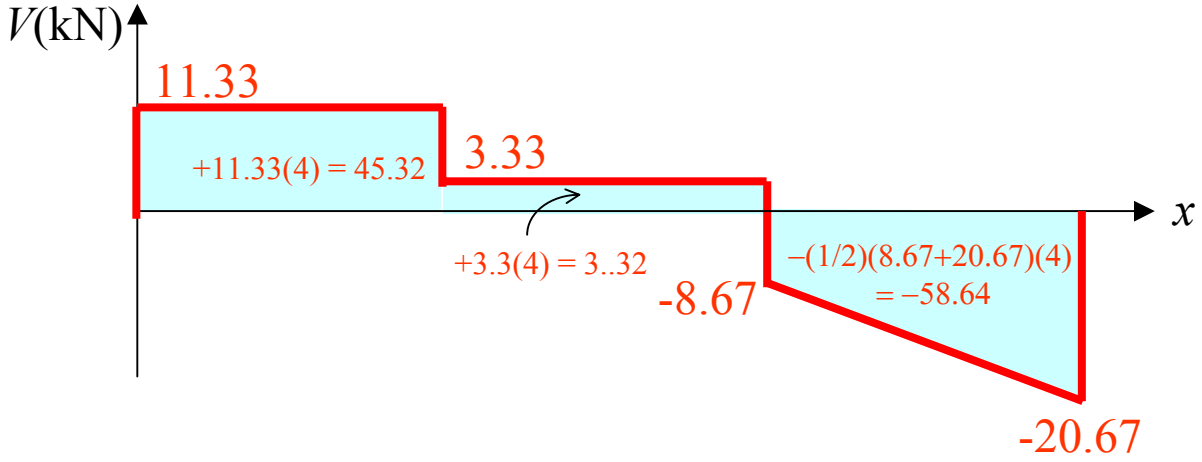
$$R_D = 20.67 \text{ kN}$$

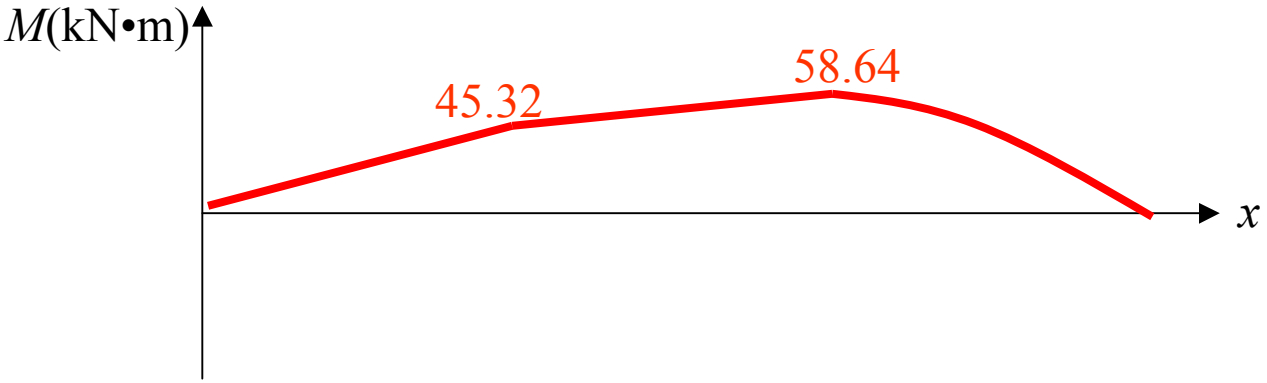
$+\uparrow \Sigma F_y = 0$:

$$R_A - 8 -12 - (3x4) + 20.67 = 0$$

$$R_A = 11.33 \text{ kN}$$

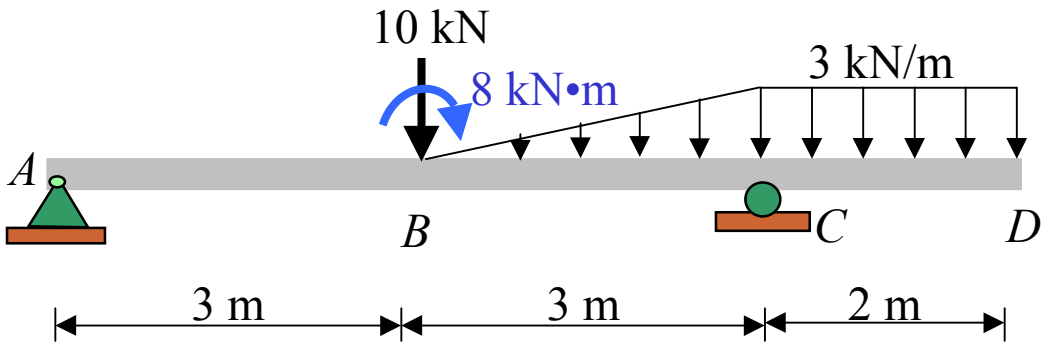

8 kN
12 kN
3(4) = 12 kN
A
B
C
D
11.33 kN
20.67 kN
4 m
4 m
4 m

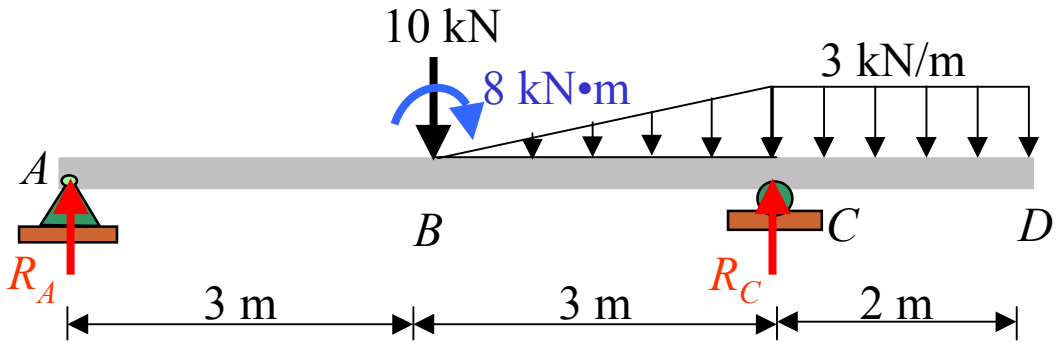

V(kN)
11.33
+11.33(4) = 45.32
3.33
+3.3(4) = 3..32
-8.67
−(1/2)(8.67+20.67)(4)
= −58.64
-20.67
x


M(kN•m)
45.32
58.64
x

Example 2

Draw the shear and moment diagram for the beam shown.





$+\circlearrowleft \Sigma M_A = 0$:

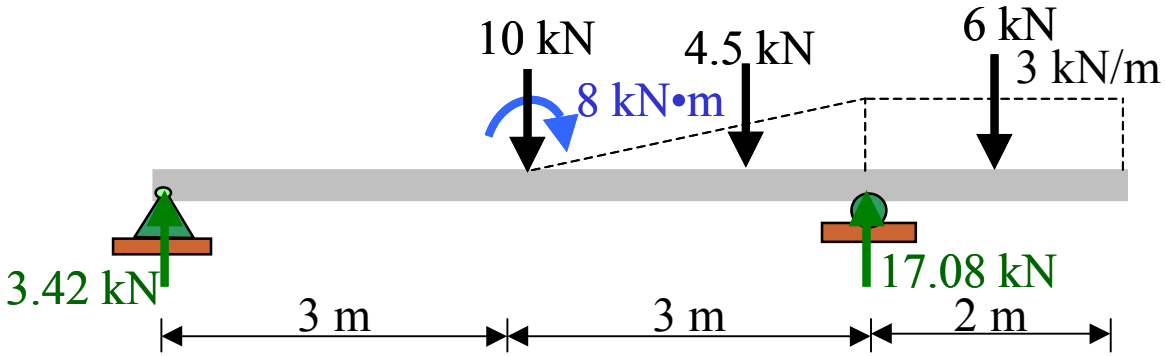
$$-10(3) - 8 - (0.5x3x3)(3+(2/3)x3) + R_C(6) - (3x2)(7) = 0$$

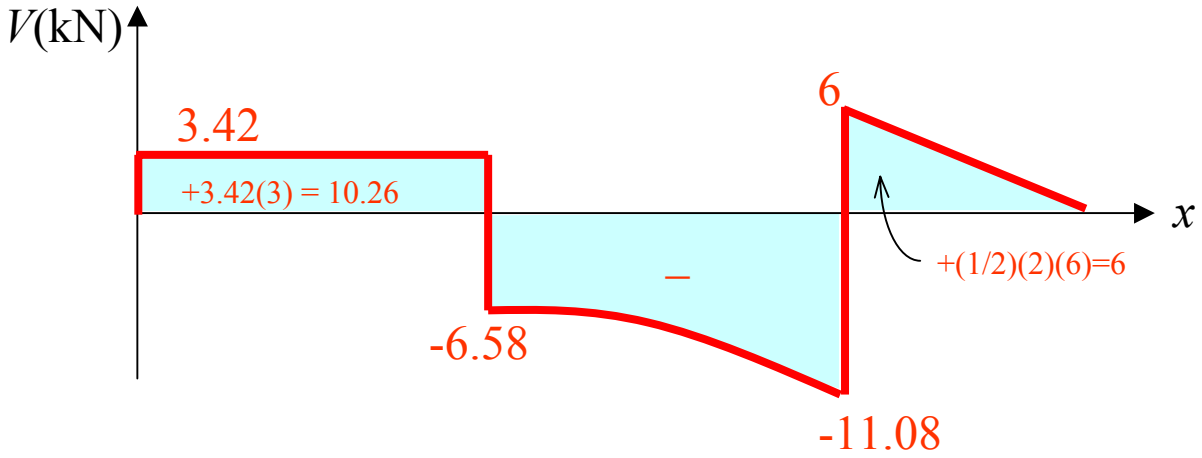
$$R_C = 17.08 \text{ kN}$$

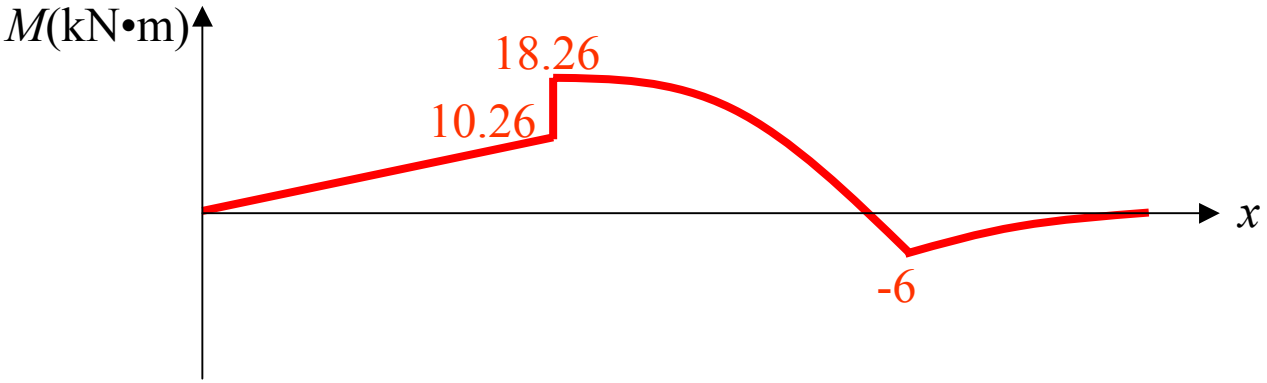
$+\uparrow \Sigma F_y = 0$:

