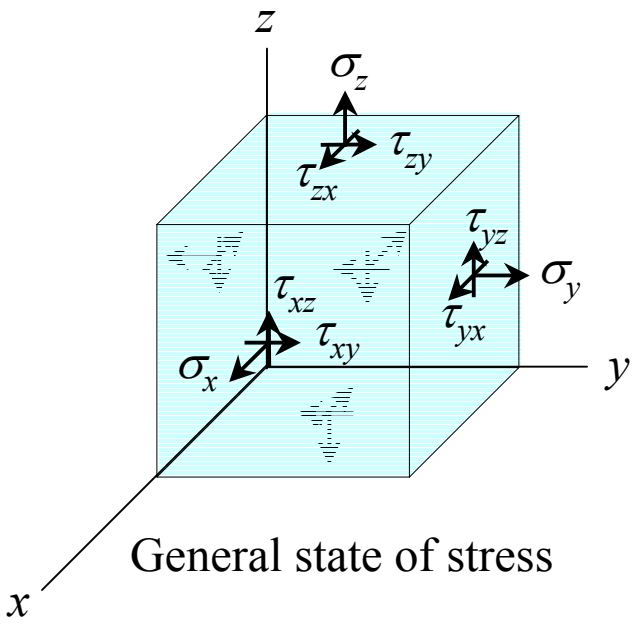
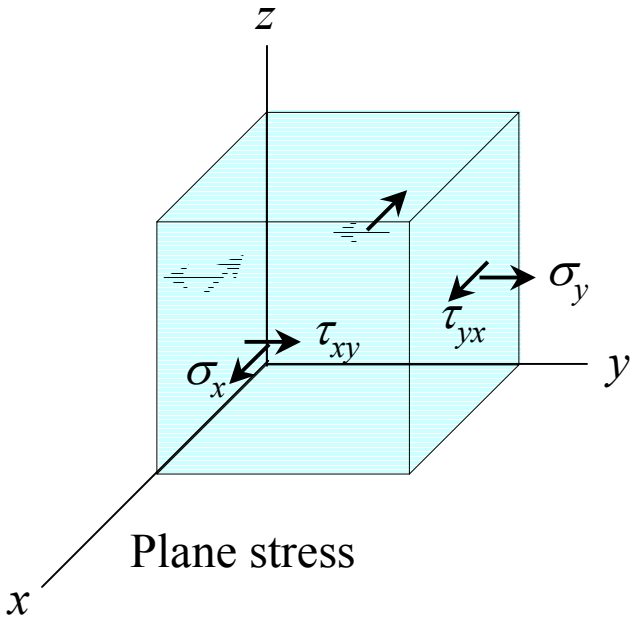
STRESS TRANSFORMATION

- Plane-Stress Transformation
- General Equations of Plane-Stress Transformation
- Principal Stresses and Maximum In-Plane Shear Stress
- Mohr's Circle-Plane Stress

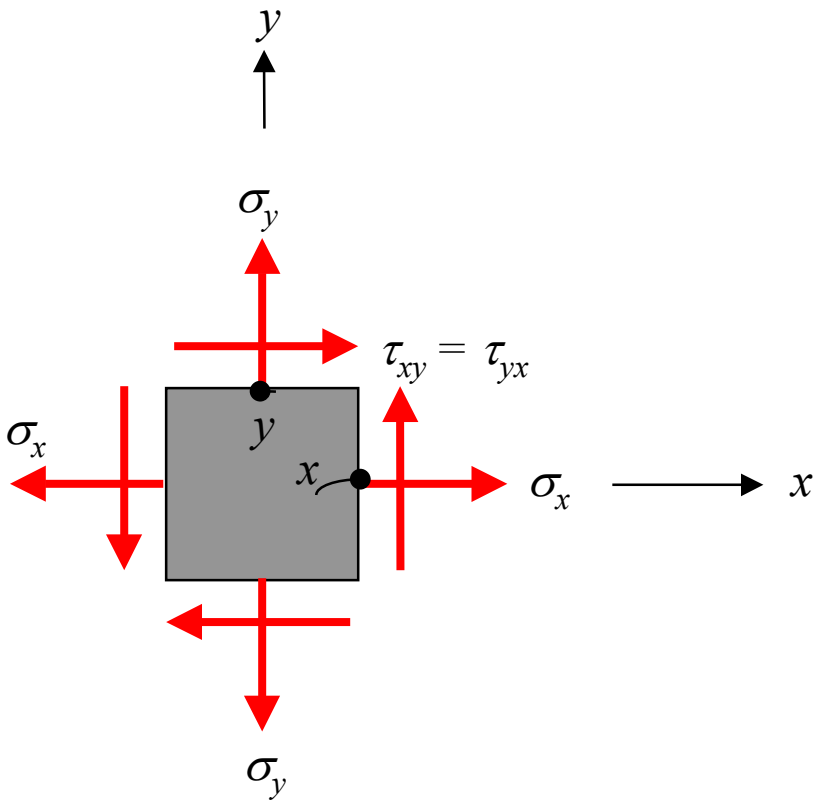
Plane-Stress Transformation



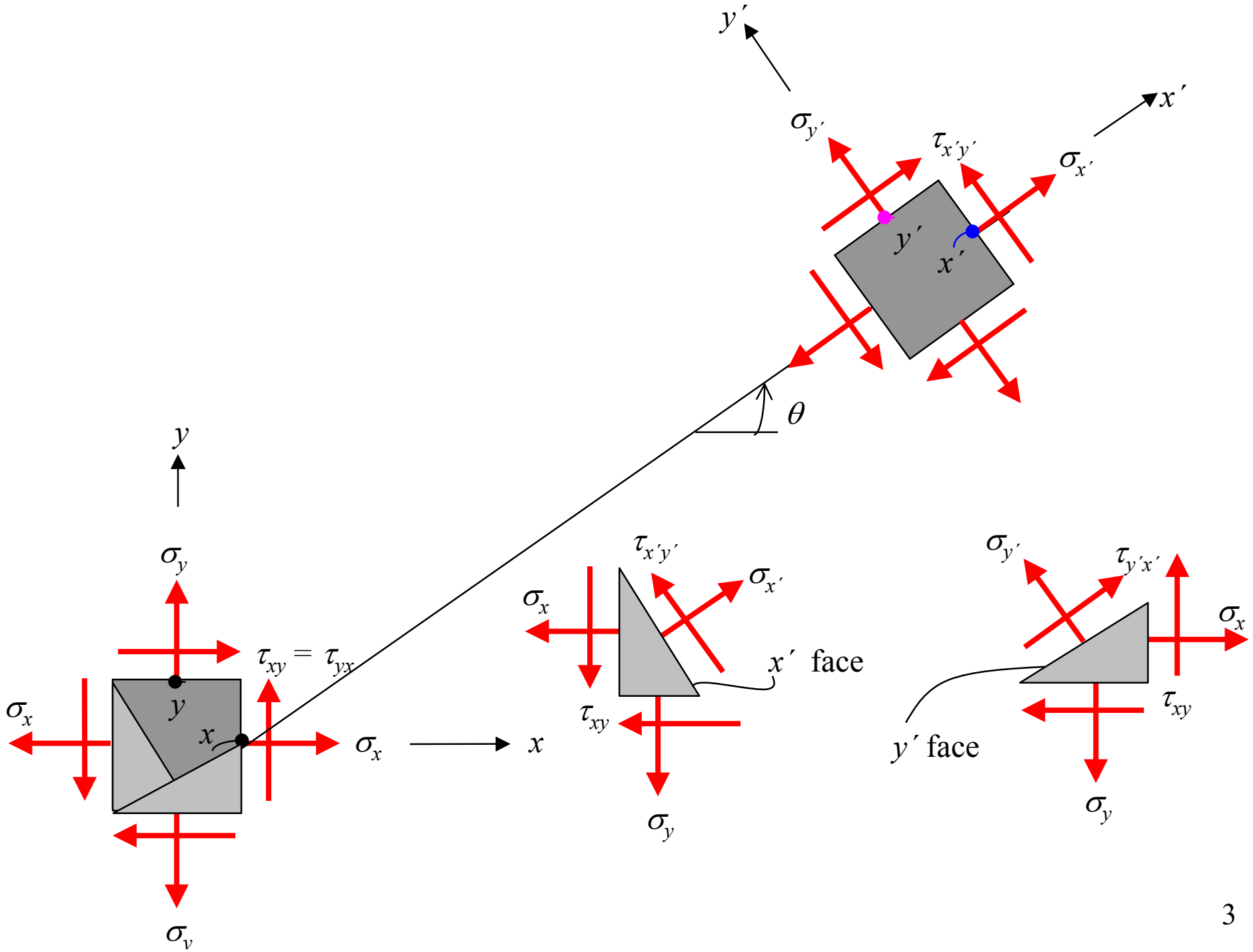
General state of stress



Plane stress

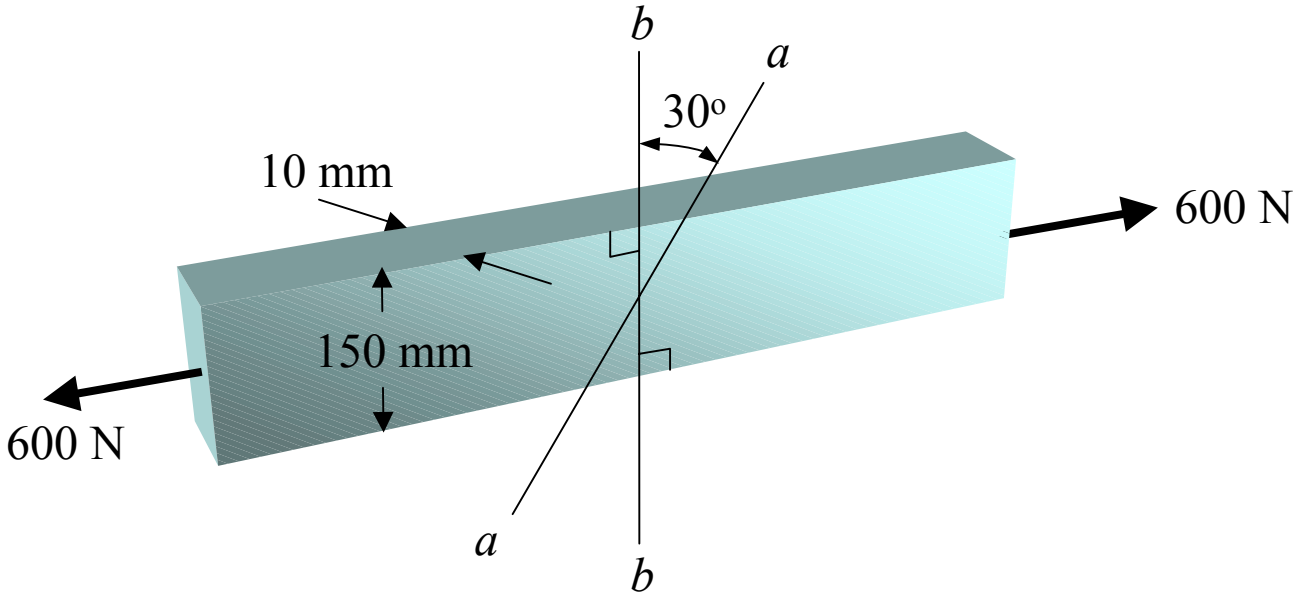


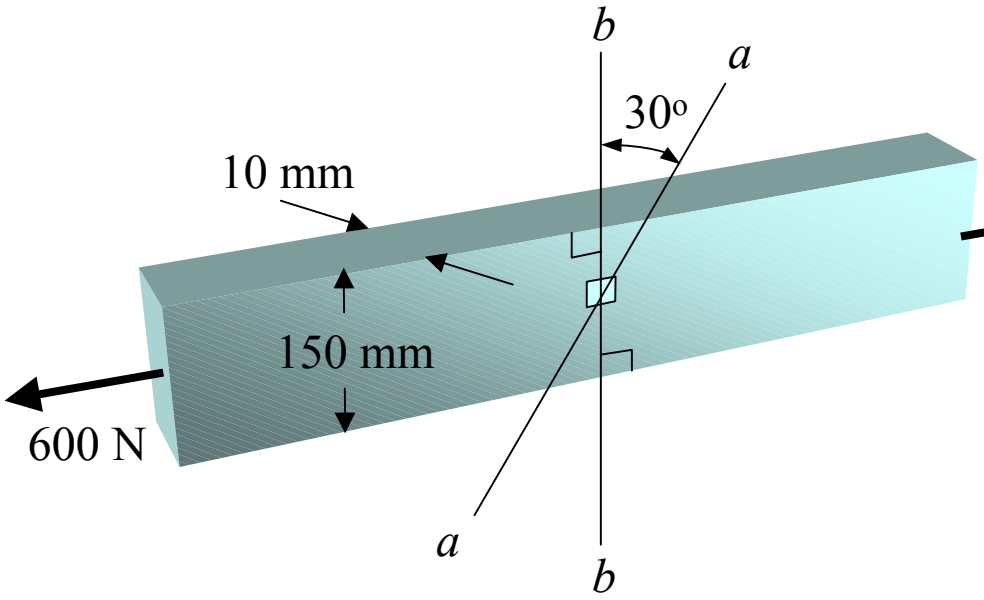
Plane stress
(two dimensional view)

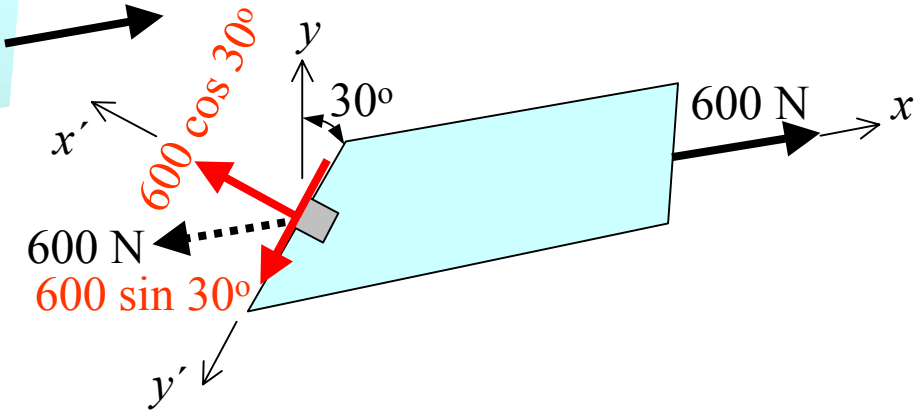

y′
x′
$\sigma_{y'}$
$\tau_{x'y'}$
$\sigma_{x'}$
y′
x′
θ
y
σ_y
$\tau_{xy} = \tau_{yx}$
σ_x
y
x
σ_x
x
σ_v
$\tau_{x'y'}$
$\sigma_{x'}$
σ_x
x′ face
τ_{xy}
σ_y
$\sigma_{y'}$
$\tau_{y'x'}$
σ_x
τ_{xy}
y′ face
σ_y

Example 1

An axial force of 600 N acts on the steel bar shown. Draw the state of stress acting on a plane defined by section *a*-*a*.