$$R_A - 10 - (0.5x3x3) + 17.08 - (3x2) = 0$$

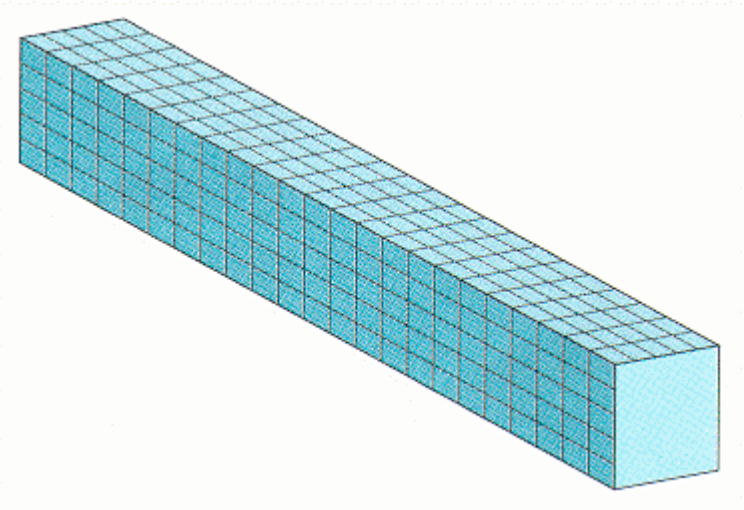
$$R_A = 3.42 \text{ kN}$$


10 kN
4.5 kN
6 kN
3 kN/m
8 kN•m
3.42 kN
17.08 kN
3 m
3 m
2 m

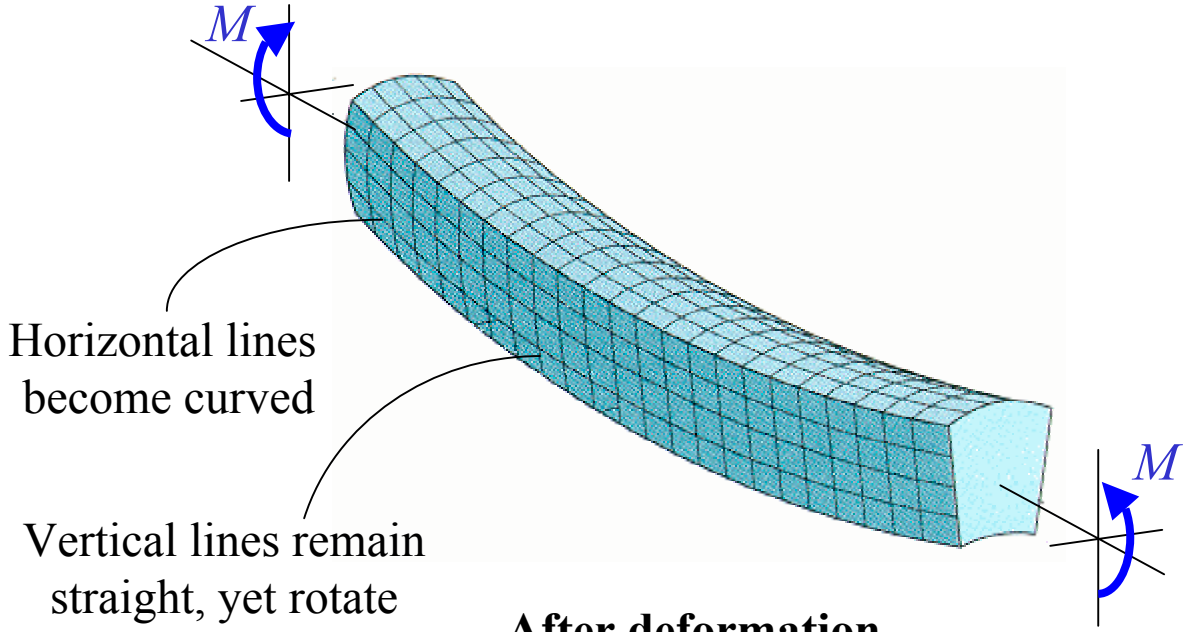

V(kN)
3.42
+3.42(3) = 10.26
6
+(1/2)(2)(6)=6
x
-6.58
-11.08


M(kN•m)
18.26
10.26
x
-6

Bending deformation of A Straight Member

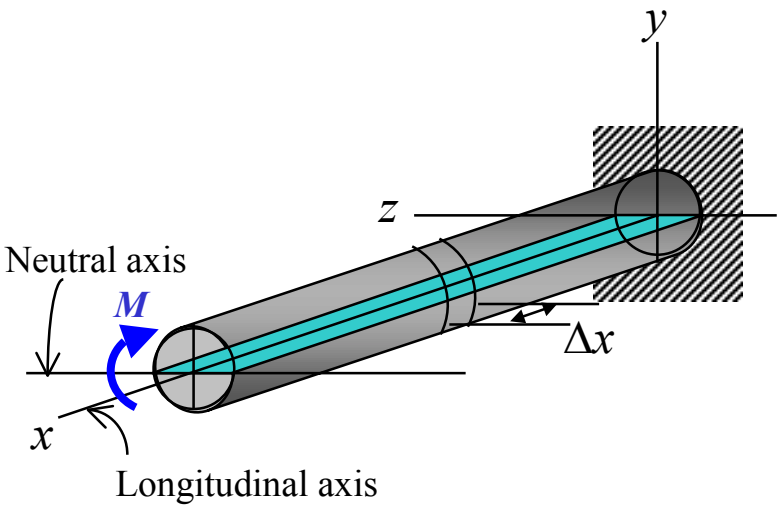


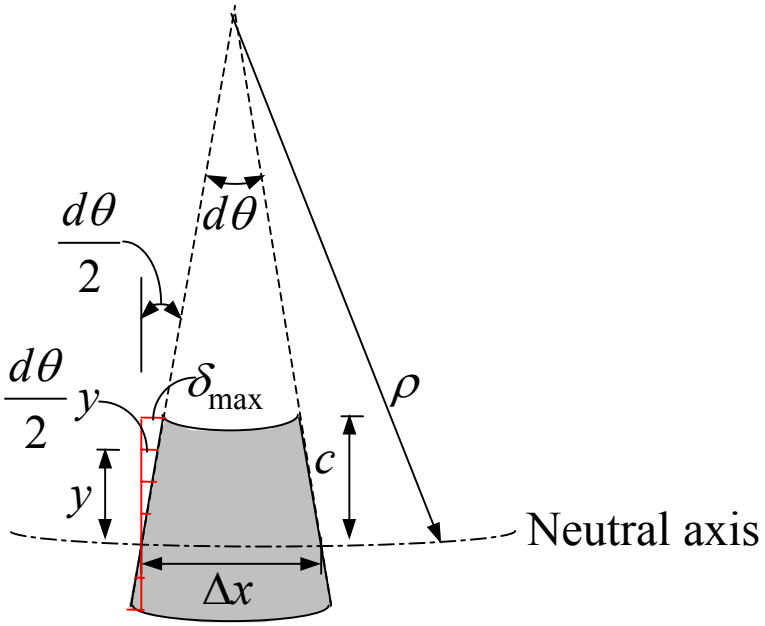
Before deformation



After deformation

- **Strain (ε)**





$$\varepsilon_{max} = \frac{-2\delta_{max}}{\Delta x}$$

$$= \frac{-2(\frac{d\theta}{2})c}{\rho d\theta}$$

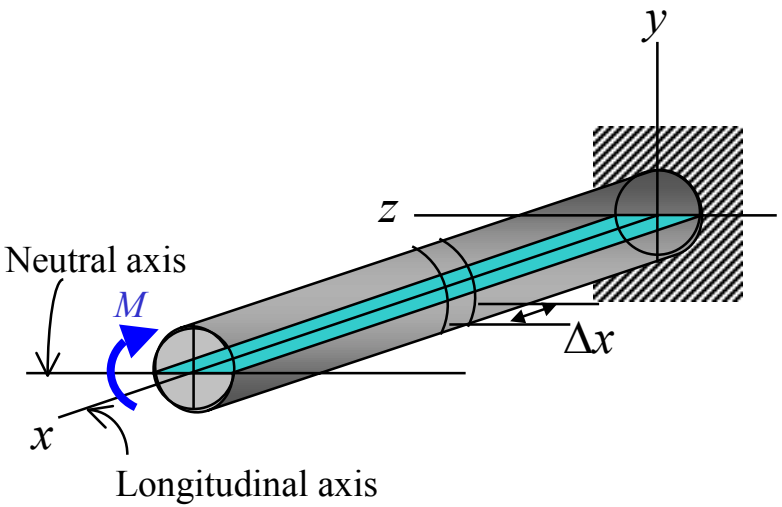
$$\varepsilon_{max} = \frac{-c}{\rho}$$

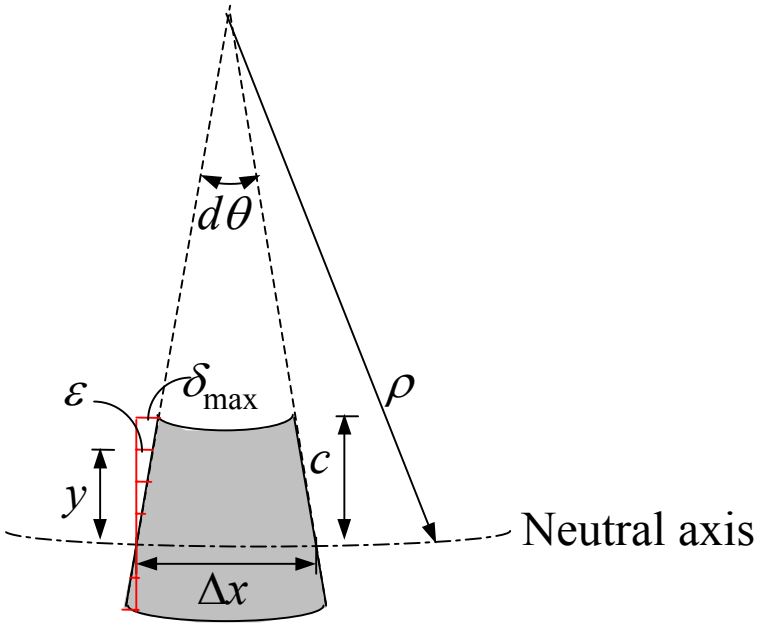
$$\frac{\varepsilon_{max}}{\varepsilon} = \frac{c}{y}$$

$$\varepsilon = \varepsilon_{max}\frac{y}{c}$$

$$\varepsilon = \frac{-c}{\rho}(\frac{y}{c})$$

$$\varepsilon = \frac{-y}{\rho}$$





$$\frac{\varepsilon}{\varepsilon_{\max}} = \frac{-y}{c}$$

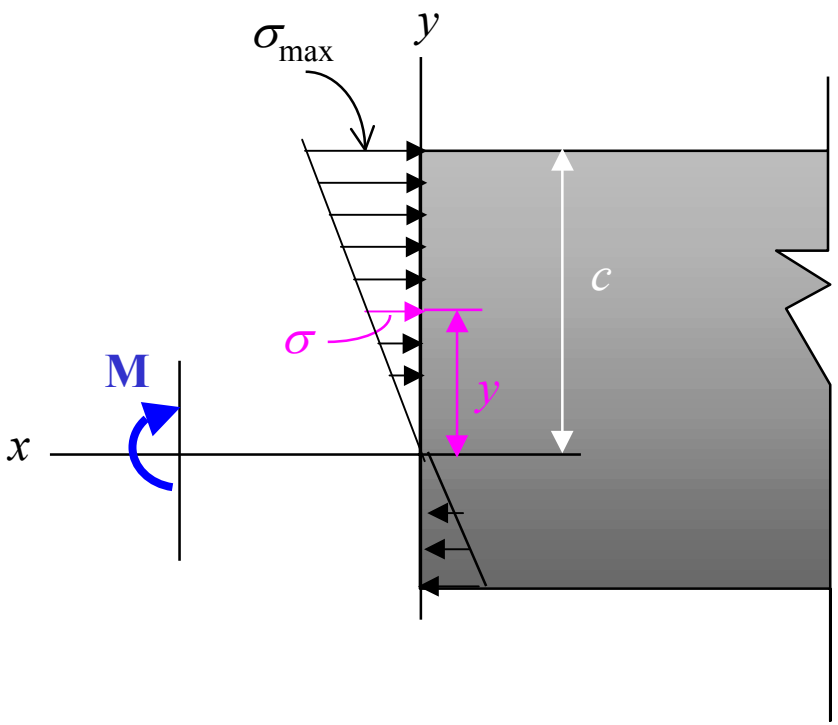
$$\varepsilon = \frac{-y}{c}\varepsilon_{\max}$$

Notes :

$\varepsilon = f(y)$

$\frac{\varepsilon_{\max}}{c} = \text{constant}$

The Flexure formula



$$\left|\frac{\sigma}{\sigma_{max}}\right| = \left|\frac{y}{c}\right|$$

$$|\sigma| = \left|y\,\frac{\sigma_{max}}{c}\right|$$

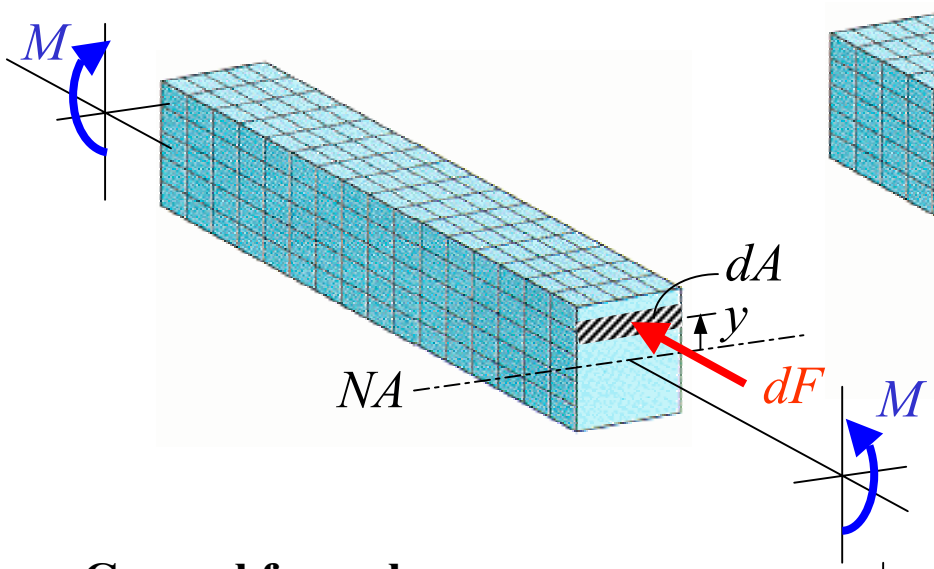
For positive bending moment M:

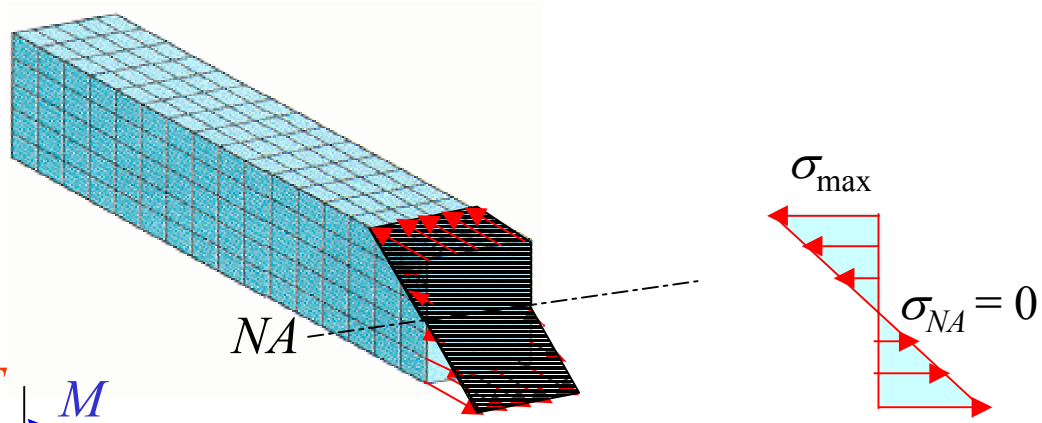
$$\sigma = -y\left|\frac{\sigma_{max}}{c}\right|$$

Note :