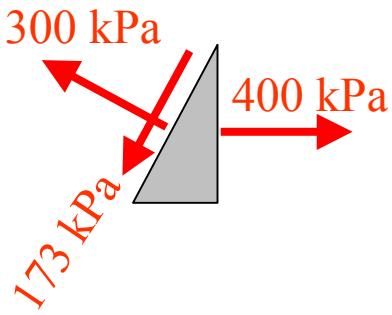


- **Section a-a: Normal and Shear Stresses**

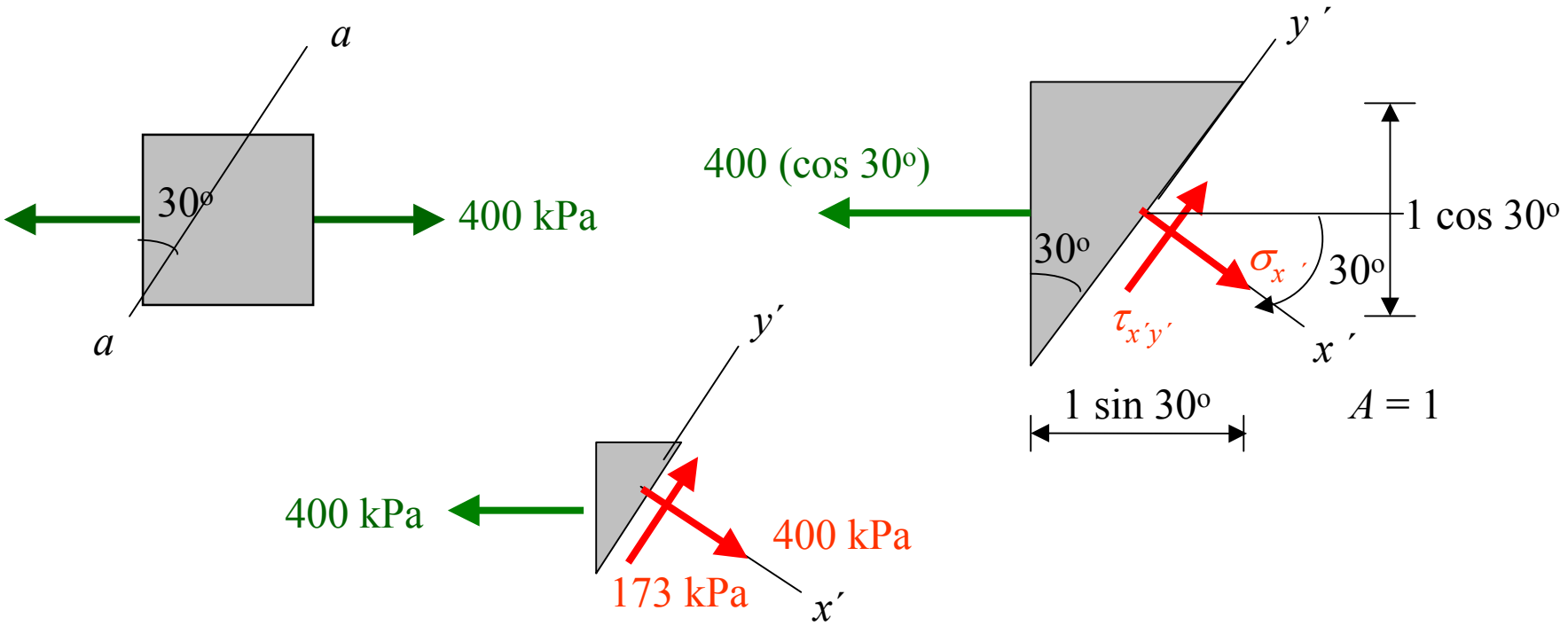
$$\sigma_x = \frac{P}{A} = \frac{600\ \text{N}}{(0.01\ \text{m})(0.15\ \text{m})} = 400\ \text{kPa}$$

$$\sigma_{aa} = \frac{600\cos 30^\circ}{(0.01\,\text{m})(\frac{0.15}{\cos 30^\circ}\,\text{m})} = 300\ \text{kPa} \Leftarrow$$

$$\tau_{aa} = \frac{600\sin 30^\circ}{(0.01\,\text{m})(\frac{0.15}{\cos 30^\circ}\,\text{m})} = 173\ \text{kPa} \Leftarrow$$



- **The Stress on plane a-a**



$$+\searrow \Sigma F_{x'} = 0;$$

$$\sigma_{x'}(1) - (400\cos 30^\circ)\cos 30^\circ = 0$$

$$\sigma_{x'} = 400 \text{ kPa}$$

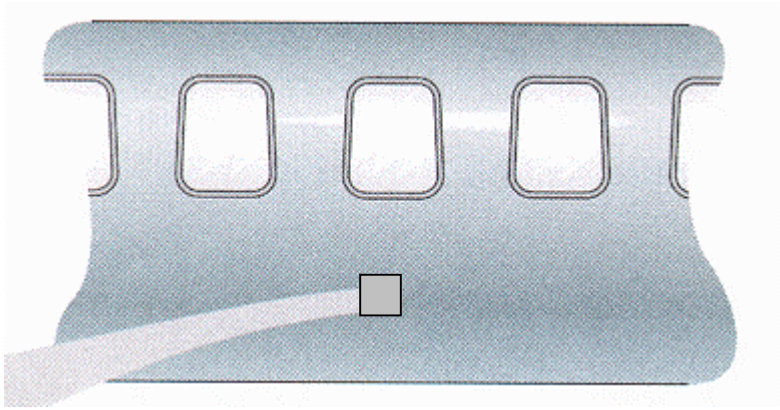
$$+\nearrow \Sigma F_{y'} = 0;$$

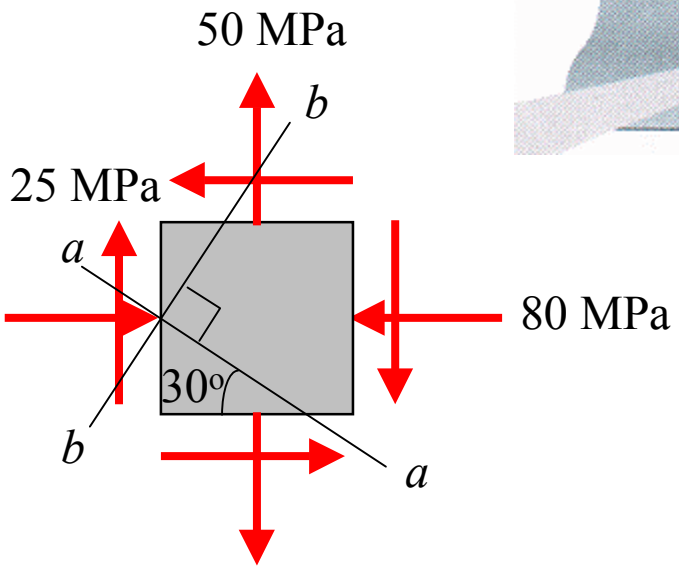
$$\tau_{x'y'}(1) - (400\cos 30^\circ)\sin 30^\circ = 0$$

$$\tau_{x'y'} = 173 \text{ kPa}$$

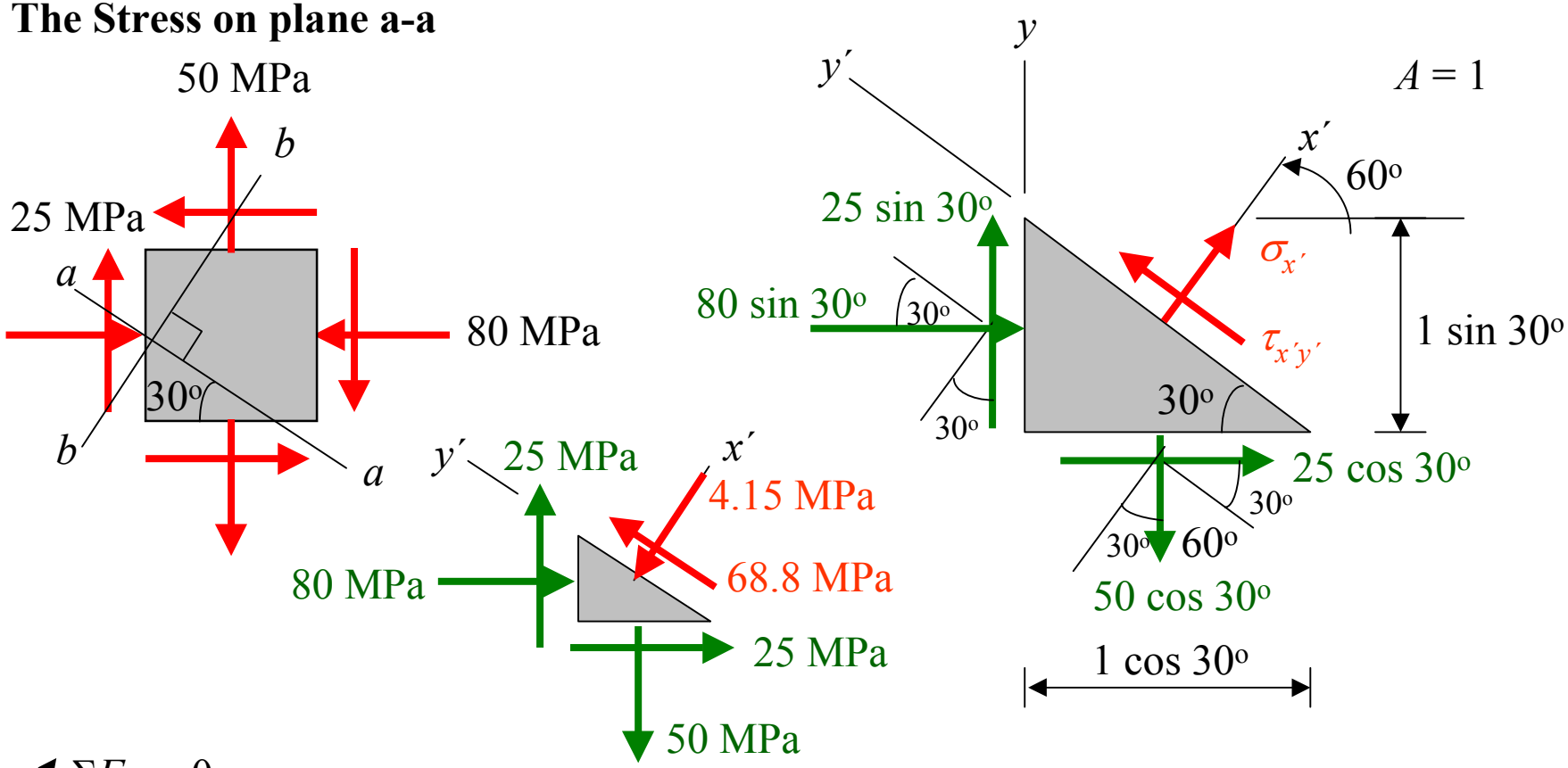
Example 2

The state of plane stress a point on the surface of the airplane fuselage is represented on the element oriented as shown. Represent the state of stress at the point on another element that is oriented 30° clockwise from the position shown.





- **The Stress on plane a-a**



$$+\nearrow \Sigma F_{x'} = 0;$$

$$\sigma_{x'}(1) - (50\cos 30^o)\cos 30^o + (25\cos 30^o)\sin 30^o + (80\sin 30^o)\sin 30^o + (25\sin 30^o)\cos 30^o = 0$$

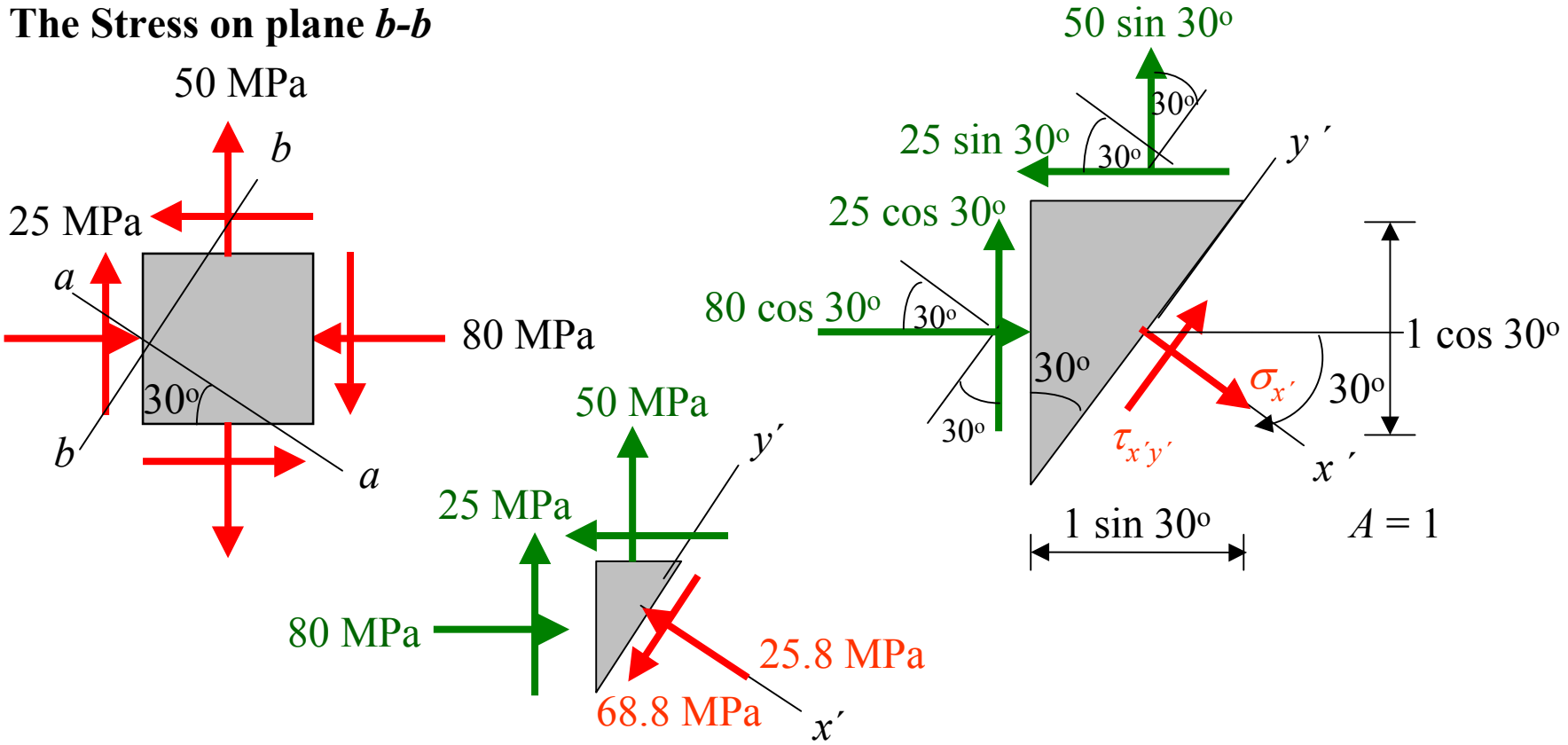
$$\sigma_{x'} = \text{-}4.15 \text{ MPa}$$

$$+\swarrow \Sigma F_{y'} = 0;$$

$$\tau_{x'y'}(1) - (50\cos 30^o)\sin 30^o - (25\cos 30^o)\cos 30^o - (80\sin 30^o)\cos 30^o + (25\sin 30^o)\sin 30^o = 0$$

$$\tau_{x'y'} = 68.8 \text{ MPa}$$

- **The Stress on plane *b-b***



$$\overset{+}{\searrow} \Sigma F_{x'} = 0;$$

$$\sigma_{x'}(1) - (50\sin 30^o)\sin 30^o - (25\sin 30^o)\cos 30^o + (80\cos 30^o)\cos 30^o - (25\cos 30^o)\sin 30^o = 0$$

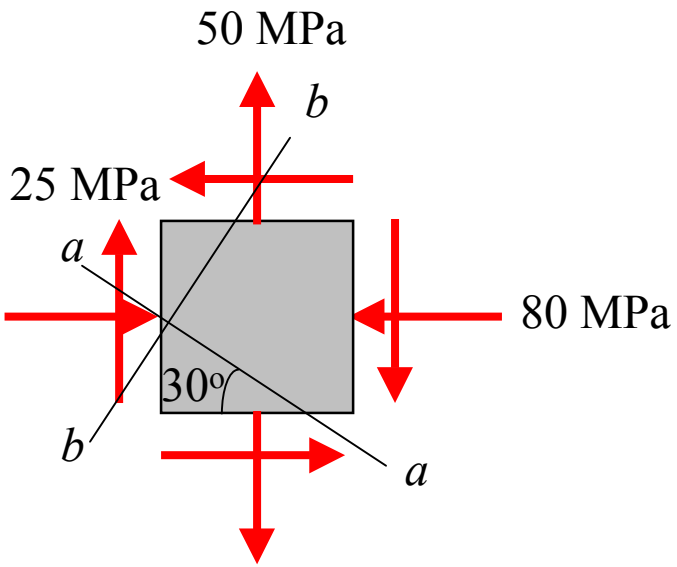
$$\sigma_{x'} = \text{-}25.8 \text{ MPa}$$

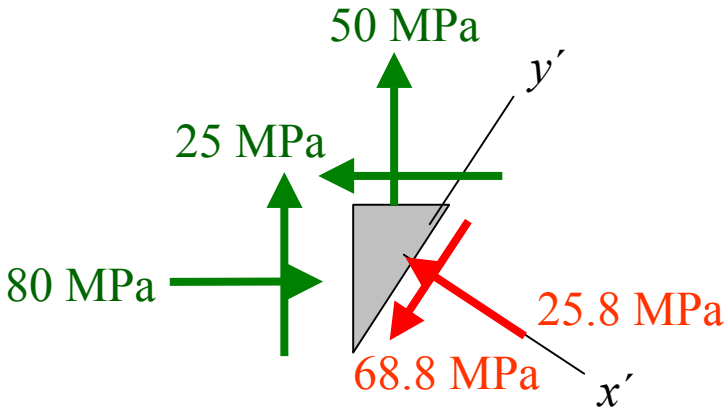
$$\overset{+}{\nearrow} \Sigma F_{y'} = 0;$$

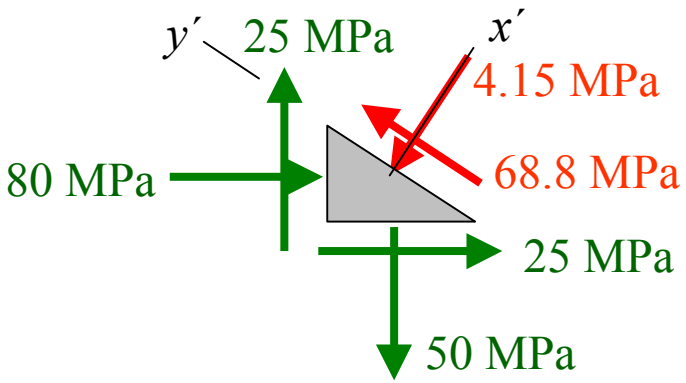
$$\tau_{x'y'}(1) + (50\sin 30^o)\cos 30^o - (25\sin 30^o)\sin 30^o + (80\cos 30^o)\sin 30^o + (25\cos 30^o)\cos 30^o = 0$$

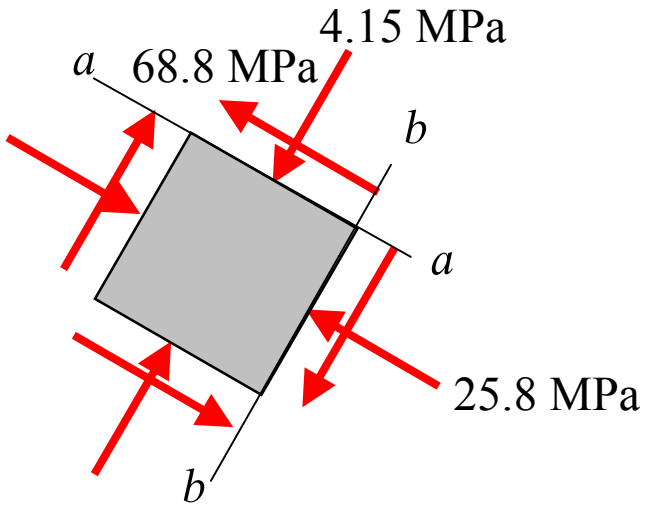
$$\tau_{x'y'} = \text{-}\ 68.8 \text{ MPa}$$

- **The State of Stress at the point**



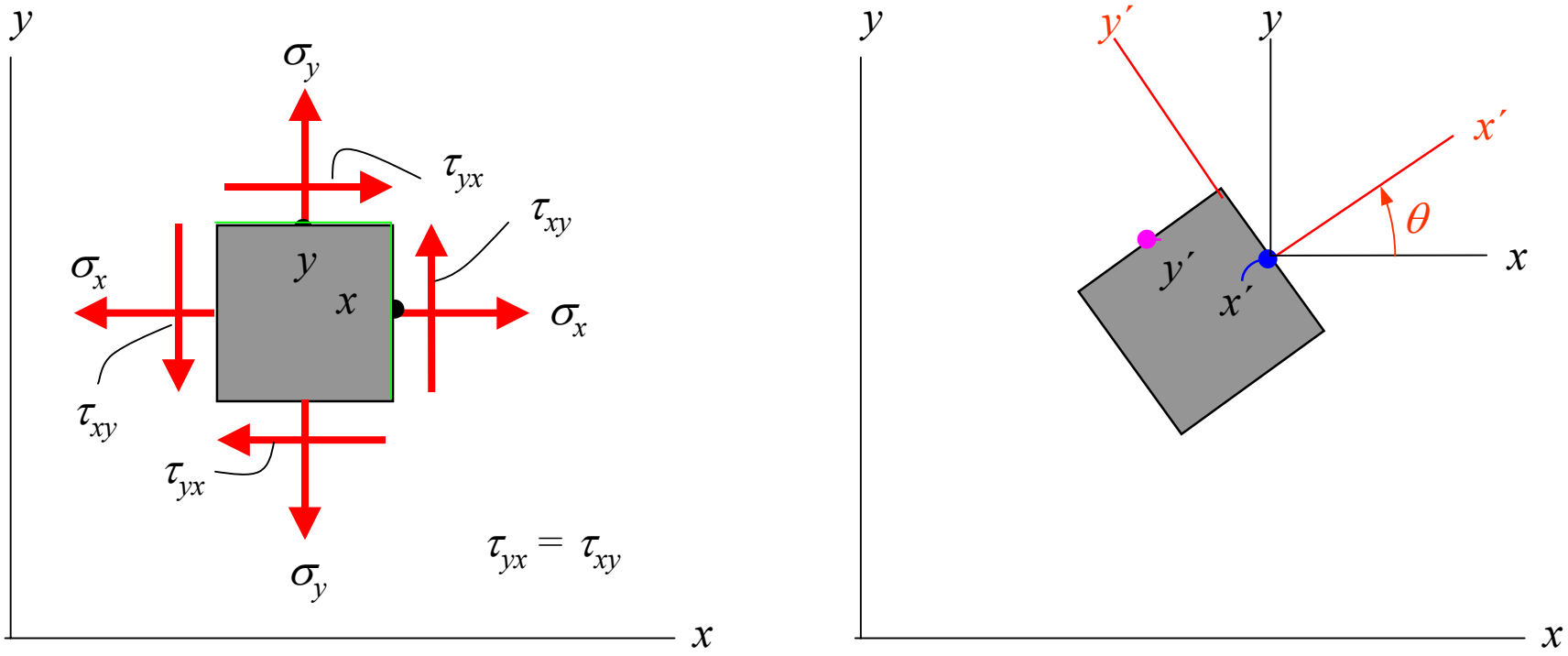






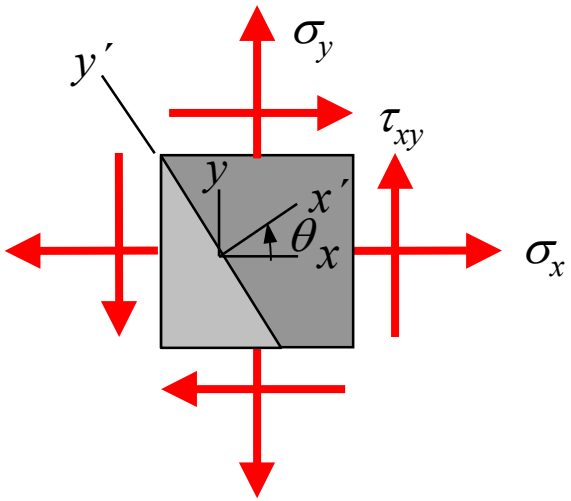
General Equations of Plane-Stress Transformation

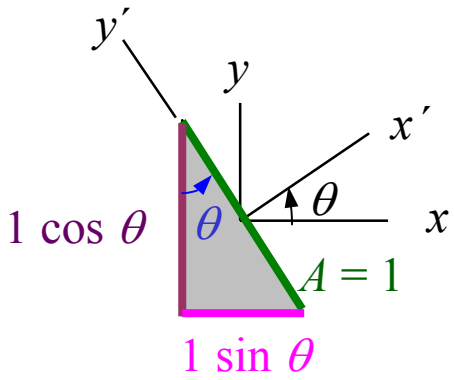
- **Sign Conventional**

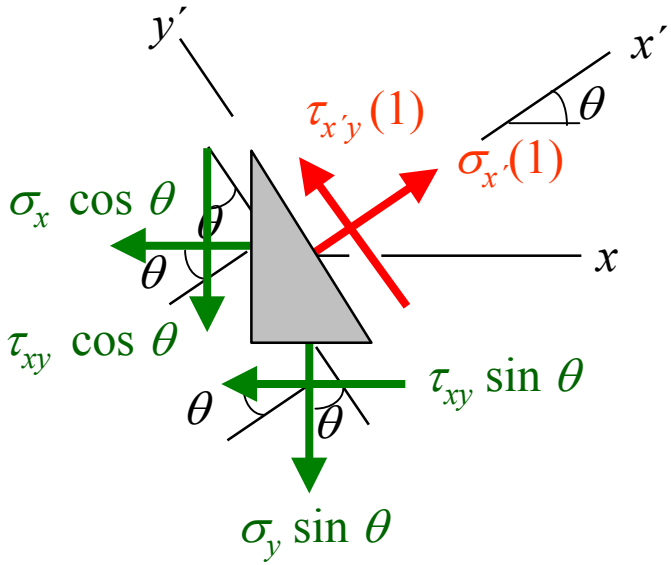


Positive Sign Convention

- **Normal and Shear Stress Components**







$+\nearrow \Sigma F_{x'} = 0$:

$$\sigma_{x'}(1) - (\tau_{xy} \sin\theta)\cos\theta - (\sigma_y \sin\theta)\sin\theta - (\tau_{xy} \cos\theta)\sin\theta - (\sigma_x \cos\theta)\cos\theta = 0$$

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)$$

$+\nwarrow \Sigma F_{y'} = 0$:

$$\tau_{x'y'}(1) + (\tau_{xy} \sin\theta)\sin\theta - (\sigma_y \sin\theta)\cos\theta - (\tau_{xy} \cos\theta)\cos\theta + (\sigma_x \cos\theta)\sin\theta = 0$$

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)$$

Using the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\sigma_{x'} = \sigma_x\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1-\cos 2\theta}{2}\right) + \tau_{xy}\sin 2\theta$$

$$\boxed{\sigma_{x'} = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta}$$

$$\boxed{\sigma_{y'} = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta}$$

Similarly,

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

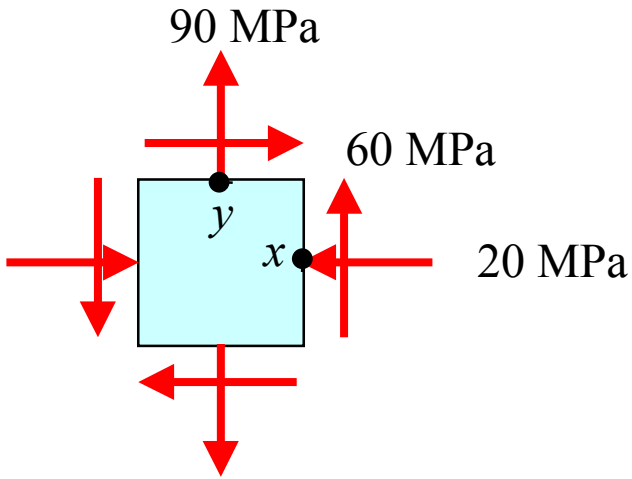
Using the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$

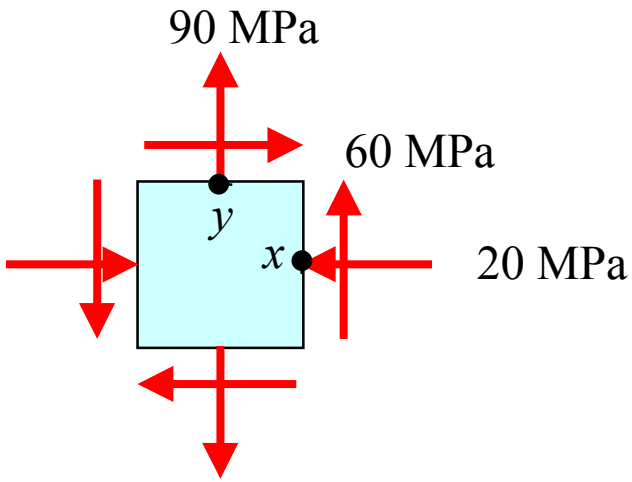
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\tau_{x'y'} = -(\frac{\sigma_x - \sigma_y}{2})\sin 2\theta + \tau_{xy}\cos 2\theta$$

Example 3

The state of plane stress at a point is represented by the element shown. Determine the state of stress at the point on another element oriented 40° counter clockwise from the position shown.