$\sigma = f(y)$

$\frac{\sigma_{max}}{c}$ = constant





Bending Stress diagram

- **General formula**

$$dM = ydF = y(\sigma dA)$$

$$M = \int_A y(-\frac{y}{c}\sigma_{\max})dA$$

$$= -\frac{\sigma_{\max}}{c}\int_A y^2 dA$$

$$= -\frac{\sigma_{\max}}{c} I_{\bar{x}} \quad = -\frac{\sigma}{y} I_{\bar{x}}$$

$$\sigma = -\frac{My}{I}$$

σ is compression when $+M$

- **Neutral axis (*NA*) : A definition**

$$\sum F_x = 0 = \int dF$$

$$0 = \int_A \sigma dA$$

$$0 = \int_A (-\frac{y}{c}\sigma_{\max})\sigma dA$$

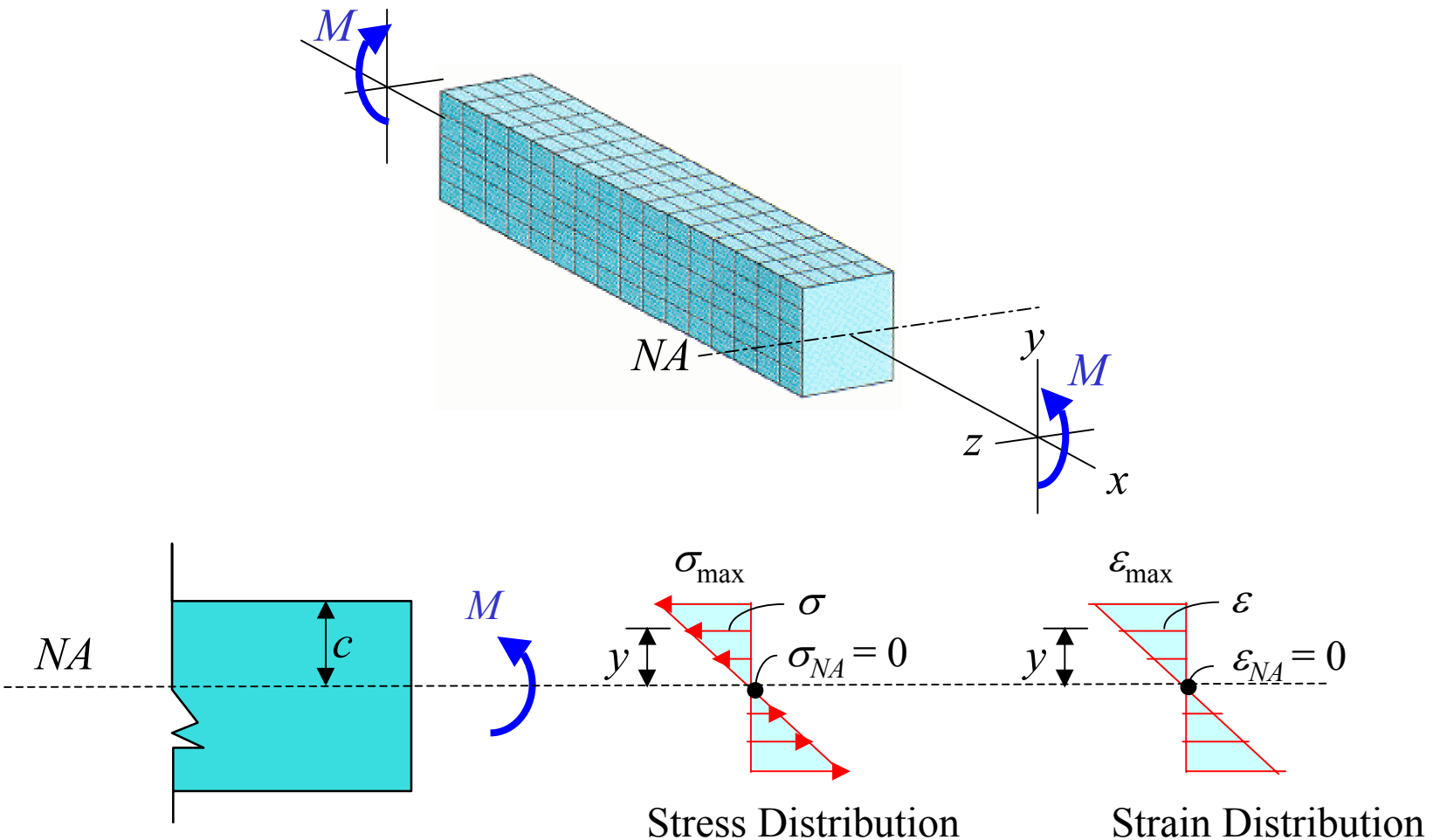
$$0 = -\frac{\sigma_{\max}}{c}\int_A y\sigma dA$$

$$\int_A y\sigma dA = 0$$

$$\bar{y}A = 0; \quad \bar{y} = 0, A \neq 0$$

Pure Bending : $\sigma = 0$ at neutral axis

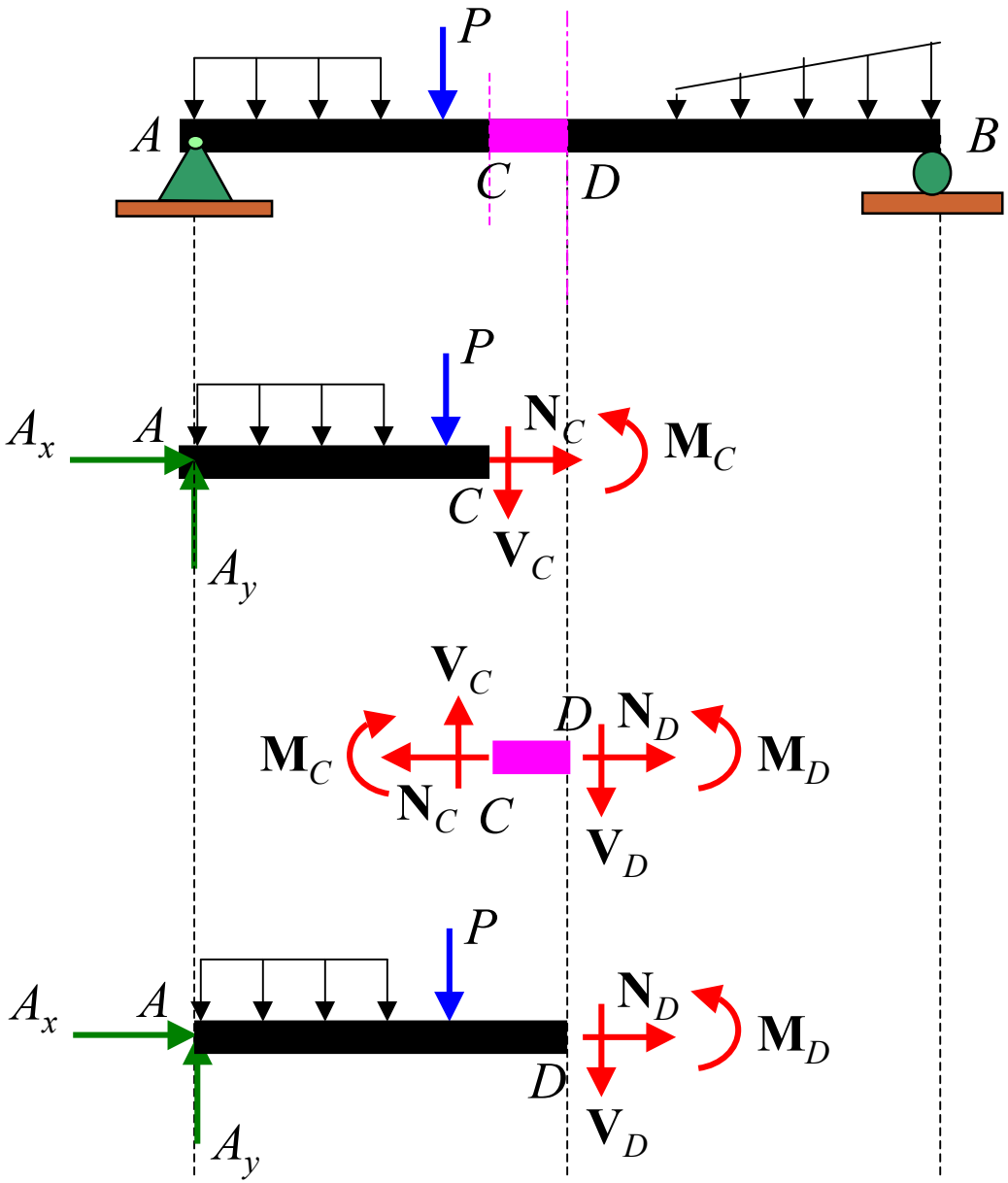
- ***Stress* and *Strain Distribution* for Pure Bending**



- $\dfrac{\sigma}{\sigma_{\max}} = \dfrac{y}{c}$

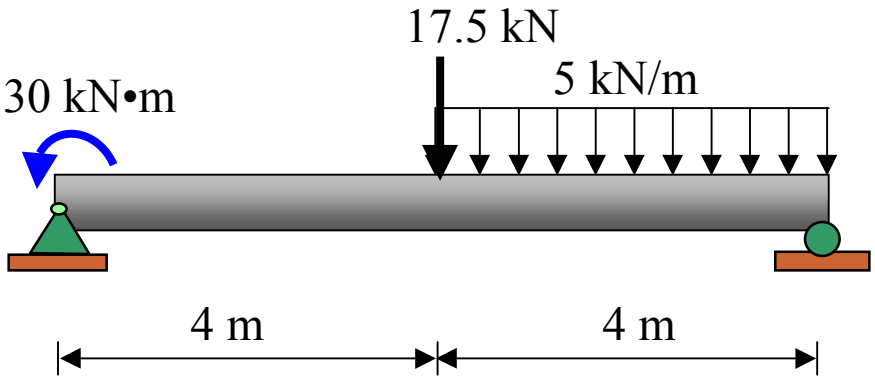
- $\dfrac{\varepsilon}{\varepsilon_{\max}} = \dfrac{y}{c}$

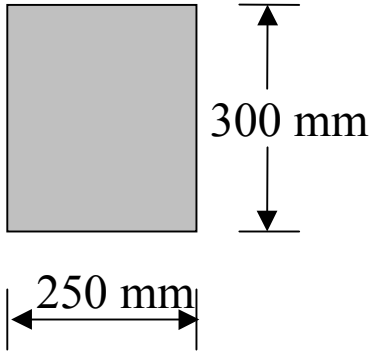
Beam


P
A
B
C
D
P
A_x
A
$\mathbf{N}_C$
$\mathbf{M}_C$
C
$\mathbf{V}_C$
A_y
$\mathbf{V}_C$
D
$\mathbf{N}_D$
$\mathbf{M}_C$
$\mathbf{M}_D$
$\mathbf{N}_C$
C
$\mathbf{V}_D$
P
A_x
A
$\mathbf{N}_D$
$\mathbf{M}_D$
D
$\mathbf{V}_D$
A_y

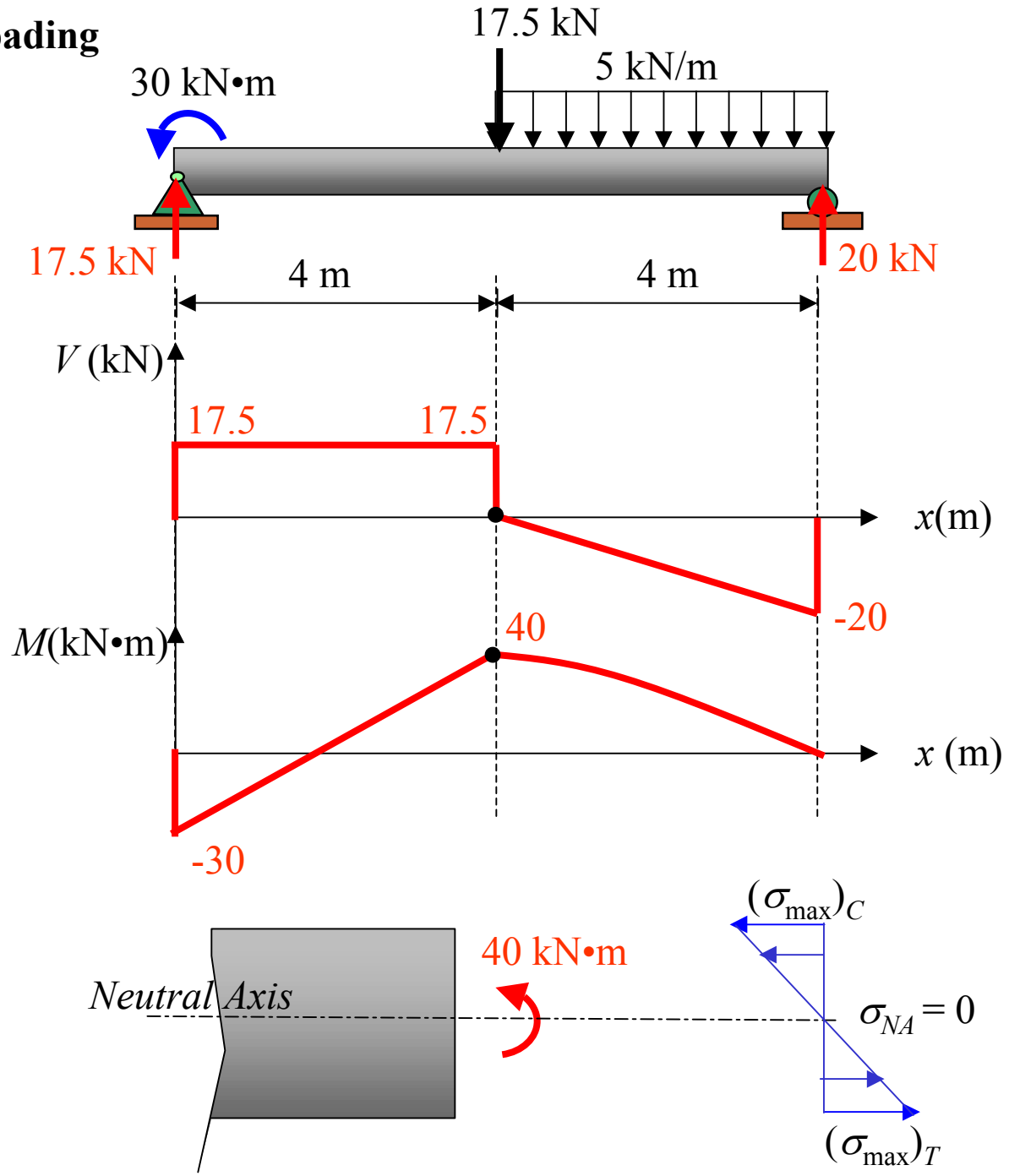
Example 3

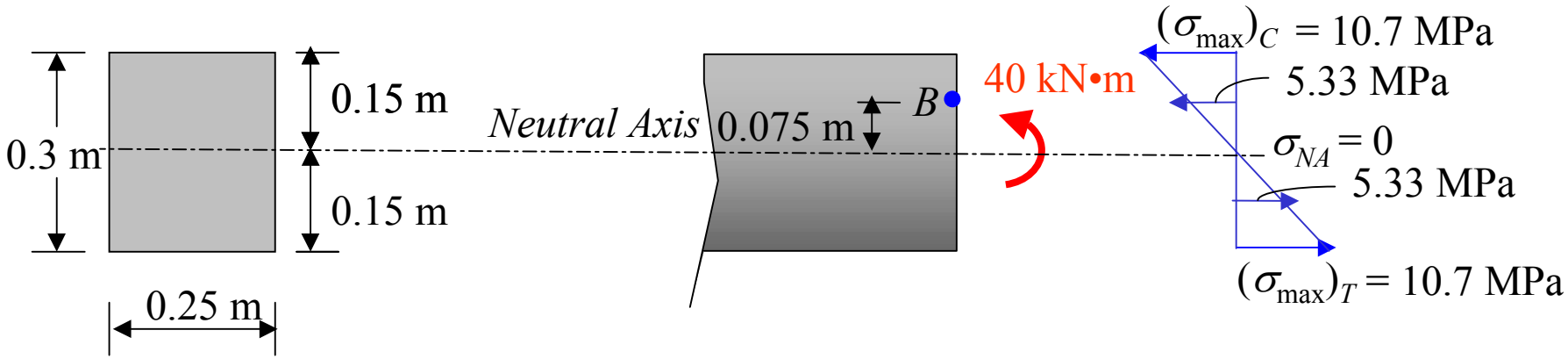
The simply supported beam has the cross-sectional area shown. Determine the absolute maximum bending stress in the beam and draw the stress distribution over the cross section at this location.





- **Internal Loading**





- **Section Property**

$$I = \sum (\bar{I} + Ad^2) = \frac{1}{12}(0.25\ \text{m})(0.3\ \text{m})^3 + 0 = 0.0005625\ \ \text{m}^4$$

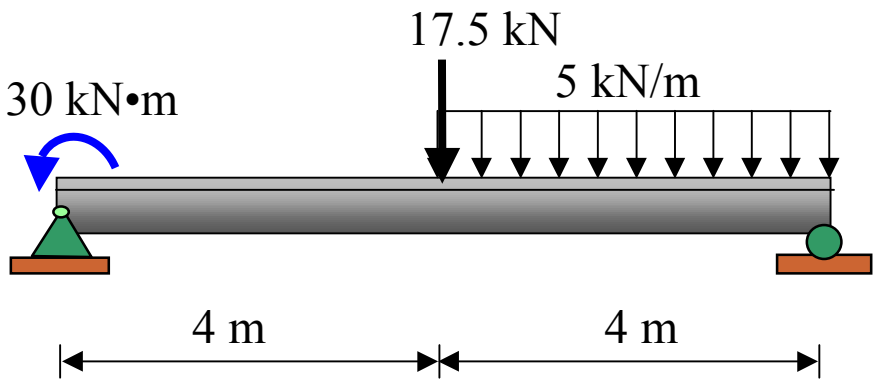
- **Bending Stress**

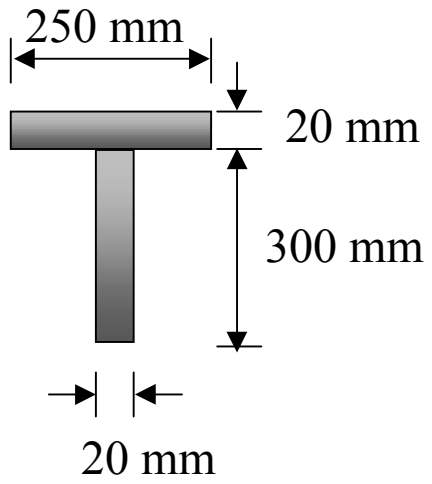
$$\sigma_{\max} = \frac{Mc}{I} = \frac{(40\ \text{kN}\bullet\text{m})(0.15\ \text{m})}{(.5625 \times 10^{-3}\ \text{m}^4)} = 10.7\ \text{MPa (C)} \Leftarrow$$

$$\sigma_B = \frac{-My_B}{I} = \frac{-(40\ \text{kN}\bullet\text{m})(0.075\ \text{m})}{(.5625 \times 10^{-3}\ \text{m}^4)} = -5.33\ \text{MPa (C)} \Leftarrow$$

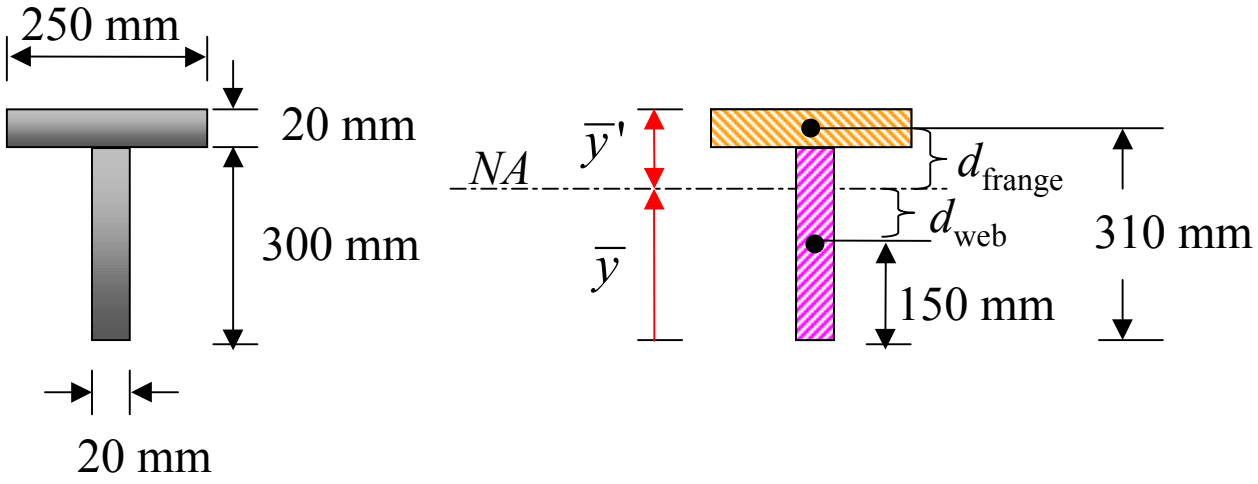
Example 4

The simply supported beam has the cross-sectional area shown. Determine the absolute maximum bending stress in tension and also in compression in the beam, and draw the stress distribution over the cross section at the mid-span.





- **Section property**



$$\bar{y} = \frac{\sum \bar{y} A}{\sum A} = \frac{(150)(300 \times 20) + (310)(250 \times 20)}{(300 \times 20) + (250 \times 20)} = 222.7 \text{ mm}$$

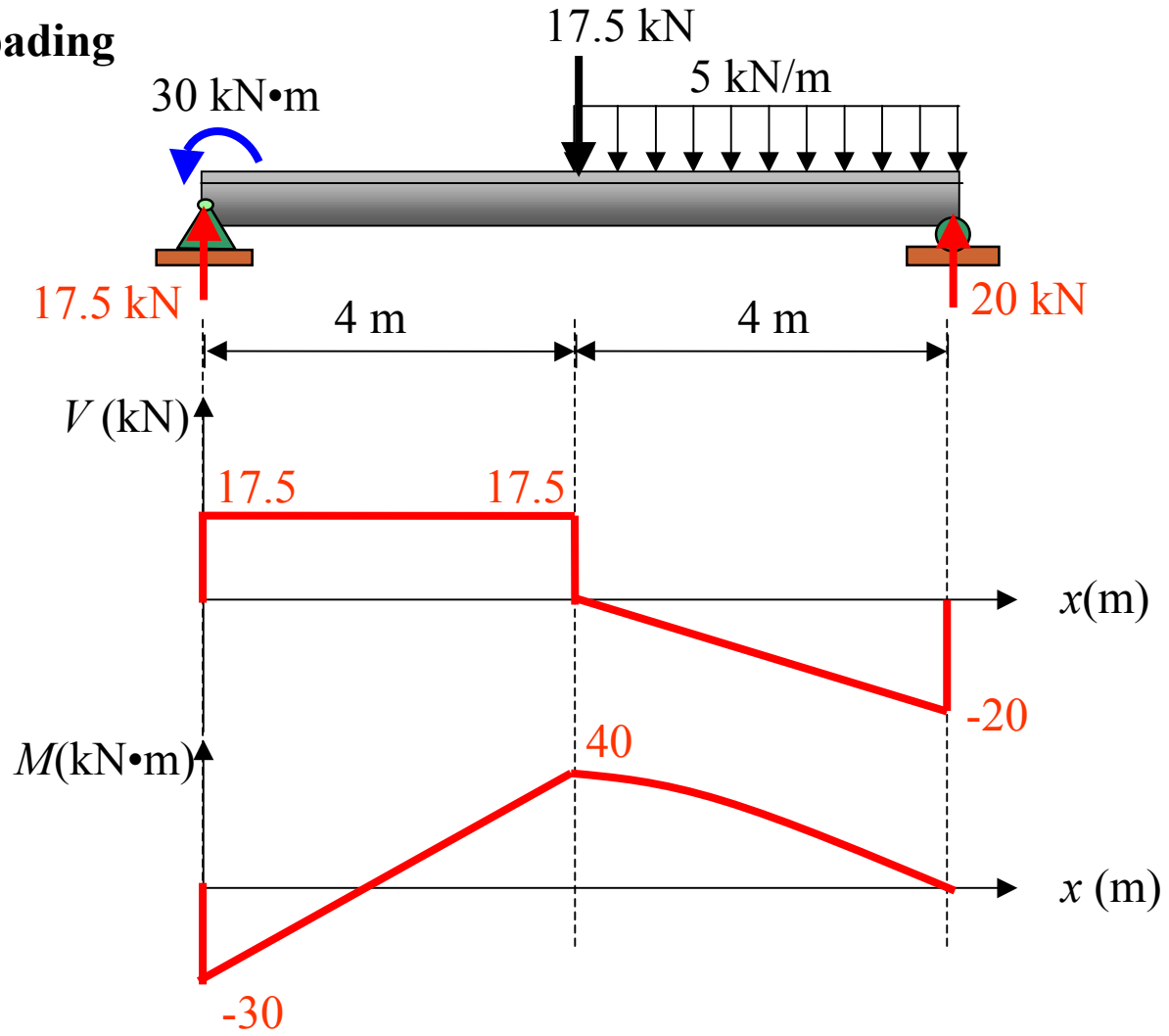
$$\bar{y}' = 320 - 222.7 = 97.3 \text{ mm}$$

$$I_{NA} = \Sigma (\bar{I} + Ad^2)$$

$$= [\frac{1}{12}(20)(300)^3 + (20 \times 300)(222.7 - 150)^2] + [\frac{1}{12}(250)(20)^3 + (250 \times 20)(310 - 222.7)^2]$$

$$= .1150 \text{x} 10^9 \text{ mm}^4 = .1150 \text{x} 10^{-3} \text{ m}^4$$

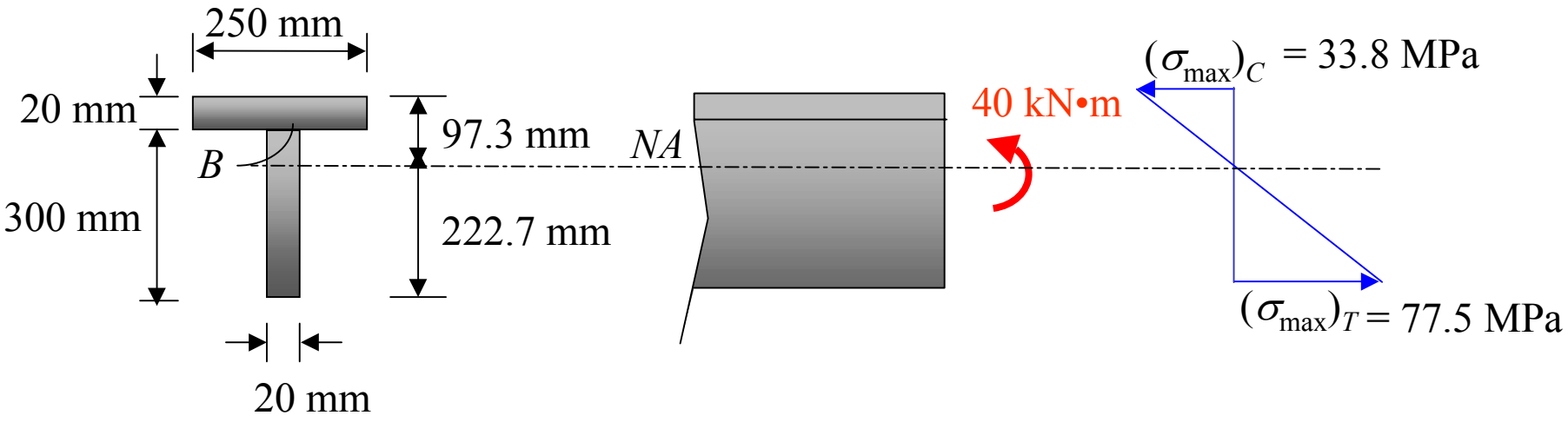
- **Internal Loading**



The positive maximum bending $M+$ = 40 kN•m

The maximum negative bending $M-$ = -30 kN•m

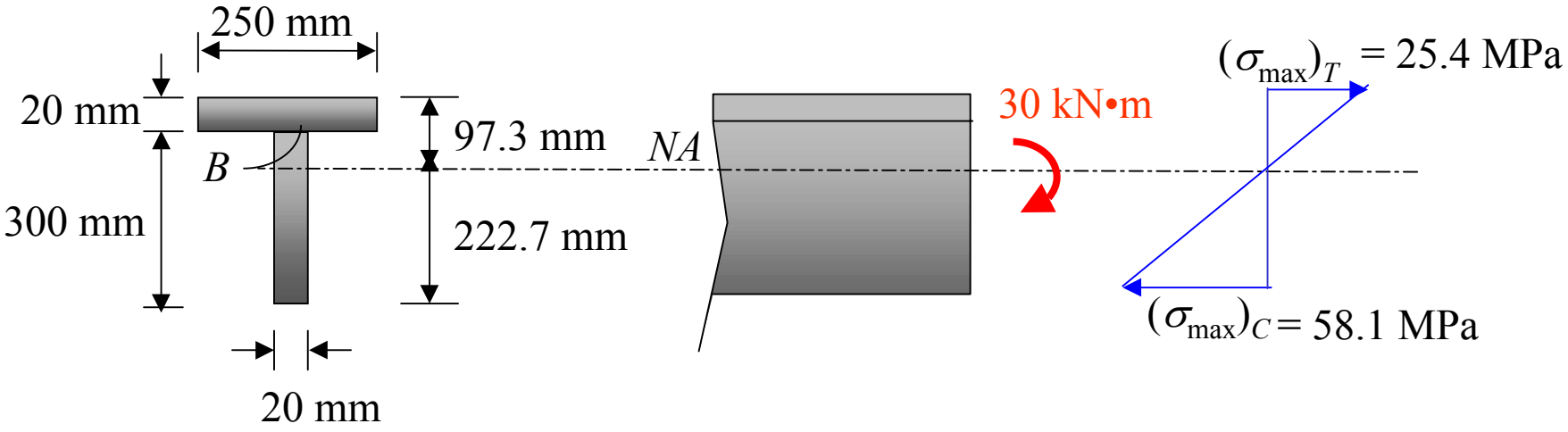
- **The Bending Stress in the Maximum Bending in Tension (M = 40 kN•m)**



$$(\sigma_{max})_C = \frac{Mc_1}{I} = \frac{(40\ \text{kN}\bullet\text{m})(0.0973\ \text{m})}{(.115\times10^{-3}\ \text{m}^4)} = 33.8\ \text{MPa (C)}$$

$$(\sigma_{max})_T = \frac{Mc_2}{I} = \frac{(40\ \text{kN}\bullet\text{m})(0.2227\ \text{m})}{(.115\times10^{-3}\ \text{m}^4)} = 77.5\ \text{MPa}\ (T)$$