σ_x = -20 MPa

σ_y = +90 MPa

τ_{xy} = +60 MPa

θ = +40°

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{x'} = \left(\frac{-20+90}{2}\right) + \left(\frac{-20-90}{2}\right)\cos(2(+40^o)) + (60)\sin(2(40^o))$$

$$= 84.5 \text{ MPa}$$

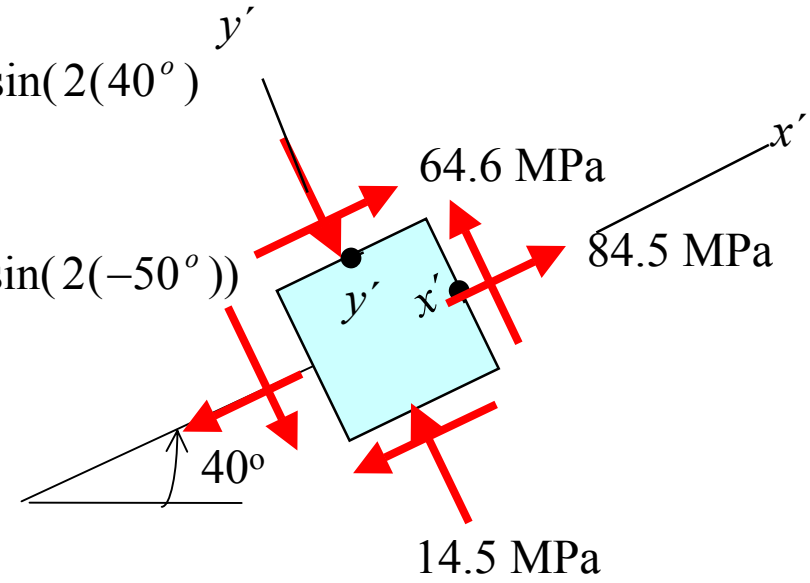
$$\sigma_{y'} = \left(\frac{-20+90}{2}\right) + \left(\frac{-20-90}{2}\right)\cos(2(-50^o)) + (60)\sin(2(-50^o))$$

$$= -14.5 \text{ MPa}$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\tau_{x'y'} = -\left(\frac{-20-90}{2}\right)\sin(2(40^o)) + (60)\cos(2(40^o))$$

$$= 64.6 \text{ MPa}$$

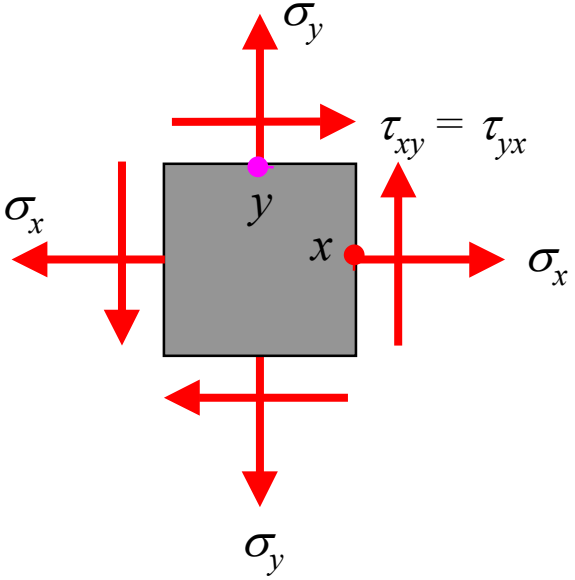


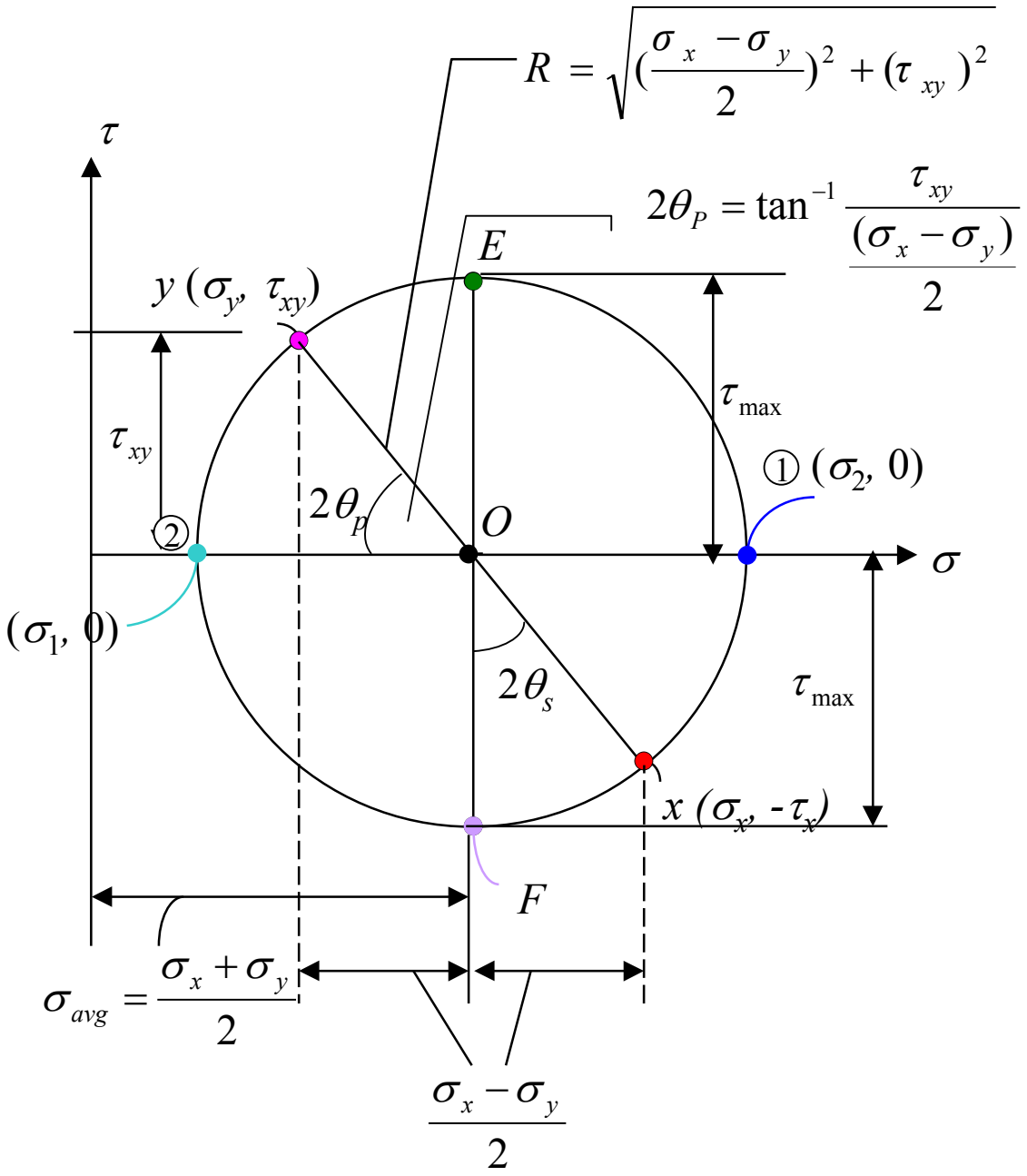
Principle Stresses

$$\frac{d\sigma_{x'}}{d\theta} = -(\frac{\sigma_x + \sigma_y}{2})(2\sin 2\theta) + \tau_{xy}\cos 2\theta = 0$$

$$\tan 2\theta_p = -\frac{\tau_{xy}}{(\frac{\sigma_x - \sigma_y}{2})}$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + (\tau_{xy})^2}$$


σ_y
$\tau_{xy} = \tau_{yx}$
σ_x
y
x
σ_x
σ_y


$R = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + (\tau_{xy})^2}$
$2\theta_P = \tan^{-1} \frac{\tau_{xy}}{\frac{(\sigma_x - \sigma_y)}{2}}$
τ
E
$y\ (\sigma_y,\ \tau_{xy})$
τ_{max}
τ_{xy}
$2\theta_p$
O
① $(\sigma_2,\ 0)$
②
σ
$(\sigma_1,\ 0)$
$2\theta_s$
τ_{max}
$x\ (\sigma_x,\ -\tau_x)$
F
$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$
$\frac{\sigma_x - \sigma_y}{2}$

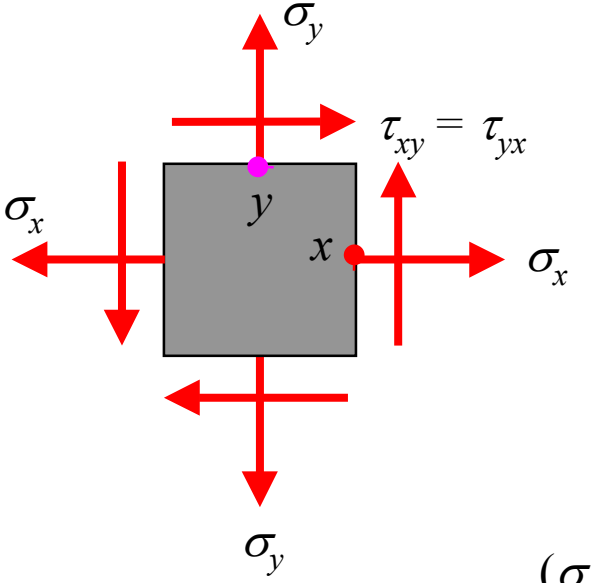
Maximum In-Plane Shear Stresses

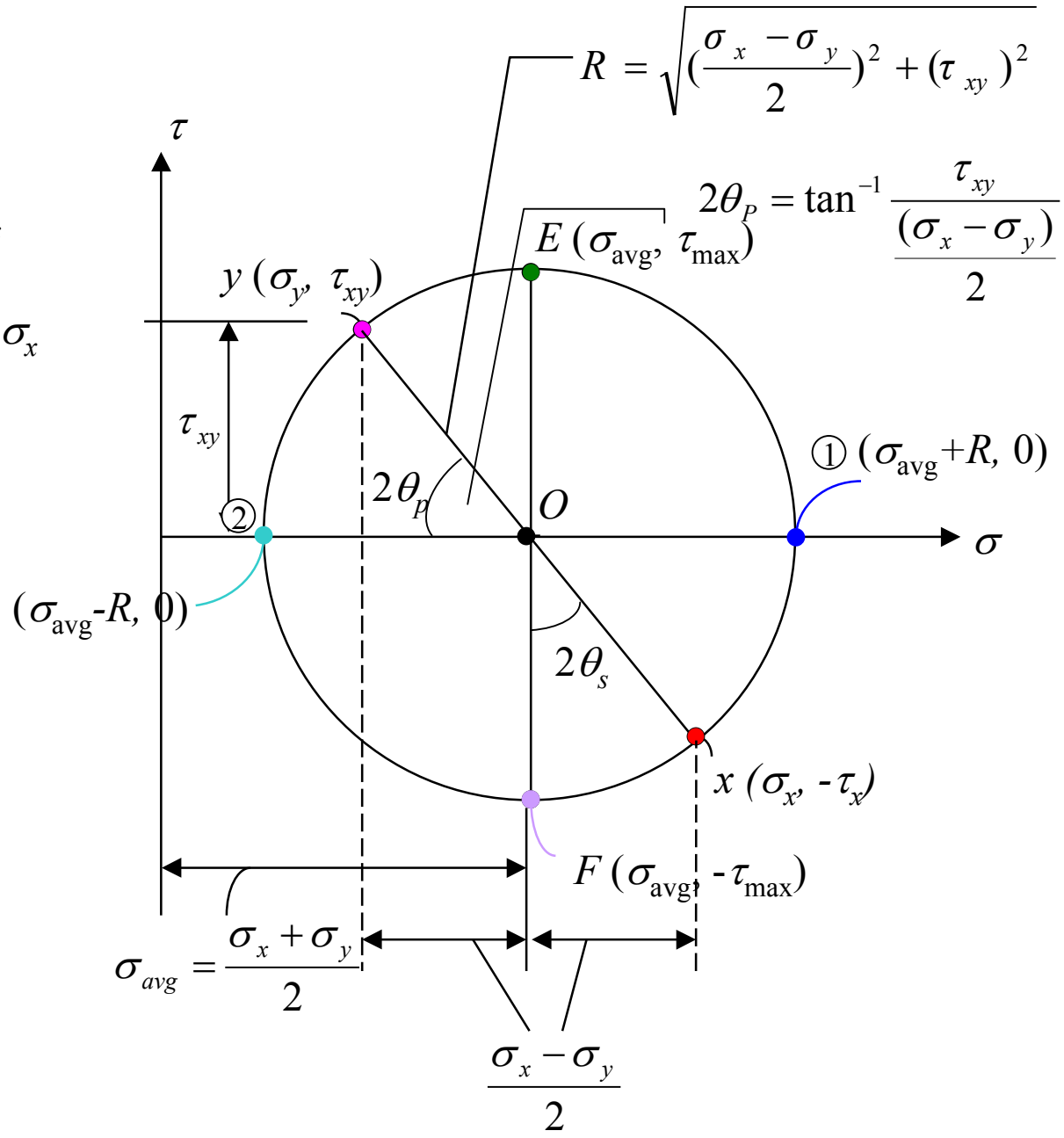
$$\frac{d\tau_{x'y'}}{d\theta} = -(\frac{\sigma_x + \sigma_y}{2})(2\cos 2\theta) - \tau_{xy}(2\sin 2\theta) = 0$$

$$\tan 2\theta_s = -\frac{(\frac{\sigma_x - \sigma_y}{2})}{\tau_{xy}}$$

$$\tau_{\max} = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + (\tau_{xy})^2}$$

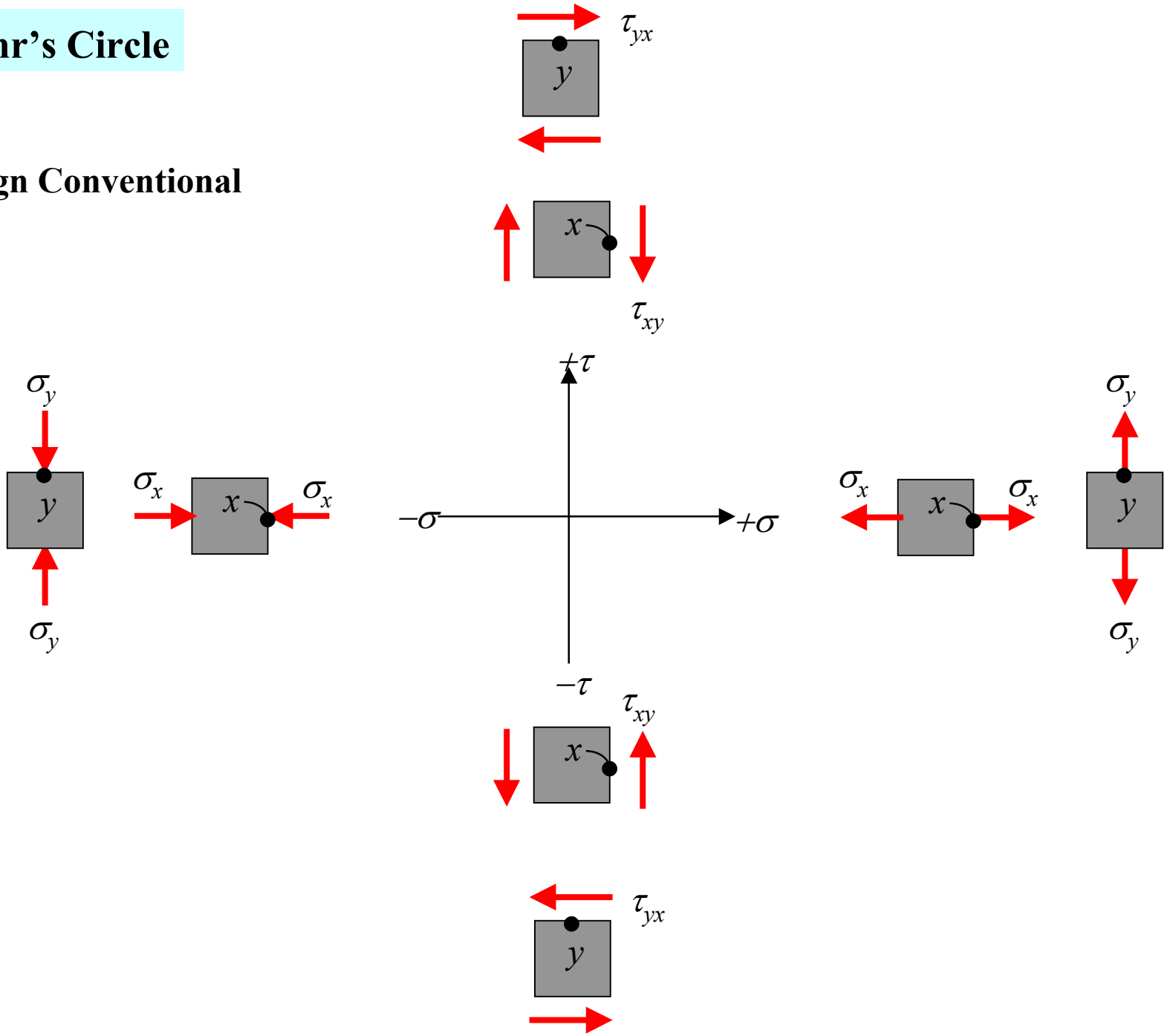
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

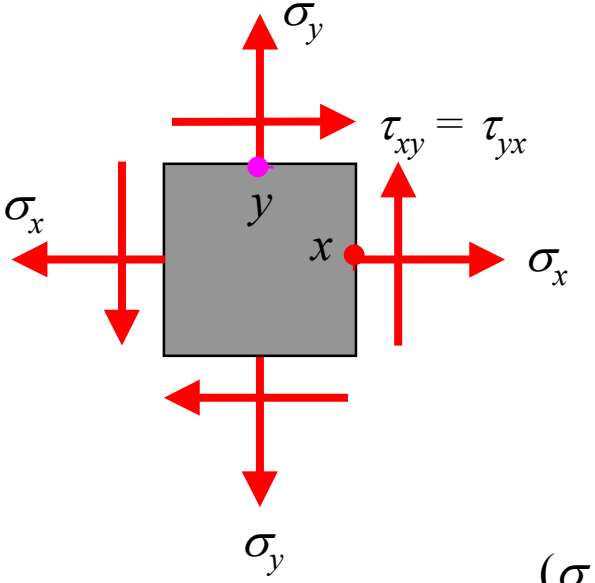

σ_y
$\tau_{xy} = \tau_{yx}$
σ_x
y
x
σ_x
σ_y

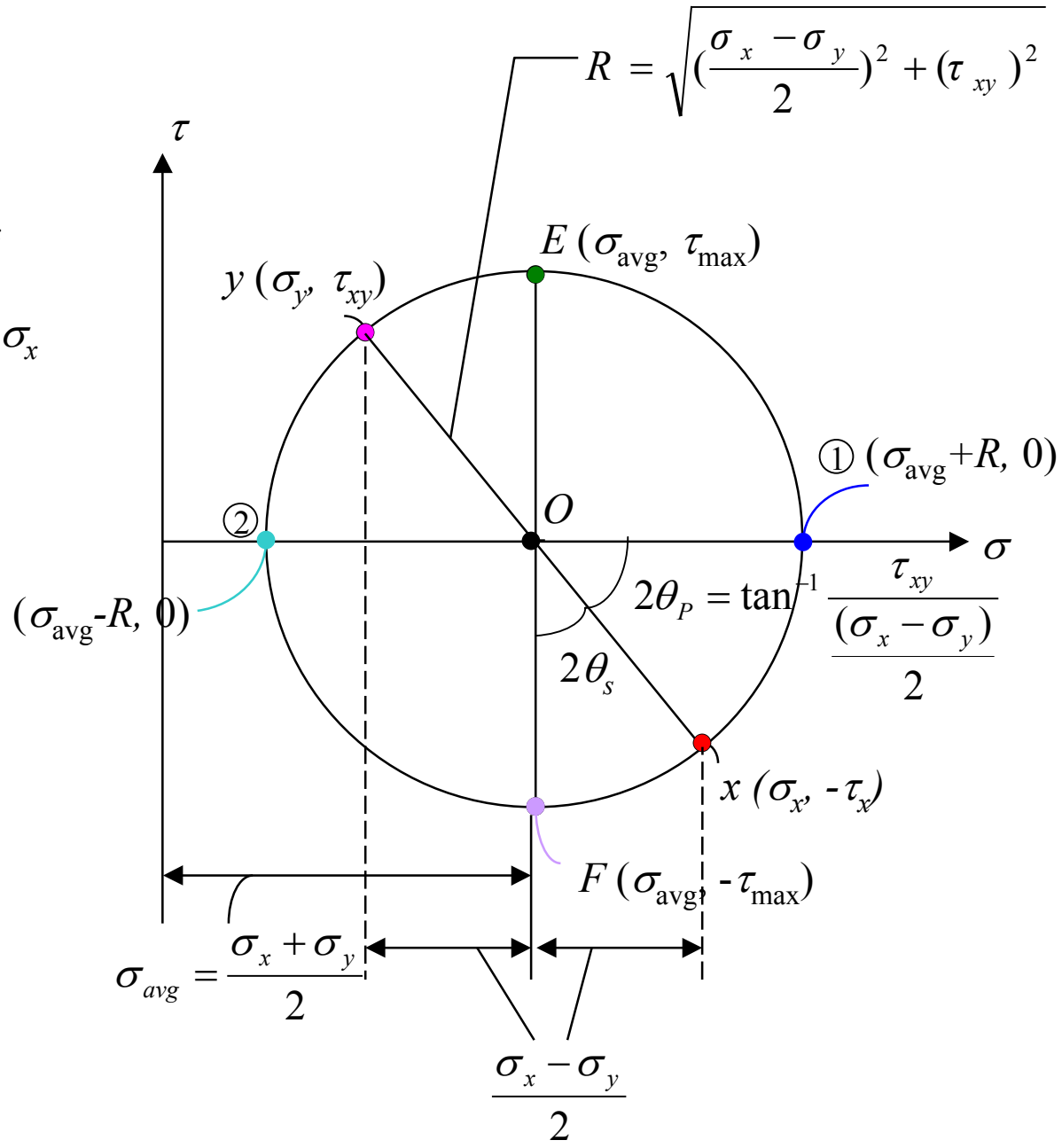

$R = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + (\tau_{xy})^2}$
$2\theta_P = \tan^{-1}\frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}$
τ
$E\ (\sigma_{avg},\ \tau_{max})$
$y\ (\sigma_y,\ \tau_{xy})$
τ_{xy}
$2\theta_p$
O
① $(\sigma_{avg}+R,\ 0)$
②
σ
$(\sigma_{avg}\text{-}R,\ 0)$
$2\theta_s$
$x\ (\sigma_x,\ -\tau_x)$
$F\ (\sigma_{avg},\ -\tau_{max})$
$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$
$\frac{\sigma_x - \sigma_y}{2}$

Mohr's Circle

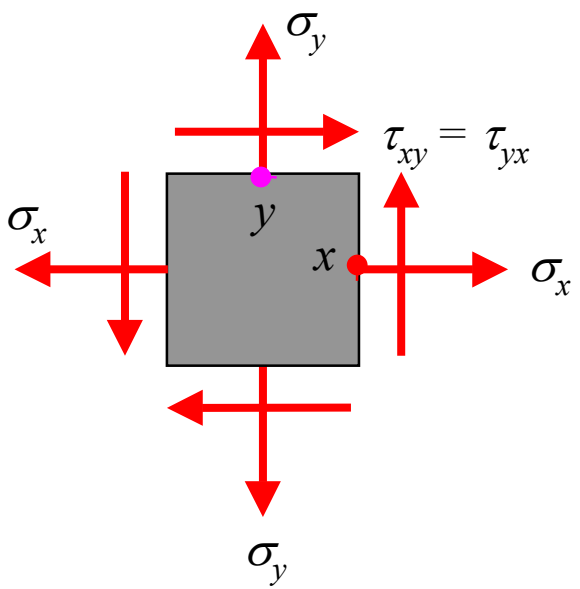
- **Sign Conventional**




σ_y
$\tau_{xy} = \tau_{yx}$
σ_x
y
x
σ_x
σ_y


$R = \sqrt{(\frac{\sigma_x - \sigma_y}{2})^2 + (\tau_{xy})^2}$
τ
$E\ (\sigma_{avg},\ \tau_{max})$
$y\ (\sigma_y,\ \tau_{xy})$
① $(\sigma_{avg}+R,\ 0)$
②
O
σ
$(\sigma_{avg}-R,\ 0)$
$2\theta_P = \tan^{-1}\frac{\tau_{xy}}{\frac{(\sigma_x - \sigma_y)}{2}}$
$2\theta_s$
$x\ (\sigma_x,\ -\tau_x)$
$F\ (\sigma_{avg},\ -\tau_{max})$
$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$
$\frac{\sigma_x - \sigma_y}{2}$

- **Procedure for Analysis**



1) Face "*x*" : $x(\sigma_x, -\tau_{xy})$

Face "*y*" : $y(\sigma_x, \tau_{xy})$

2) $\sigma_{avg} = \dfrac{\sigma_x + \sigma_y}{2}$

3) $R : \tau_{\max}$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

4) Principle Stress σ_1, σ_2

$$\sigma_{1,2} = \sigma_{avg} \pm R$$

$$|\sigma_1| > |\sigma_2|$$

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

5) Plane of principle stress θ_P

$$2\theta_P = \tan^{-1}\left|\frac{\tau_{xy}}{\left(\frac{\sigma_x - \sigma_y}{2}\right)}\right|$$

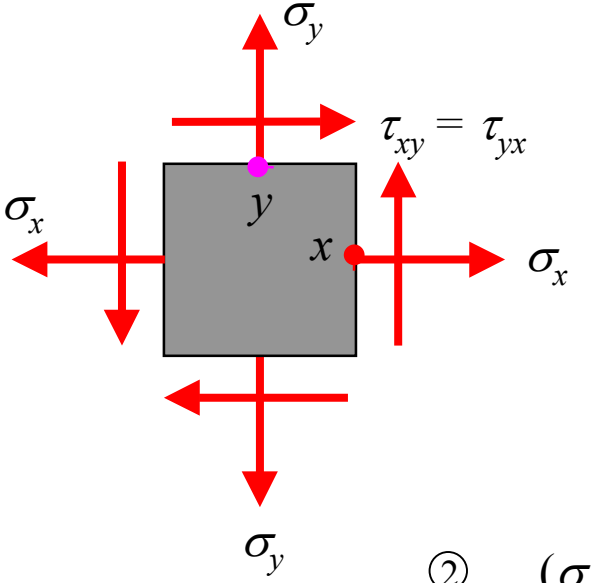
$$2\theta_s = 90^o - 2\theta_P$$

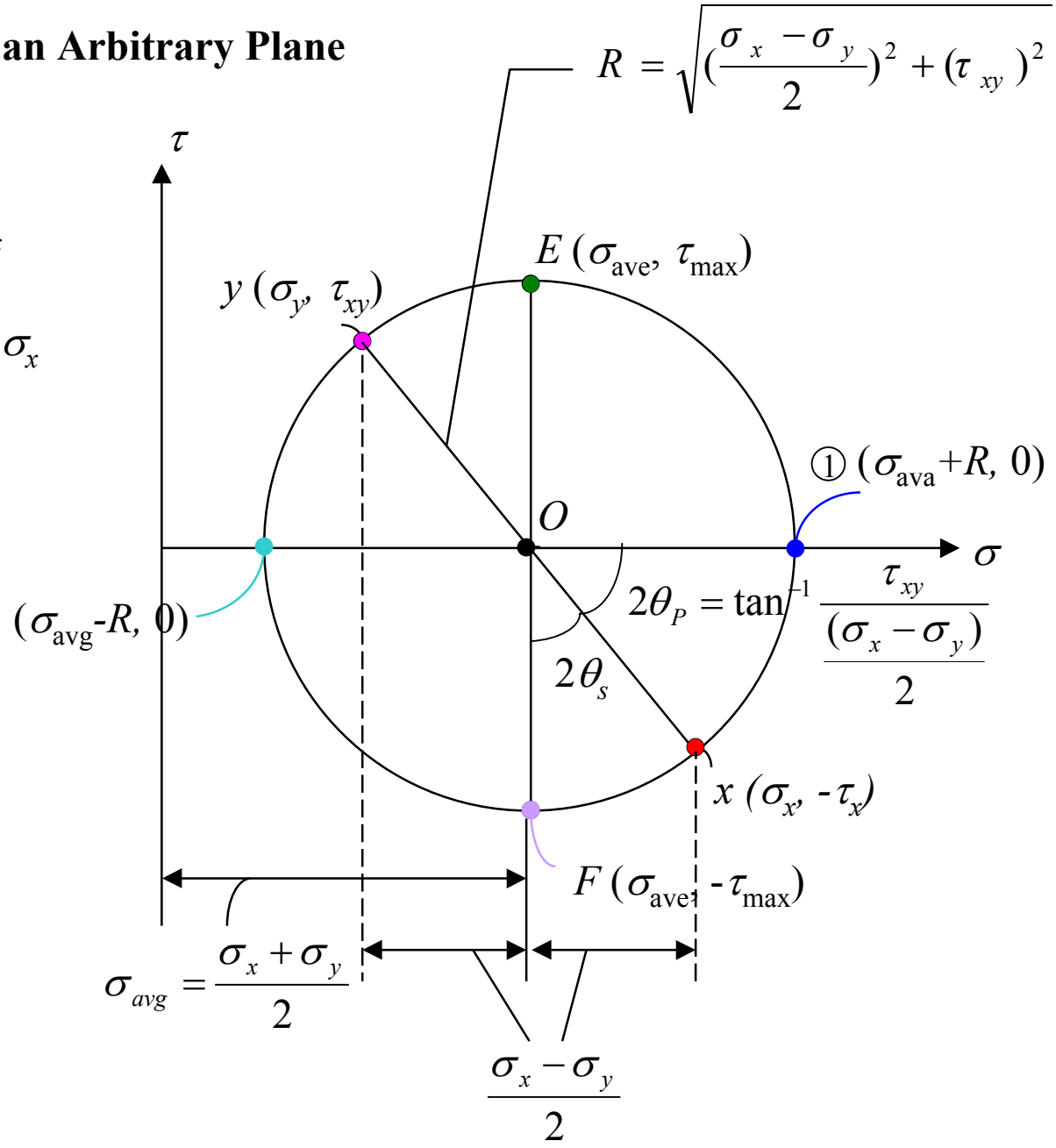
$$= \tan^{-1}\left|\frac{\left(\frac{\sigma_x - \sigma_y}{2}\right)}{\tau_{xy}}\right|$$