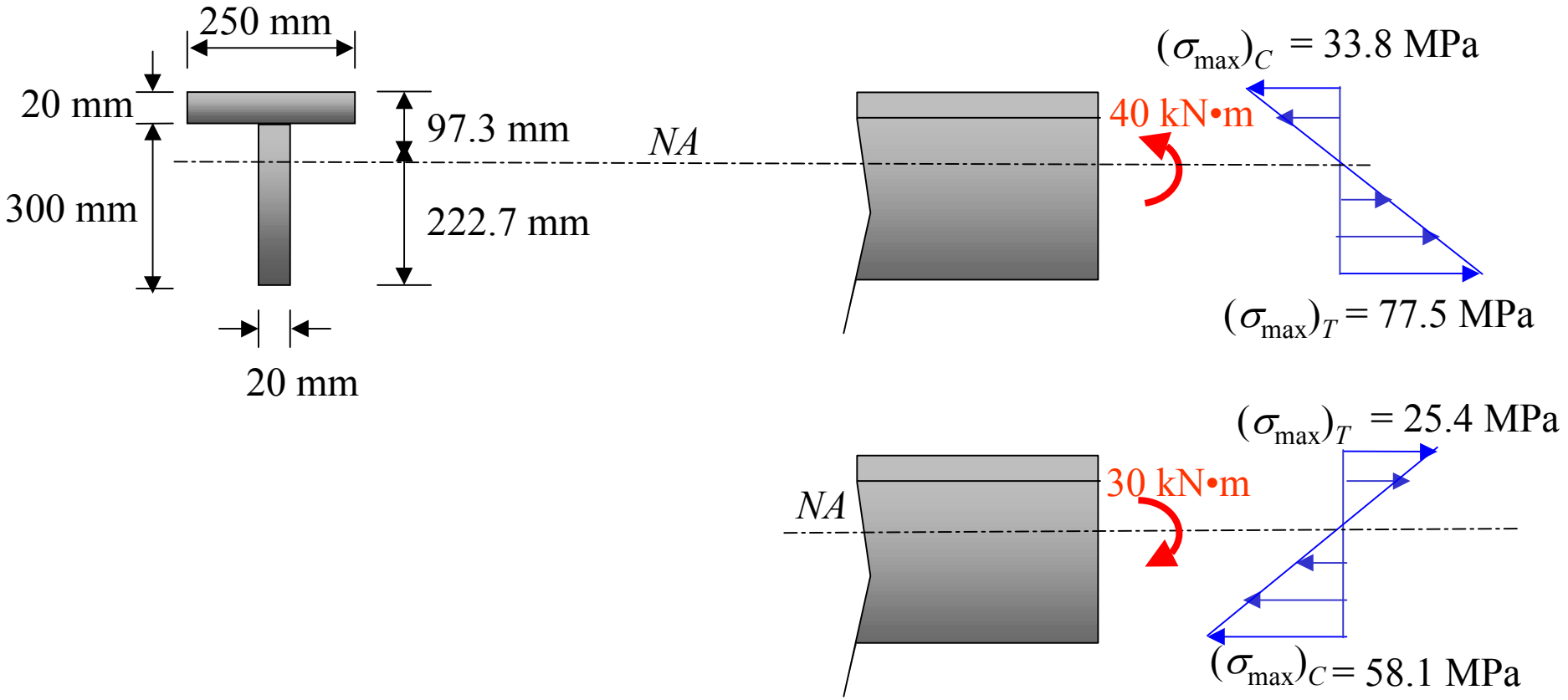
- **The Bending Stress in Maximum Bending in Compression (M = -30 kN•m)**



$$(\sigma_{max})_T = \frac{Mc_1}{I} = \frac{(30\ \text{kN}\bullet\text{m})(0.0973\ \text{m})}{(.115\times10^{-3}\ \text{m}^4)} = 25.4\ \text{MPa (T)}$$

$$(\sigma_{max})_C = \frac{Mc_2}{I} = \frac{(30\ \text{kN}\bullet\text{m})(0.2227\ \text{m})}{(.115\times10^{-3}\ \text{m}^4)} = 58.1\ \text{MPa (C)}$$

- **The absolute maximum bending stress in tension and also in compression in the beam**



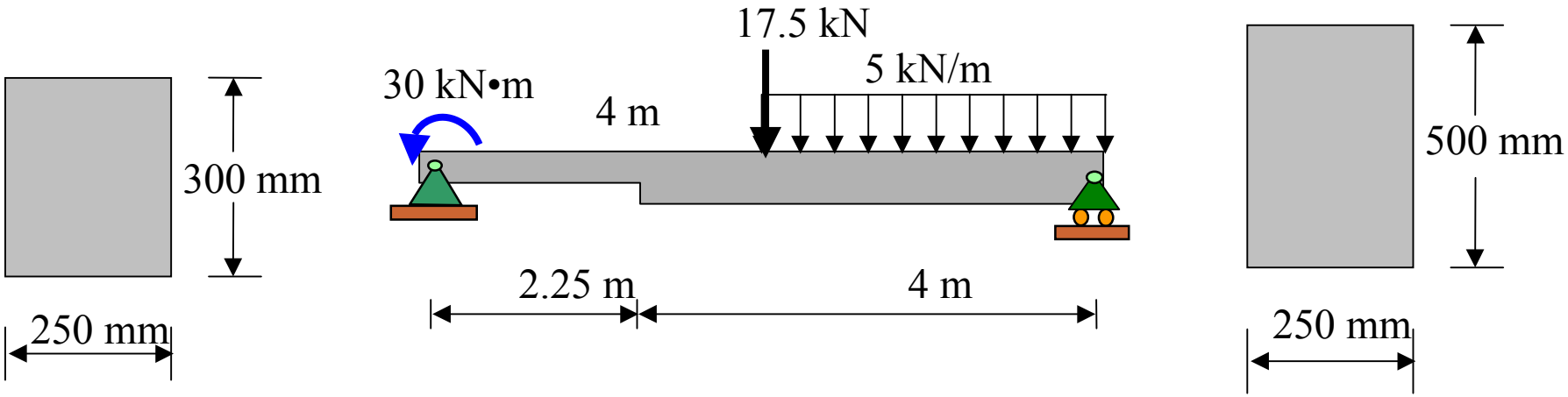
By comparison,

The maximum bending stress in *tension*; $(\sigma_{max})_T = 77.5$ MPa (T)

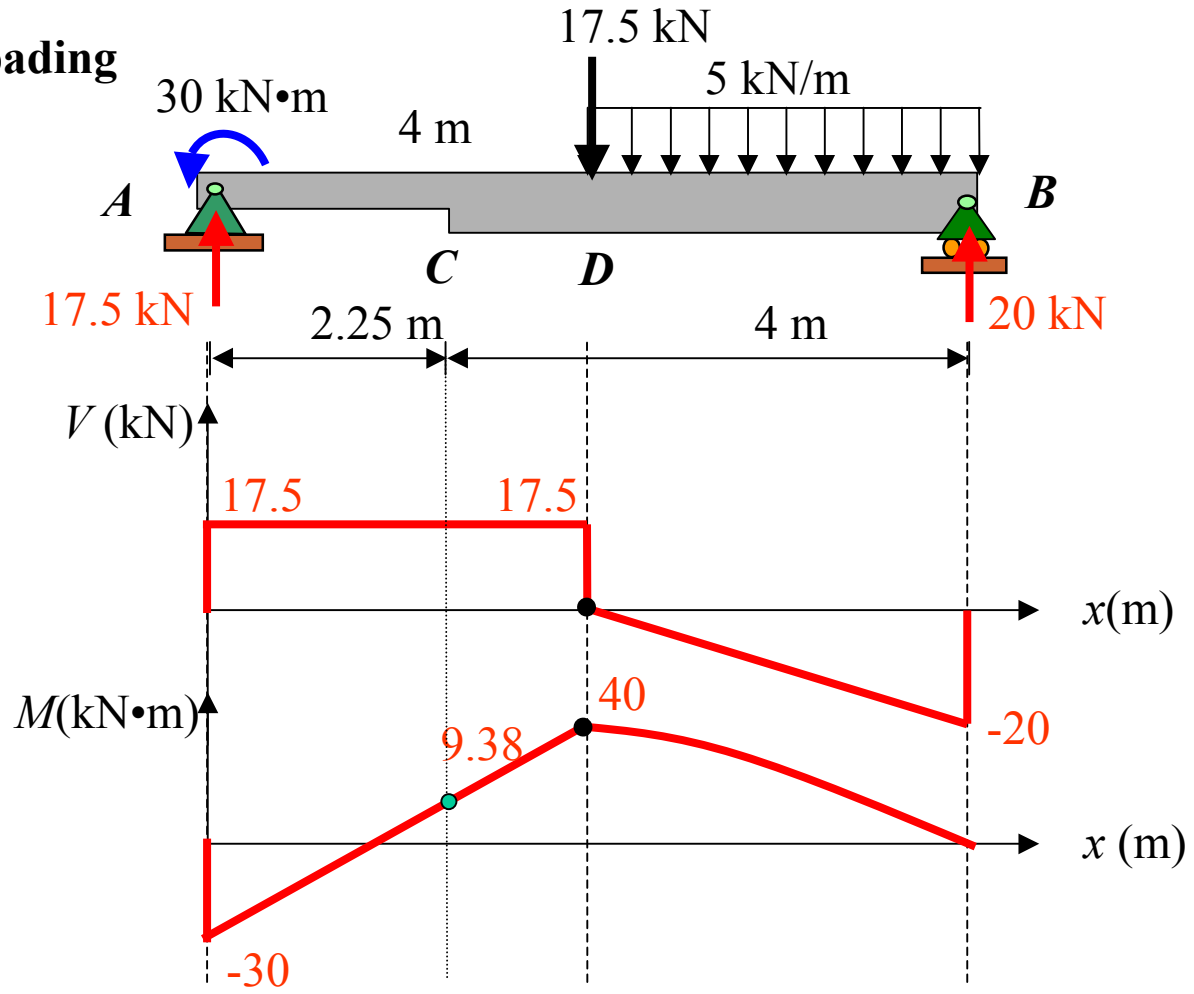
The maximum bending stress in *compression*; $(\sigma_{max})_C = 58.1$ MPa (C)

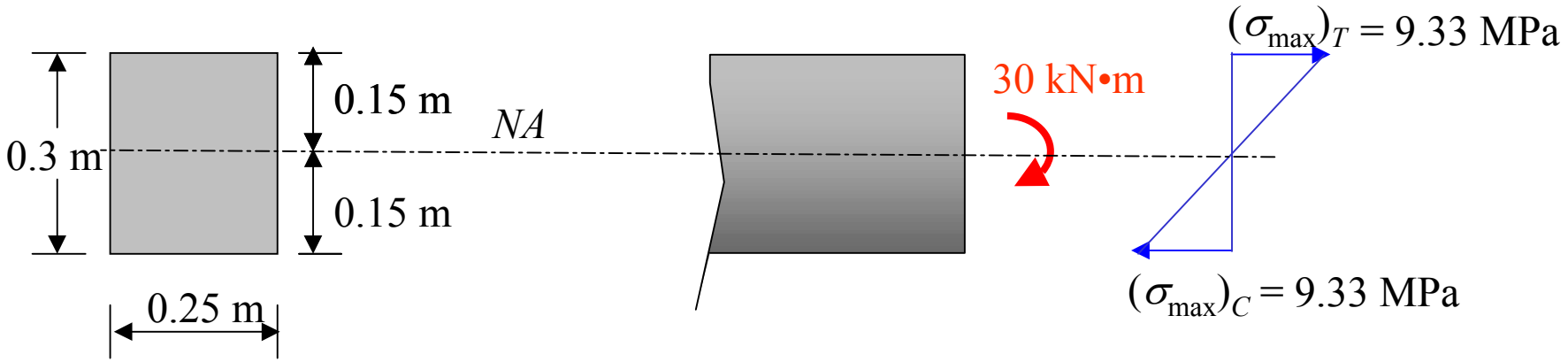
Example 5

The simply supported beam has the cross-sectional area shown. Determine the absolute maximum bending stress in the beam and draw the stress distribution over the cross section at this location.



- **Internal Loading**



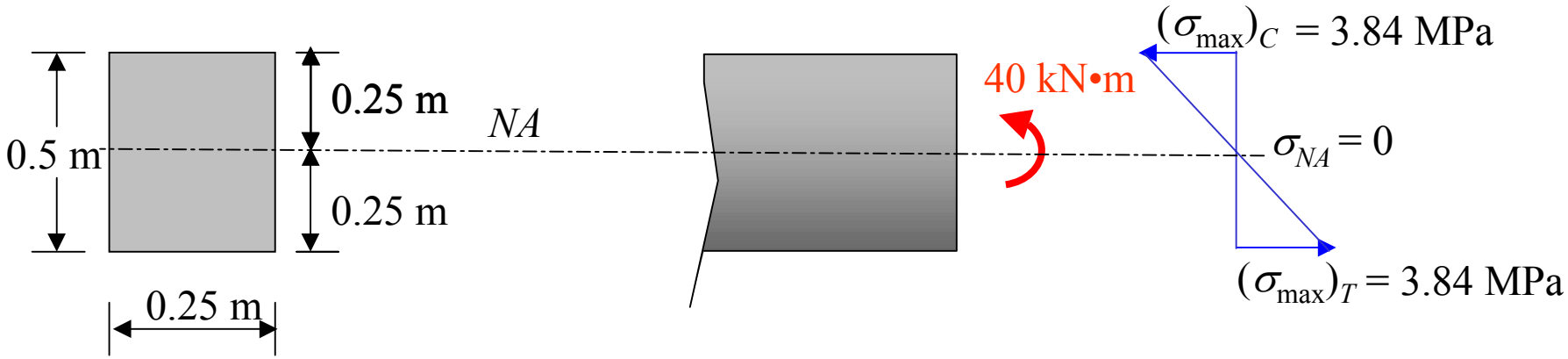


- **Section Property**

$$I = \sum (\bar{I} + Ad^2) = \frac{1}{12}(0.25\ \text{m})(0.3\ \text{m})^3 + 0 = 0.0005625\ \ \text{m}^4$$

- **Bending Stress at *A***

$$\sigma_{max} = \frac{Mc}{I} = \frac{(30\ \text{kN}\bullet\text{m})(0.15\ \text{m})}{(.5625 \times 10^{-3}\ \text{m}^4)} = 9.33\ \text{MPa}$$