Note : Self-checked using

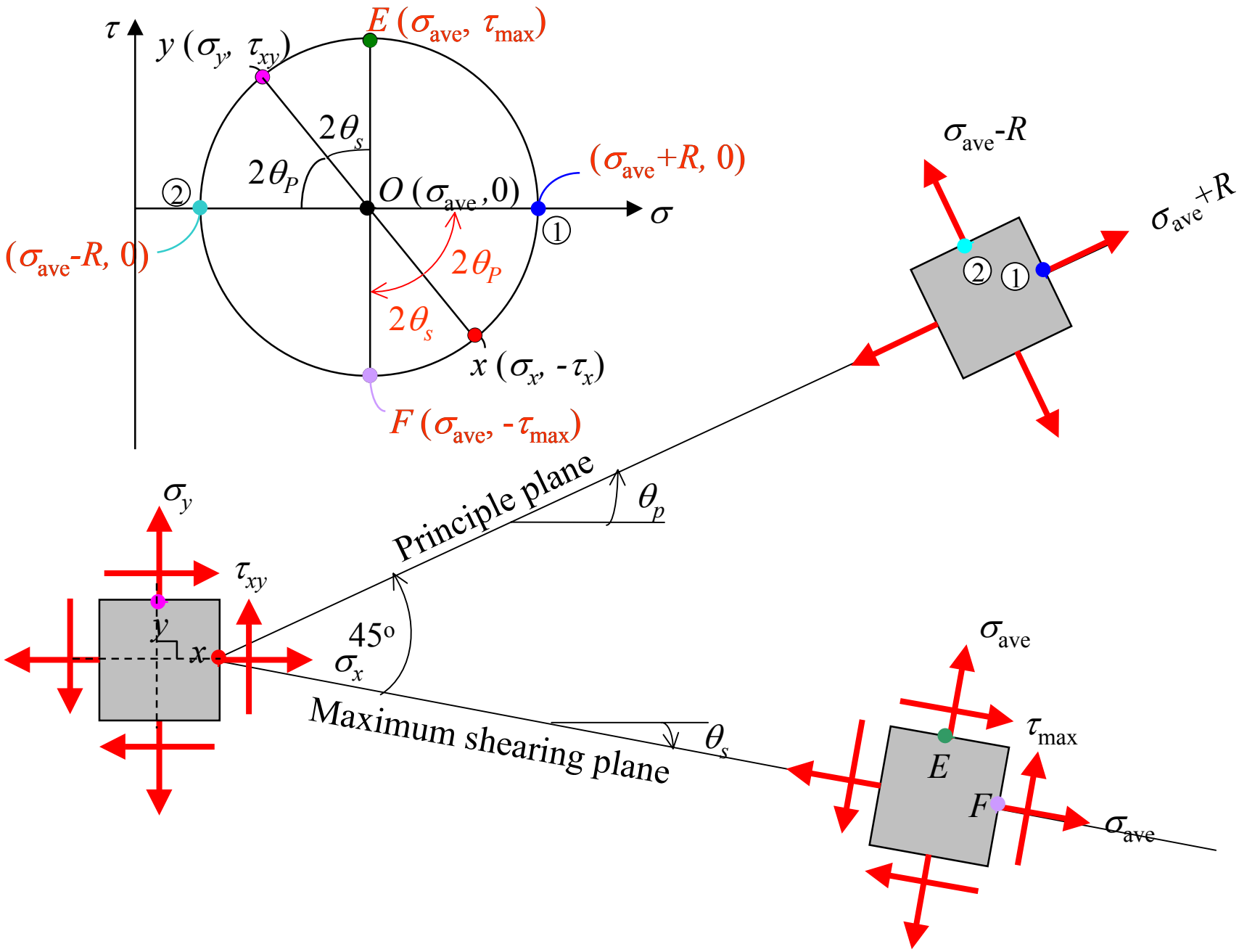
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_1 + \sigma_2}{2}$$

- **Stress Components on an Arbitrary Plane**

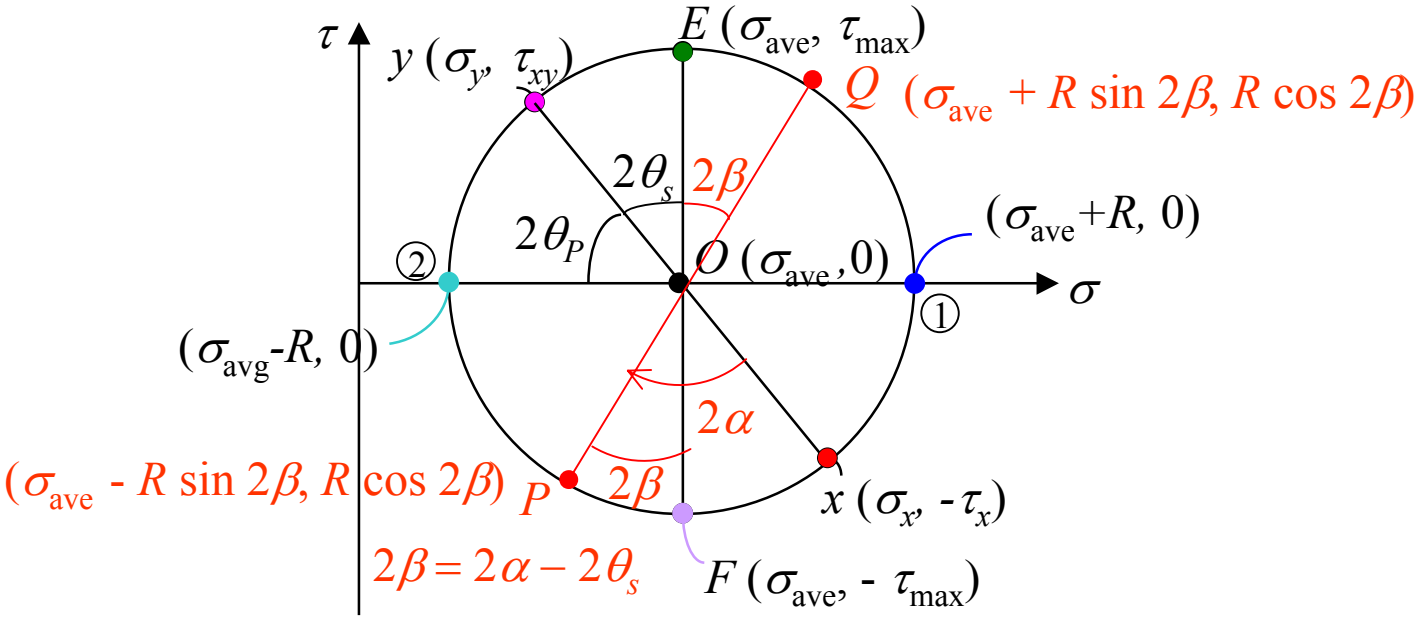


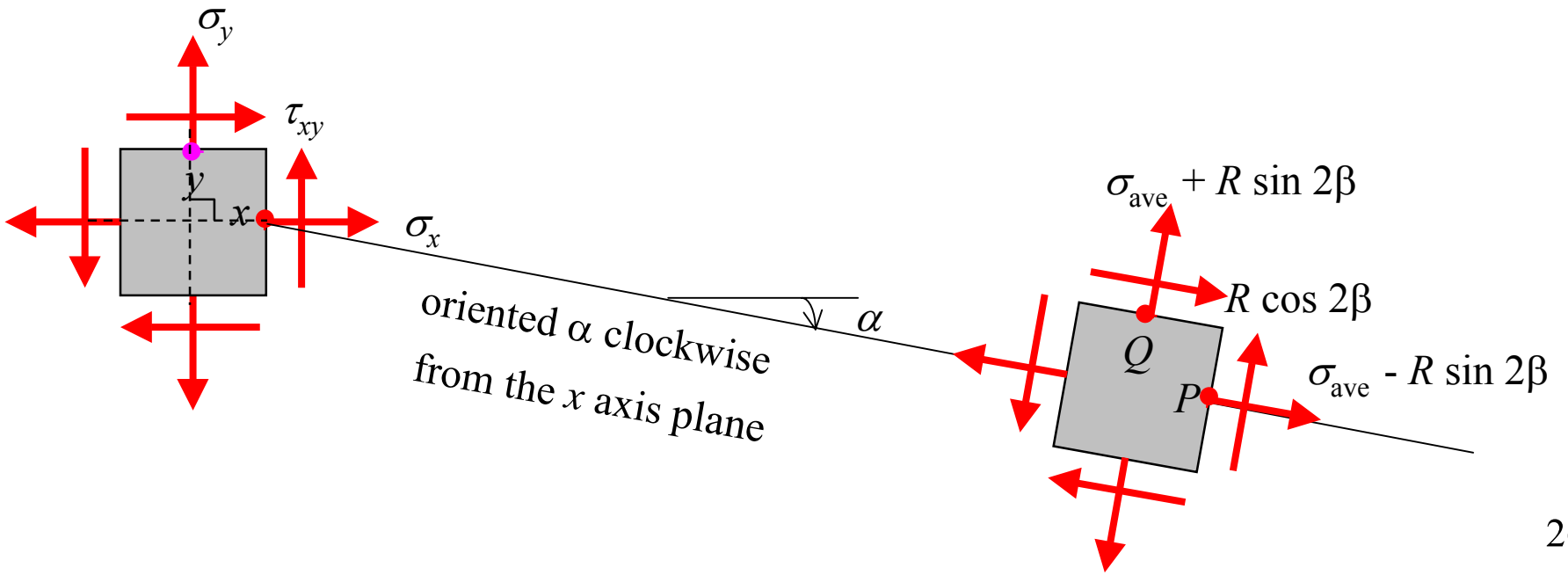


- **Principle Stresses and Maximum In-Plane Shear Stress**



• **Stress Components on an *Oriented Plane***



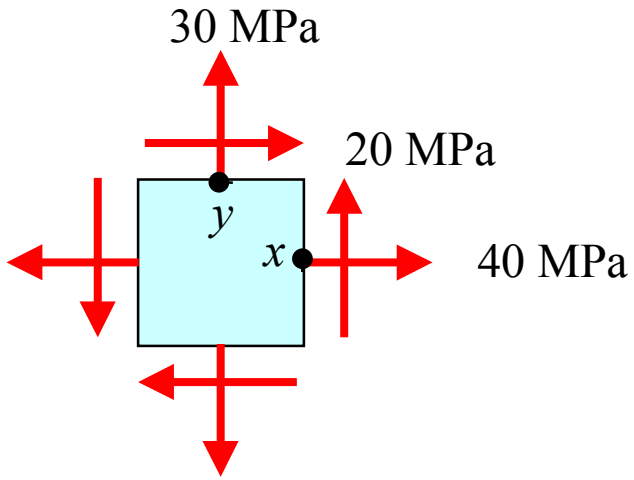


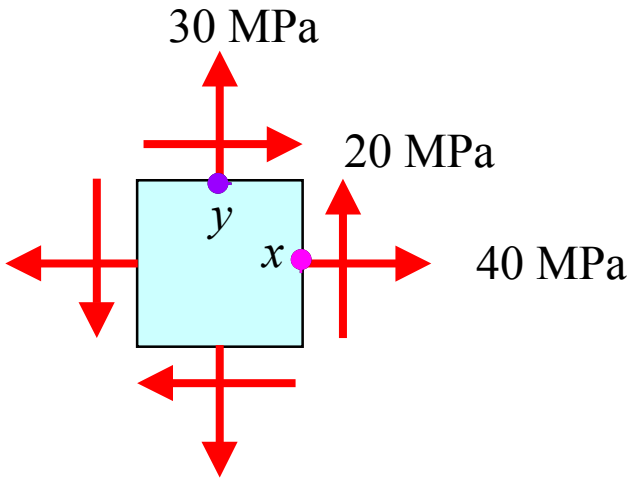
Example 4

The state of plane stress has $\sigma_x = 40$ MPa, $\sigma_y = 30$ MPa, $\tau_{xy} = 20$ MPa as shown.

(a) Represent the state of stress at the point in terms of the in-plane principal stresses.

(b) Represent the state of stress at the point in terms of the maximum in-plane shear stress



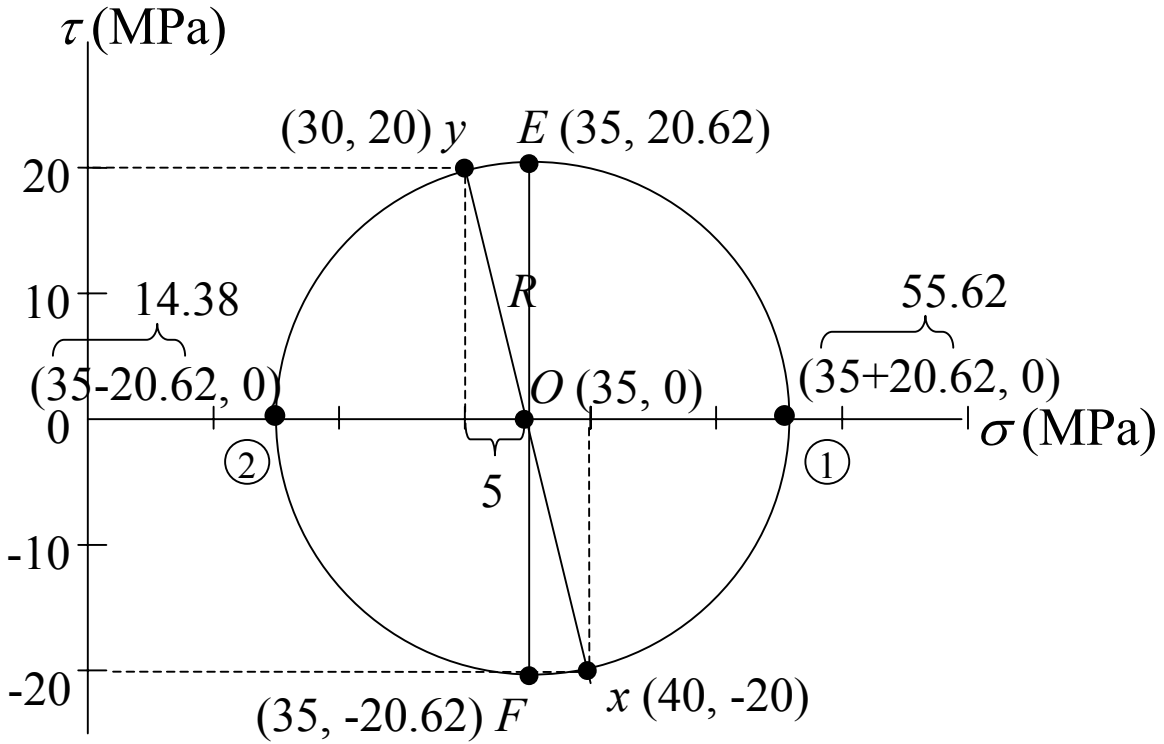


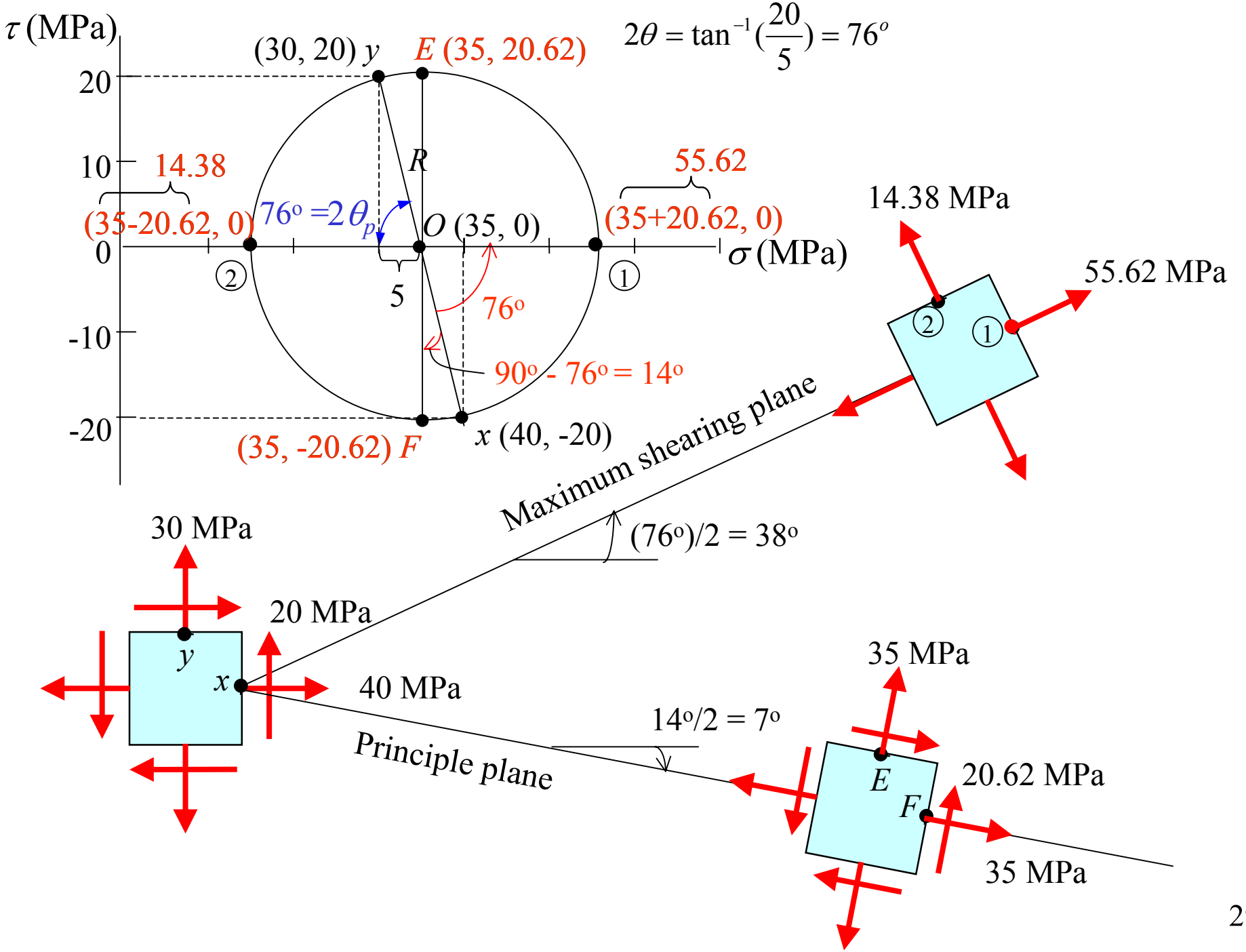
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{40+30}{2} = 35 \text{ MPa}$$

$$\frac{\sigma_x - \sigma_y}{2} = \frac{40-30}{2} = 5 \text{ MPa}$$

$$R = \sqrt{5^2 + 20^2} = 20.62 = \tau_{\max}$$

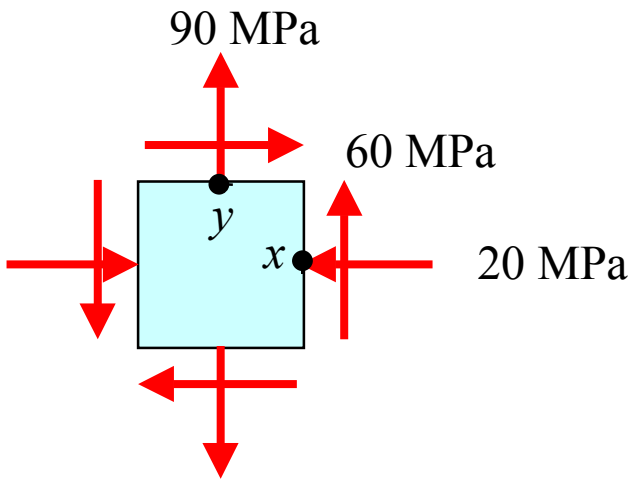
$$\sigma_{1,2} = 35 \pm 20.62 \text{ MPa}$$

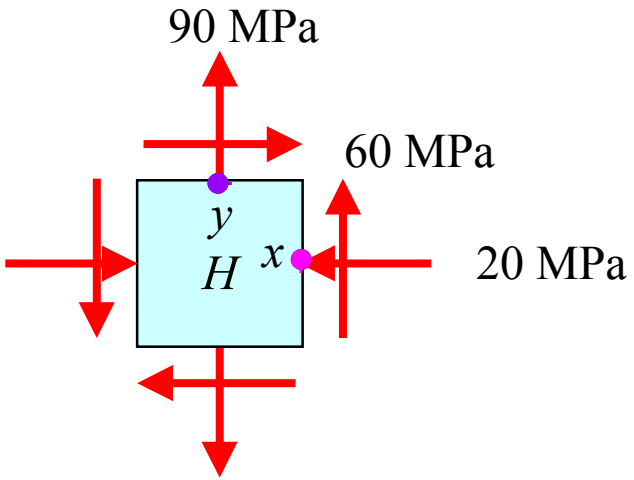



τ (MPa)
(30, 20) y
E (35, 20.62)
$2\theta = \tan^{-1}(\frac{20}{5}) = 76^o$
20
10
14.38
R
55.62
(35-20.62, 0)
76° =2 θp
O (35, 0)
(35+20.62, 0)
0
σ (MPa)
②
①
5
76°
-10
90° - 76° = 14°
-20
(35, -20.62) F
x (40, -20)
14.38 MPa
55.62 MPa
Maximum shearing plane
(76°)/2 = 38°
30 MPa
20 MPa
y
x
40 MPa
14°/2 = 7°
Principle plane
35 MPa
E
F
20.62 MPa
35 MPa

Example 5

The state of plane stress at a point on a body is shown.
(a) Represent the state of stress at the point in terms of the in-plane principal stresses.
(b) Represent the state of stress at the point in terms of the maximum in-plane shear stress
(c) Represent the state of stress at the point that oriented 40^{o} counter-clockwise from the x axis