- **Section Property**

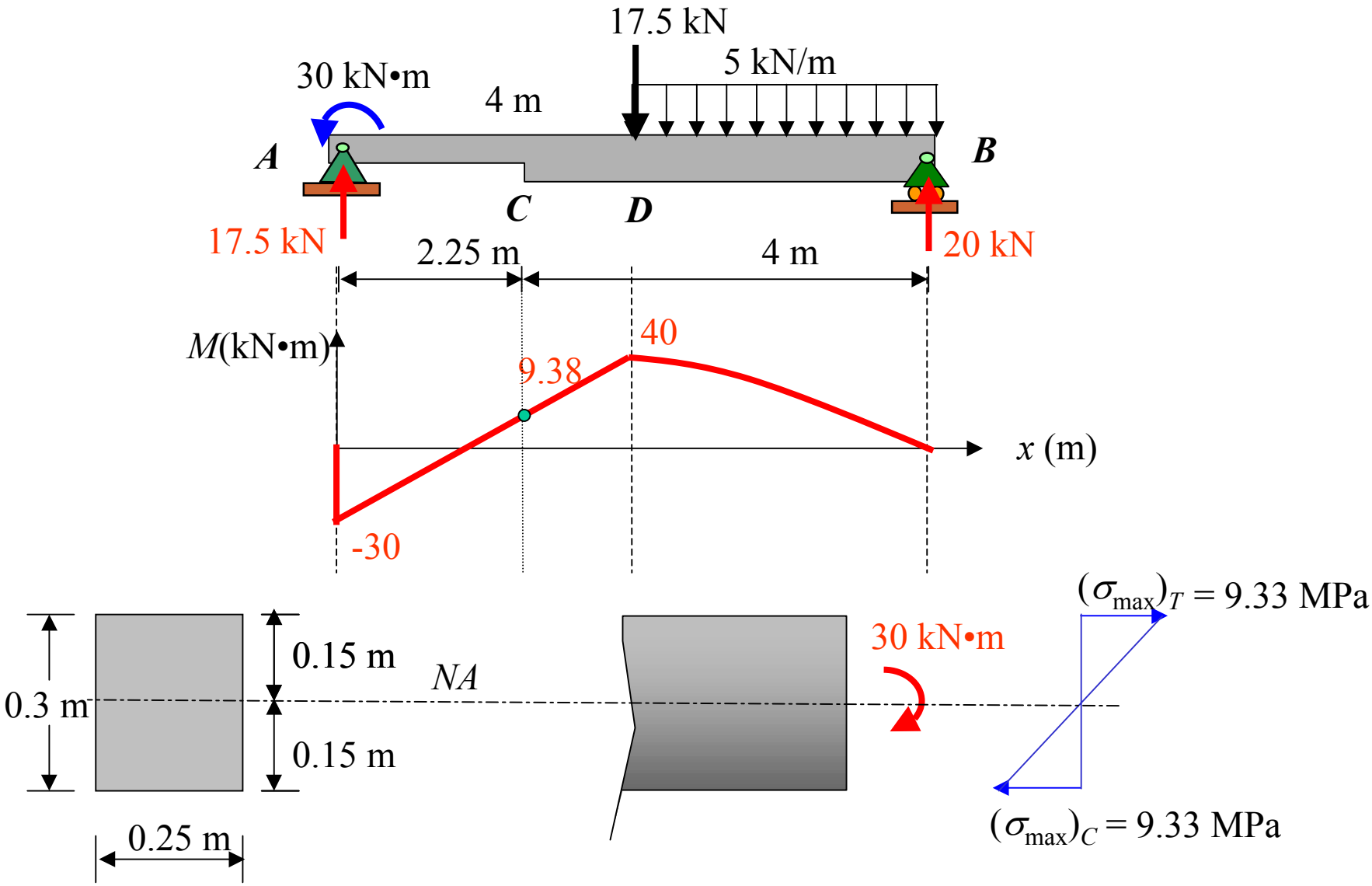
$$I = \sum (\bar{I} + Ad^2) = \frac{1}{12}(0.25\ \text{m})(0.5\ \text{m})^3 + 0 = 0.002604\ \ \text{m}^4$$

- **Bending Stress at *D***

$$\sigma_{max} = \frac{Mc}{I} = \frac{(40\ \text{kN} \bullet \text{m})(0.25\ \text{m})}{(2.604 \times 10^{-3}\ \text{m}^4)} = 3.84\ \text{MPa}$$

Maximum bending stress occurs at *D*

$$\sigma_{max} = 9.33\ \text{MPa}$$

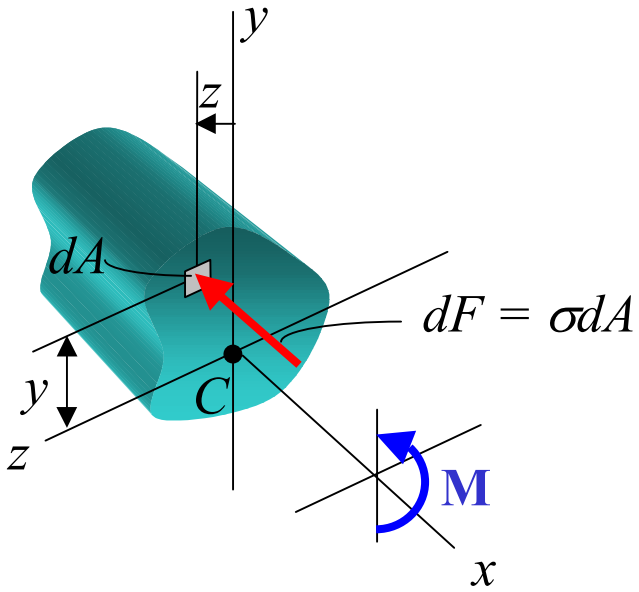


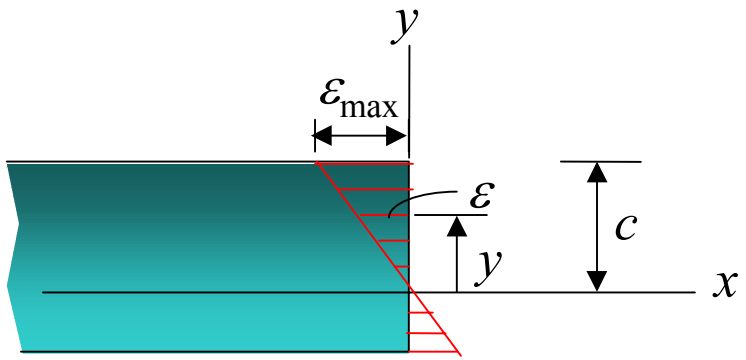
Maximum bending stress occurs at *A*

$$\sigma_{max} = 9.33 \text{ MPa}$$

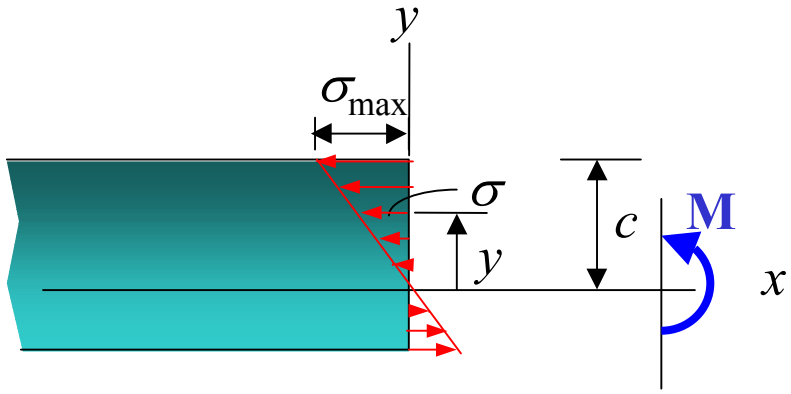
Unsymmetric Bending

- **Moment Applied Along Principal Axis**



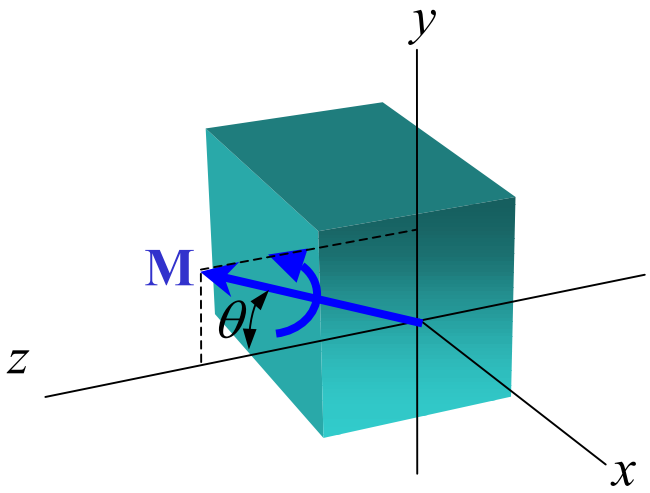


Normal Strain Distribution
(Profile View)

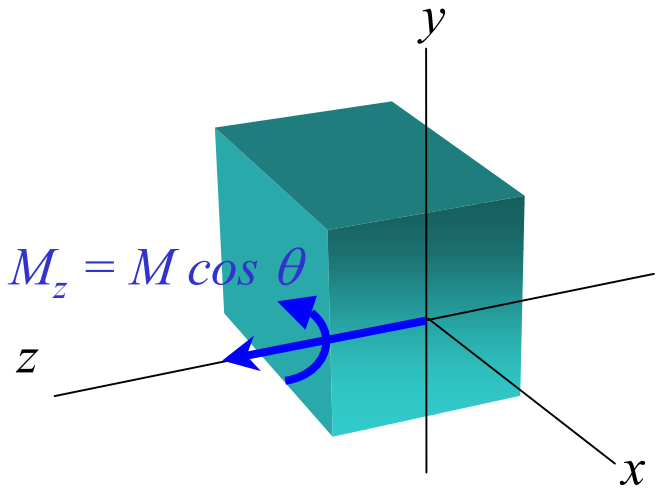


Bending Stress Distribution
(Profile View)

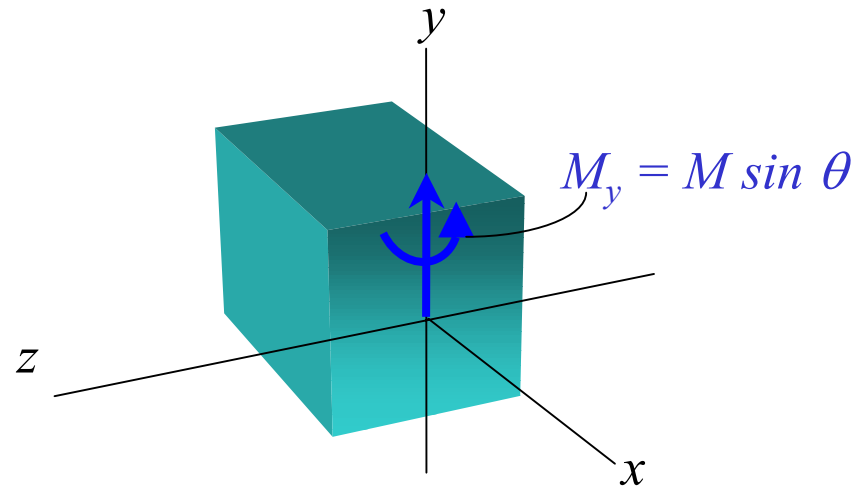
• **Moment Arbitrarily Applied**



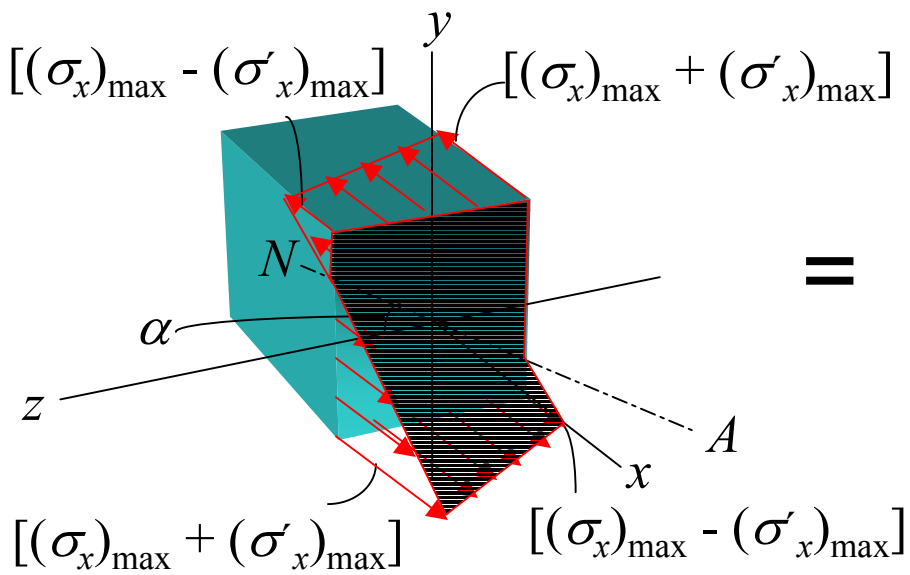
=



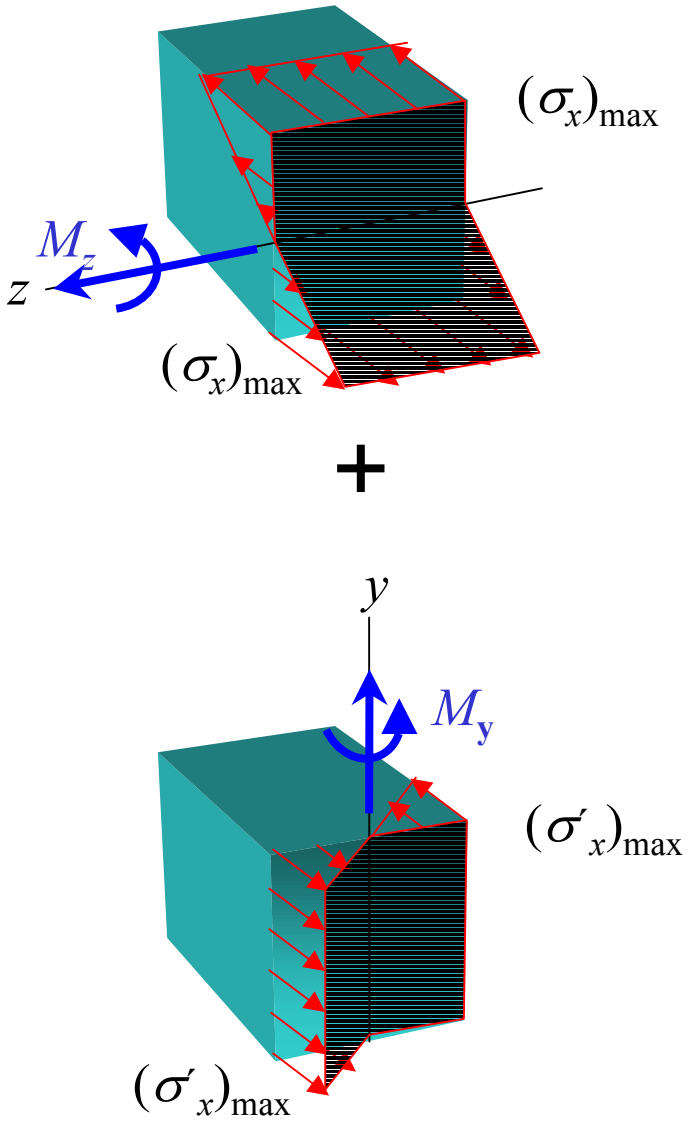
+



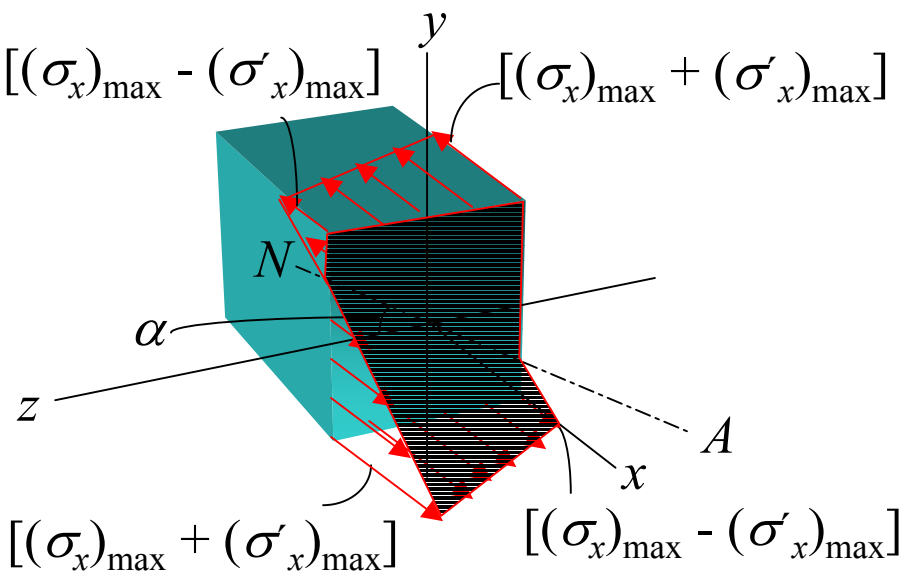
$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

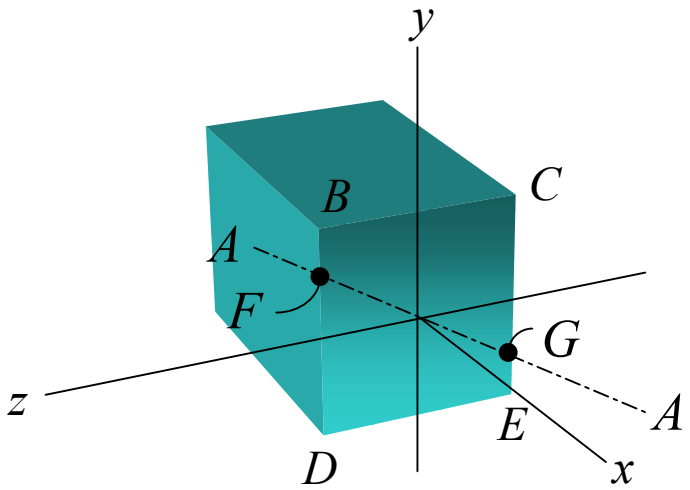

$[(\sigma_x)_{max} - (\sigma'_x)_{max}]$
y
$[(\sigma_x)_{max} + (\sigma'_x)_{max}]$
N
α
z
A
x
$[(\sigma_x)_{max} + (\sigma'_x)_{max}]$
$[(\sigma_x)_{max} - (\sigma'_x)_{max}]$

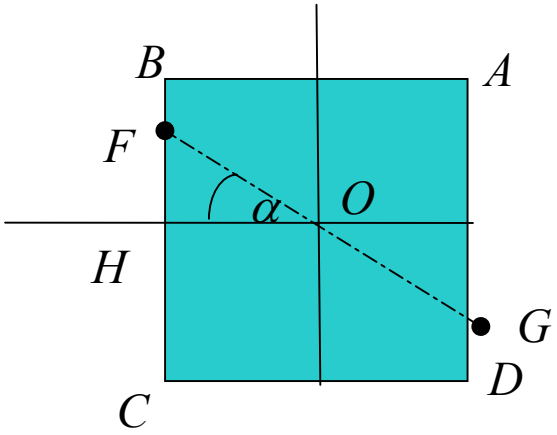
=


$(\sigma_x)_{max}$
M_z
z
$(\sigma_x)_{max}$
+
y
M_y
$(\sigma'_x)_{max}$
$(\sigma'_x)_{max}$

• **Orientation of the Neutral Axis (α)**



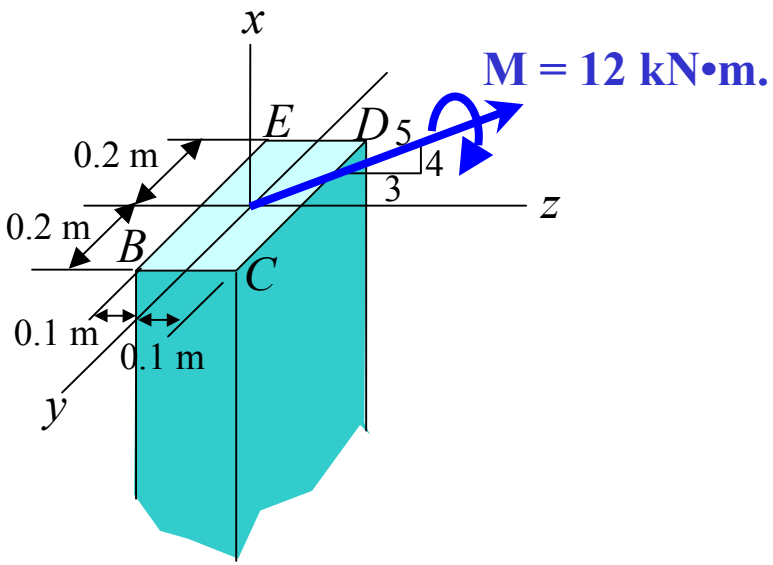


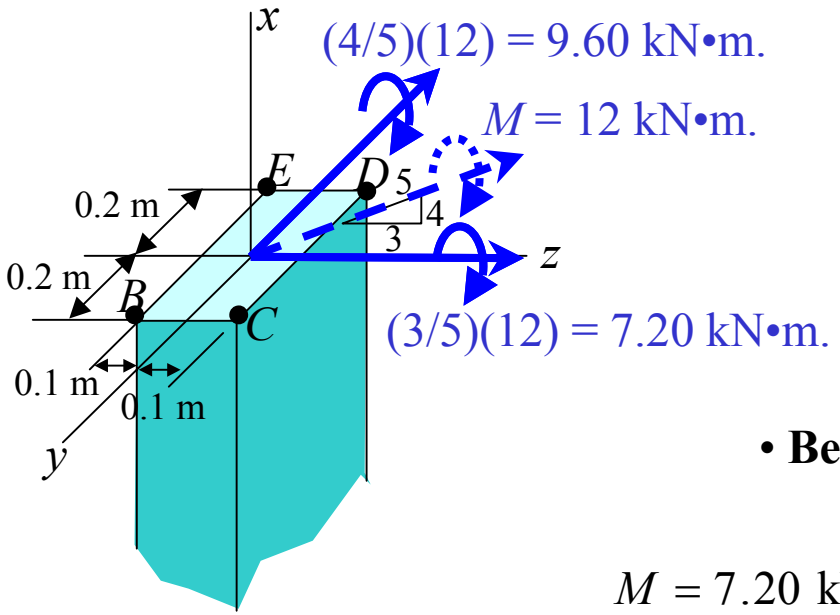


$$\alpha = \tan^{-1}\left(\frac{FH}{HO}\right)$$

Example 6

The rectangular cross section shown is subjected to a bending moment of $M = 12$ kN•m. Determine the normal stress developed at each corner of the section, and specify the orientation of the neutral axis.





• **Section Properties**

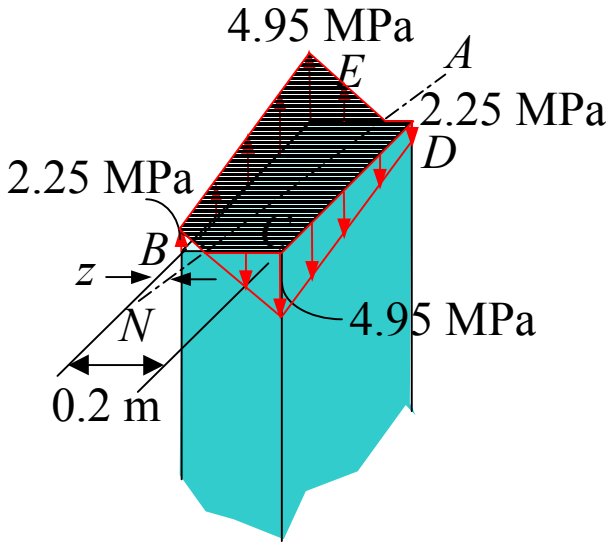
$$I_y = \frac{1}{12}(0.4\text{ m})(0.2\text{ m})^3 = 0.2667\,(10^{-3})\text{ m}^4$$

$$I_z = \frac{1}{12}(0.2\text{ m})(0.4\text{ m})^3 = 1.067\,(10^{-3})\text{ m}^4$$

• **Bending Stress** : $\sigma = \dfrac{My}{I}$

$$M = 7.20\text{ kN}\bullet\text{m}: \quad \sigma = \frac{7.2(10^3)\text{ N}\bullet\text{m}(0.2\text{ m})}{1.067\,(10^{-3})\text{m}^4} = \pm 1.35\text{ MPa}$$

$$M = 9.60\text{ kN}\bullet\text{m}: \quad \sigma = \frac{9.60(10^3)\text{ N}\bullet\text{m}(0.1\text{ m})}{0.2667\,(10^{-3})\text{ m}^4} = \pm 3.60\text{ MPa}$$



$M = 7.2$ kN•m $\Downarrow$ $M = 9.6$ kN•m $\Downarrow$

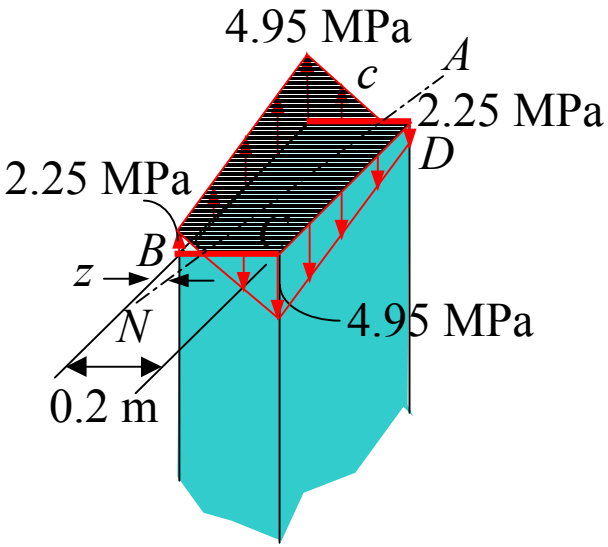
$$\sigma_B = (-\ 1.35\text{ MPa}) + (+\ 3.60\text{ MPa}) = 2.25\text{ MPa}\ (T)$$

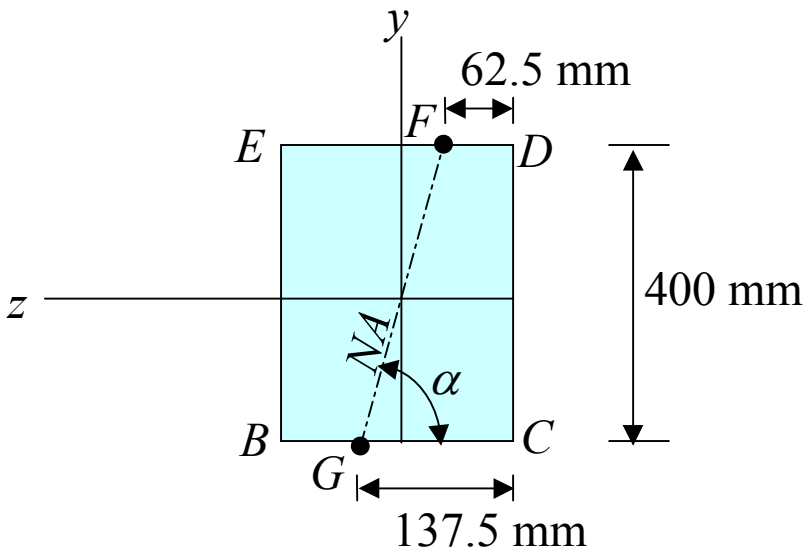
$$\sigma_C = (-\ 1.35\text{ MPa}) + (-\ 3.60\text{ MPa}) = -4.95\text{ MPa}\ (C)$$

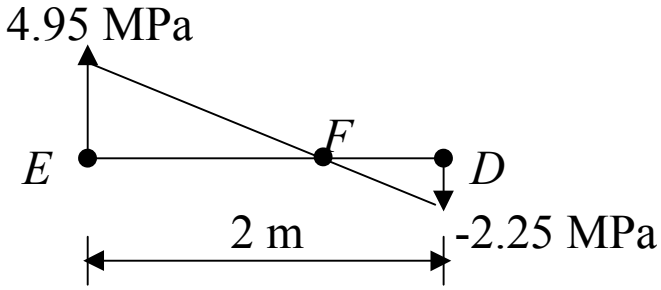
$$\sigma_D = (+\ 1.35\text{ MPa}) + (-\ 3.60\text{ MPa}) = -2.25\text{ MPa}\ (C)$$

$$\sigma_E = (+\ 1.35\text{ MPa}) + (+\ 3.60\text{ MPa}) = 4.95\text{ MPa}\ (T)$$

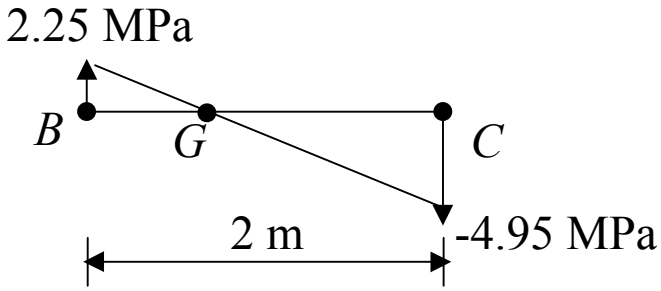
• **Orientation of Neutral Axis**







$$\frac{FD}{2.25} = \frac{2}{(4.95 + 2.25)}; FD = 0.625 \text{ m}$$

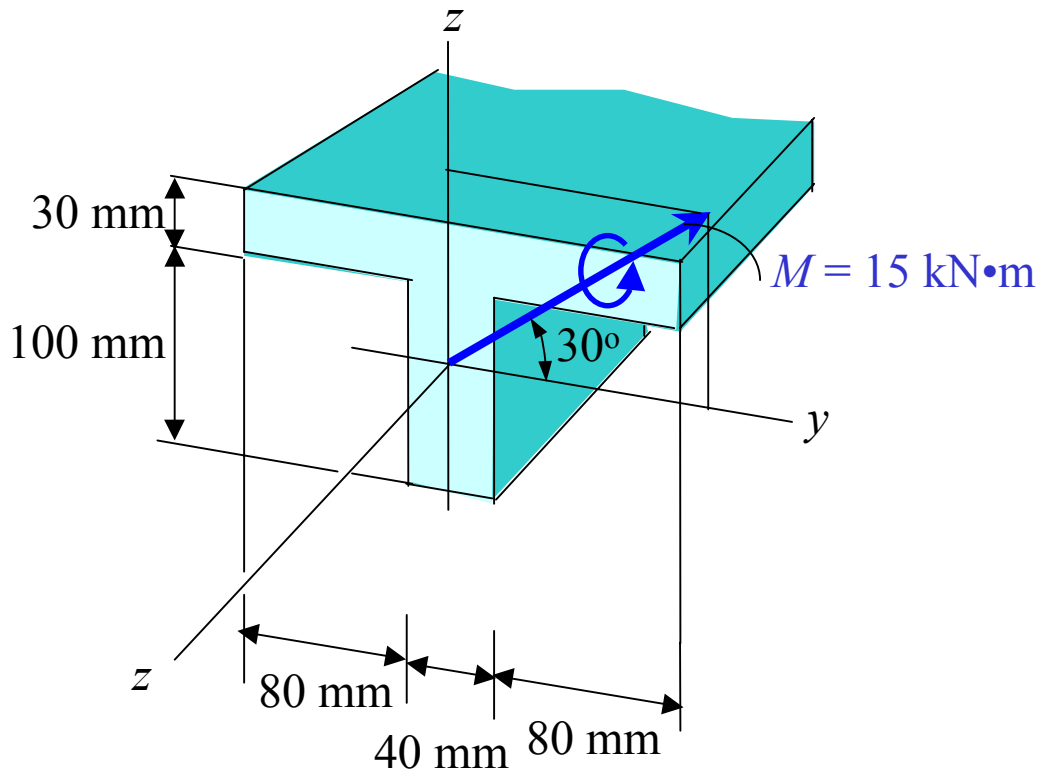


$$\frac{GC}{4.95} = \frac{2}{(4.95 + 2.25)}; GC = 1.375 \text{ m}$$

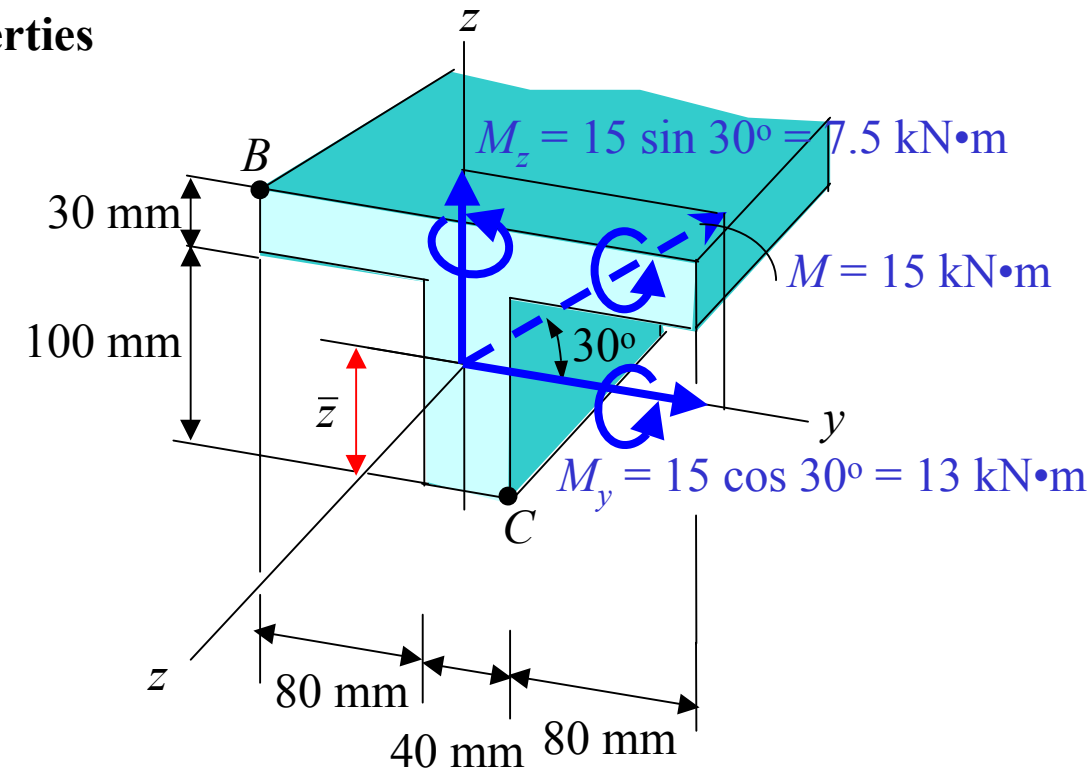
$$\alpha = \tan^{-1}\left(\frac{400}{137.5 - 62.5}\right) = 79.4^{o}$$

Example 7

A T-beam is subjected to the bending moment of 15 kN•m as shown. Determine the maximum normal stress in the beam and the orientation of the neutral axis.



- **Section Properties**



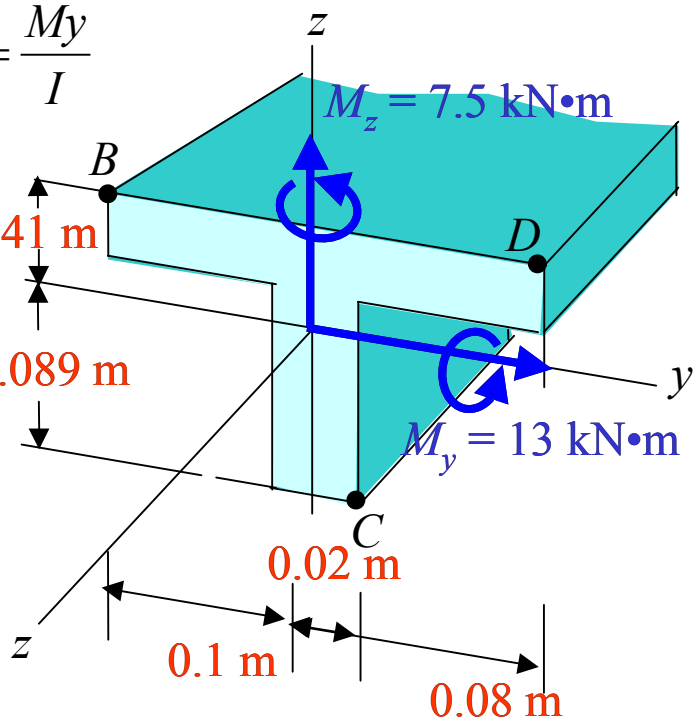
$$\bar{z} = \frac{\sum \bar{z}A}{\sum A} = \frac{(0.05\ \text{m})(0.1\ \text{m} \times 0.04\ \text{m}) + (0.115\ \text{m})(0.03\ \text{m} \times 0.2\ \text{m})}{(0.1\ \text{m} \times 0.04\ \text{m}) + (0.03\ \text{m} \times 0.2\ \text{m})} = 0.0890\ \text{m}$$

$$I_z = \frac{1}{12}(0.10\ \text{m})(0.04\ \text{m})^3 + \frac{1}{12}(0.03\ \text{m})(0.20\ \text{m})^3 = 20.53(10^{-6})\ \text{m}^4$$

$$I_y = [\frac{1}{12}(0.04\ \text{m})(0.10\ \text{m})^3 + (0.04\ \text{m} \times 0.10\ \text{m})(0.089\ \text{m} - 0.05\ \text{m})^2]$$

$$+ [\frac{1}{12}(0.20\ \text{m})(0.03\ \text{m})^3 + (0.2\ \text{m} \times 0.03\ \text{m})(0.115\ \text{m} - 0.089\ \text{m})^2] = 13.92(10^{-6})\ \text{m}^4$$

• **Maximum Normal Stress,** $: \sigma = \dfrac{My}{I}$



$I_z = 20.53(10^{-6})\ \text{m}^4$

$I_y = 13.92(10^{-6})\ \text{m}^4$

$M_z = 7.5$ kN•m ⇩ $\qquad$ $M_y = 13$ kN•m ⇩

$$\sigma_B = \frac{(+7.5\ \text{kN•m})(0.10\ \text{m})}{20.53(10^{-6})\ \text{m}^4} + \frac{(+13\ \text{kN•m})(0.041\ \text{m})}{13.92(10^{-6})\ \text{m}^4} = 74.8\ \text{MPa}\ (T)$$

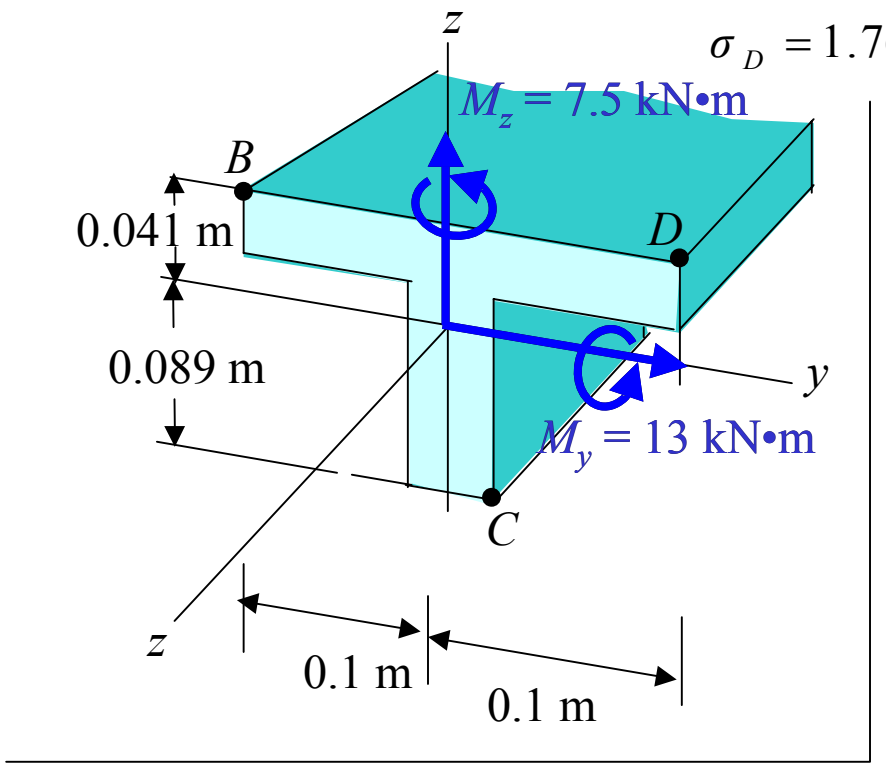
$$\sigma_C = \frac{(-7.5\ \text{kN•m})(0.02\ \text{m})}{20.53(10^{-6})\ \text{m}^4} + \frac{(-13\ \text{kN•m})(0.089\ \text{m})}{13.92(10^{-6})\ \text{m}^4} = -90.4\ \text{MPa}\ (C) \Leftarrow$$

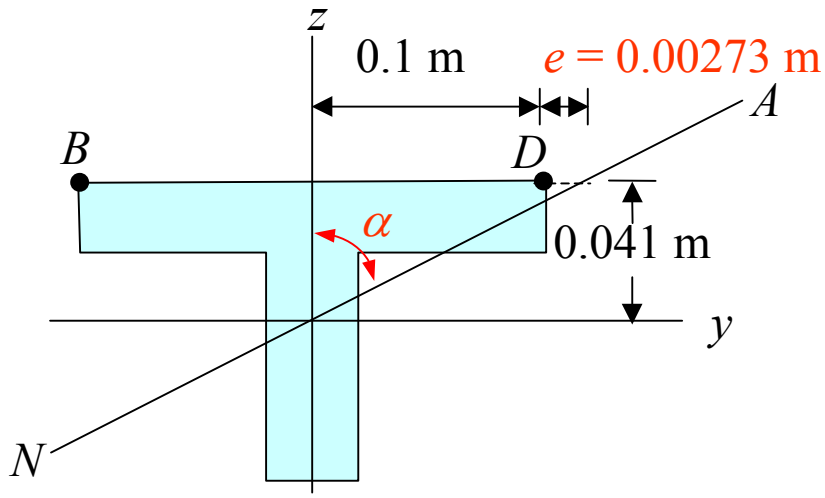
$$\sigma_D = \frac{(-7.5\ \text{kN•m})(0.10\ \text{m})}{20.53(10^{-6})\ \text{m}^4} + \frac{(+13\ \text{kN•m})(0.041\ \text{m})}{13.92(10^{-6})\ \text{m}^4} = 1.76\ \text{MPa}\ (T)$$

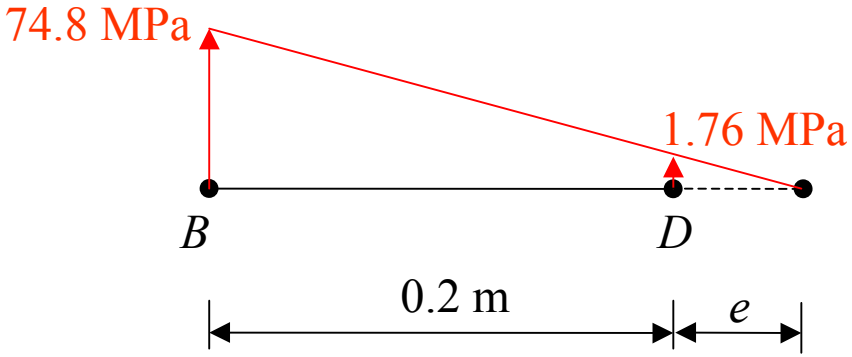
- **Orientation of Neutral Axis**

$\sigma_B = 74.8$ MPa

$\sigma_D = 1.76$ MPa







$$\frac{e}{1.76} = \frac{0.2 + e}{74.8}, \quad e = 2.738(10^{-3}) \text{ m}$$

$$\tan\alpha = \frac{(0.10 \text{ m} + 0.002738 \text{ m})}{0.0410 \text{ m}}, \quad \alpha = 68^o$$