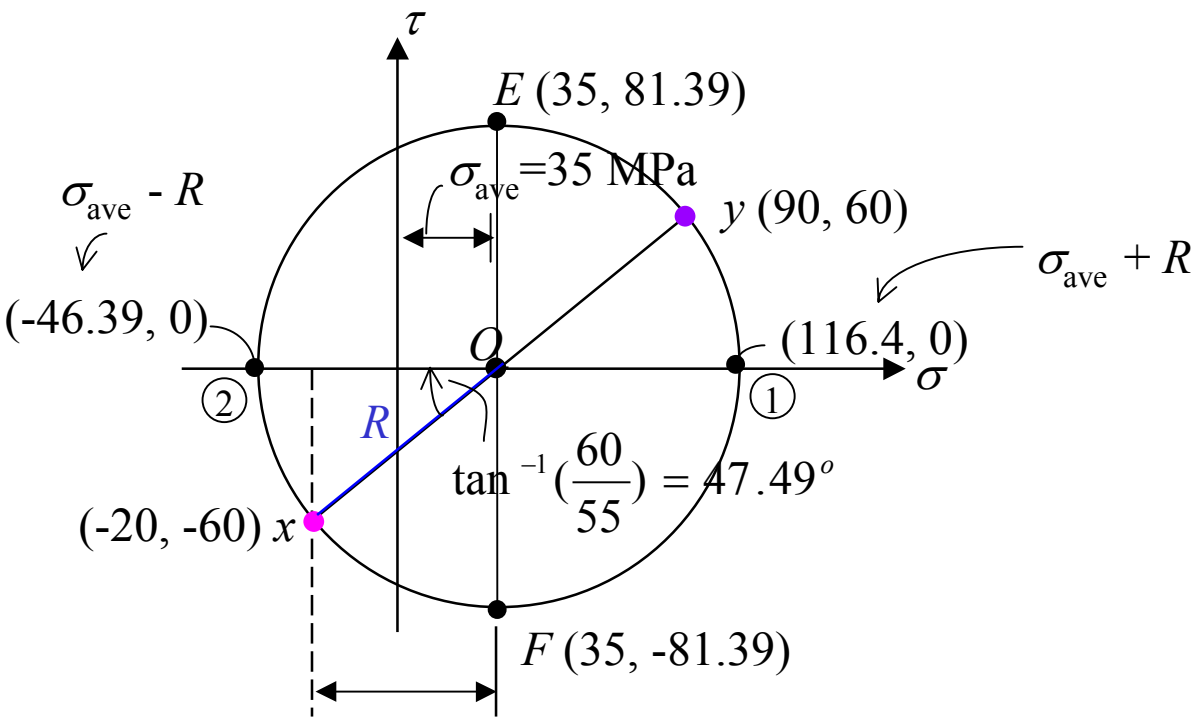


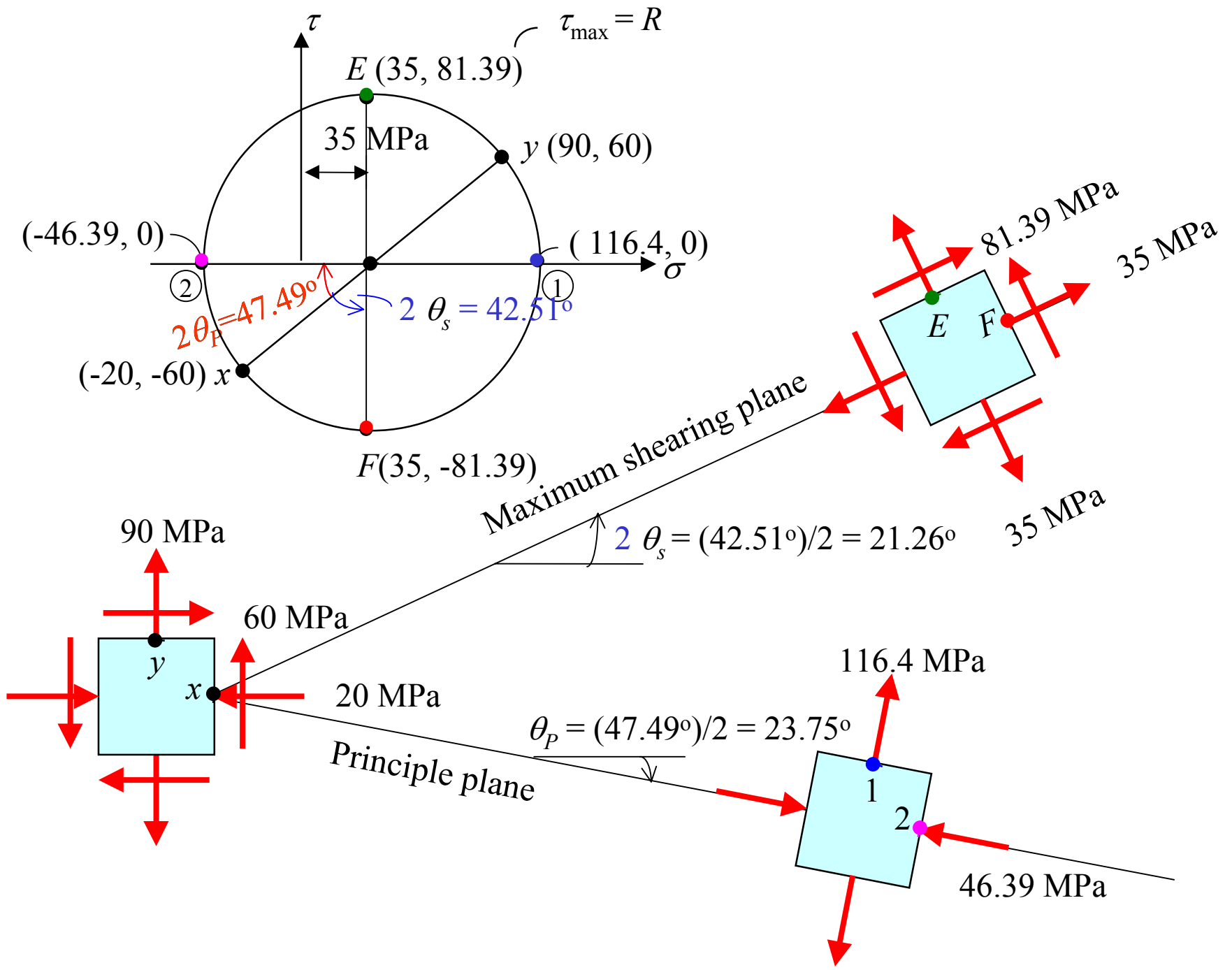


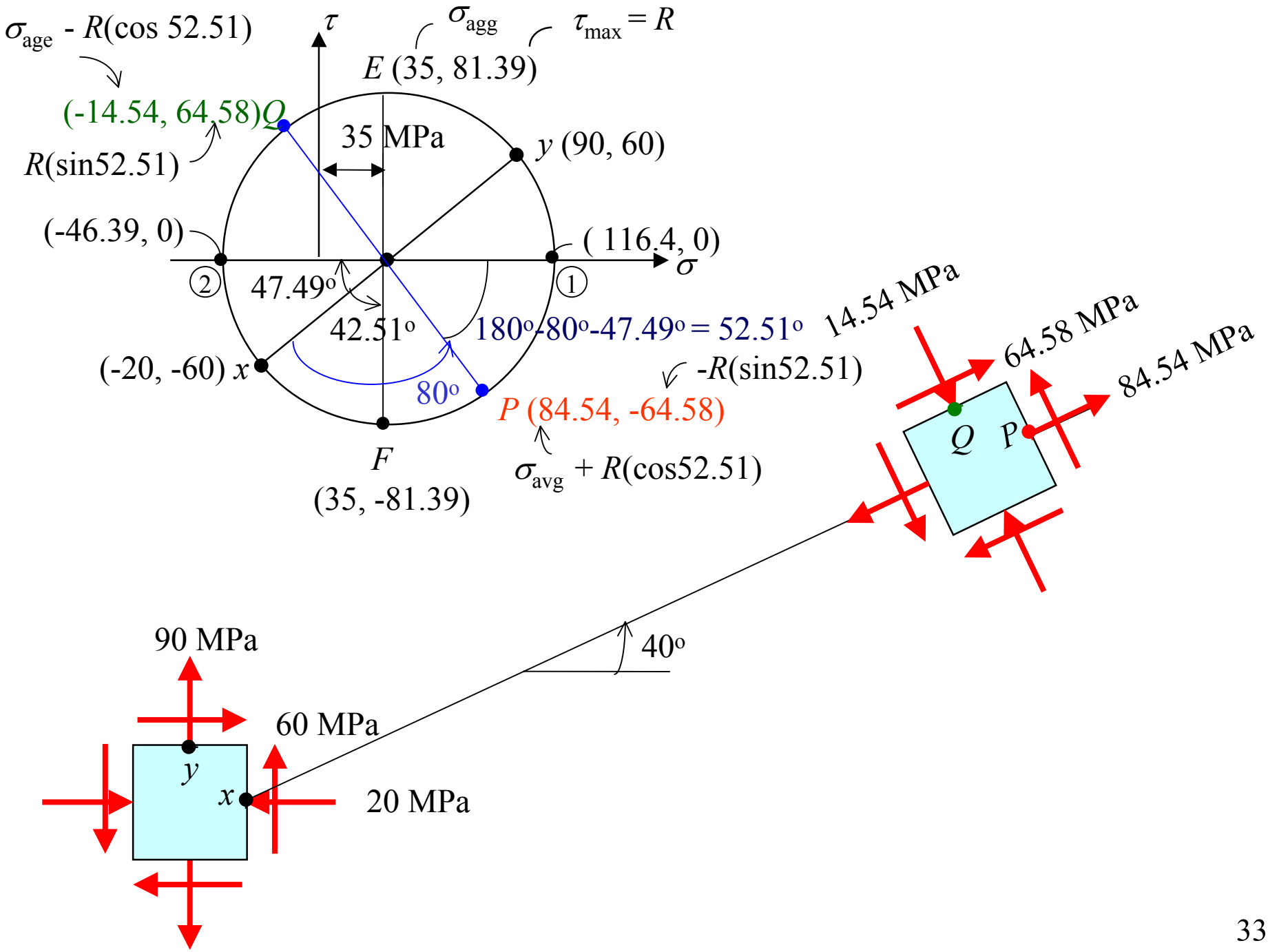
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{-20 + 90}{2} = 35 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

$$= \sqrt{\left(\frac{-20 - 90}{2}\right)^2 + (-60)^2} = \pm 81.39 \text{ MPa}$$

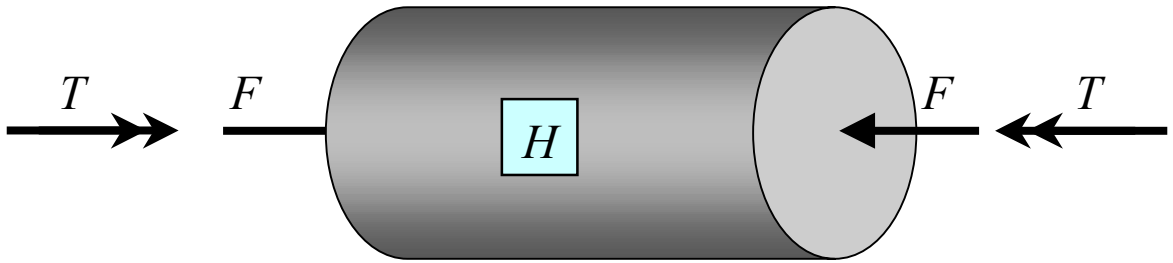


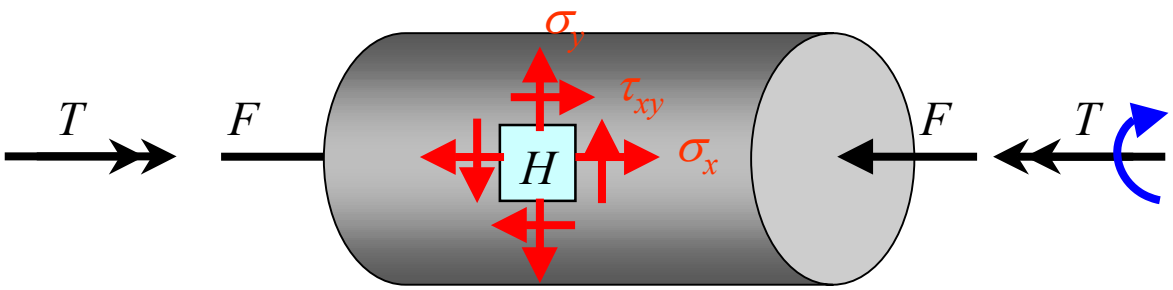

τ
σ
τmax = R
E (35, 81.39)
35 MPa
y (90, 60)
(-46.39, 0)
(116.4, 0)
2
1
2θP=47.49°
2 θs = 42.51°
(-20, -60) x
F(35, -81.39)
81.39 MPa
35 MPa
E
F
35 MPa
Maximum shearing plane
2 θs = (42.51°)/2 = 21.26°
90 MPa
60 MPa
y
x
20 MPa
116.4 MPa
θP = (47.49°)/2 = 23.75°
Principle plane
1
2
46.39 MPa


σ_{age} - R(cos 52.51)
τ
σ_{agg}
$\tau_{max} = R$
E (35, 81.39)
(-14.54, 64.58)Q
35 MPa
y (90, 60)
R(sin52.51)
(-46.39, 0)
(116.4, 0)
σ
2
1
47.49°
42.51°
180°-80°-47.49° = 52.51°
14.54 MPa
64.58 MPa
84.54 MPa
(-20, -60) x
80°
-R(sin52.51)
P (84.54, -64.58)
F
(35, -81.39)
σ_{avg} + R(cos52.51)
Q
P
40°
90 MPa
60 MPa
y
x
20 MPa

Example 6

A thin walled cylindrical has an outer diameter of 45 mm and a thickness of 1 mm,has the internal pressure of 3.5 MPa, has compression force of 8500 N and has torsion 100 N•m. From the state of stress at point *H* ,
(a) Represent the state of stress at the point of the maximum shearing stress, and the state of stress at the point of the principle stresses (σ_1, σ_2).
(b) Represent the state of stress at the point oriented 30^o clockwise from the horizontal axis

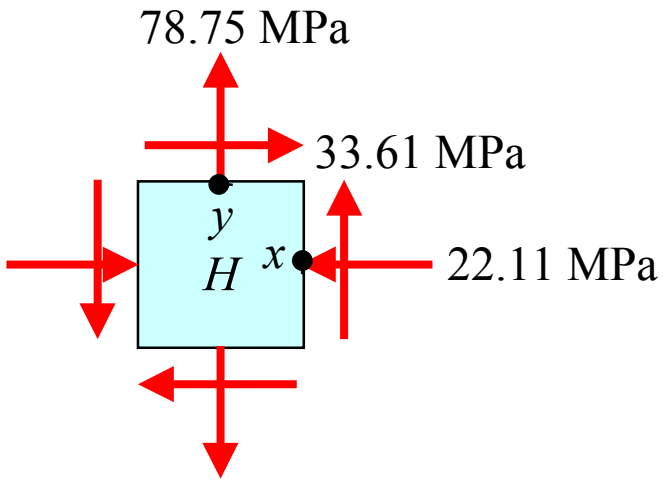




$$\sigma_x = \frac{Pr}{2t} + \frac{F}{A}$$

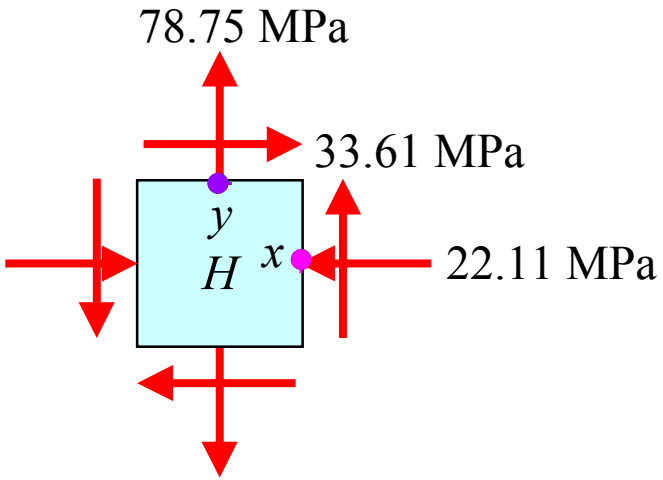
$$= \frac{(3.5\times10^6)(0.0225)}{2(0.001)} + \frac{(-8500)}{\pi\,(0.0225^2 - 0.0215^2)}$$

$$= +39.38 - 61.49 = -22.11 \text{ MPa}$$



$$\sigma_y = \frac{Pr}{t} = \frac{(3.5\times10^6)(0.0225)}{0.001} = 78.75 \text{ MPa}$$

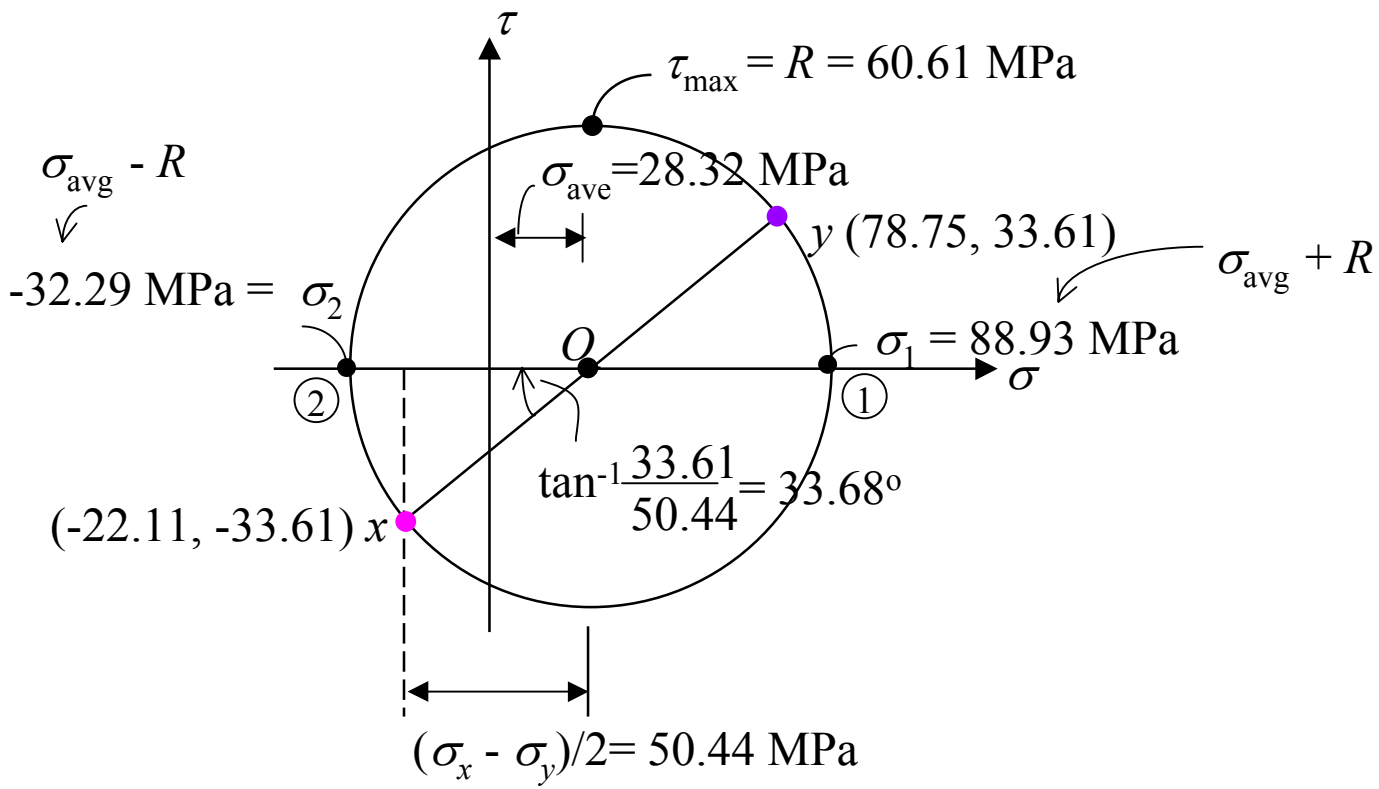
$$\tau_{xy} = \frac{Tc}{J} = \frac{(100)(0.0225)}{(\pi/2)(0.0225^4 - 0.0215^4)} = 33.61 \text{ MPa}$$

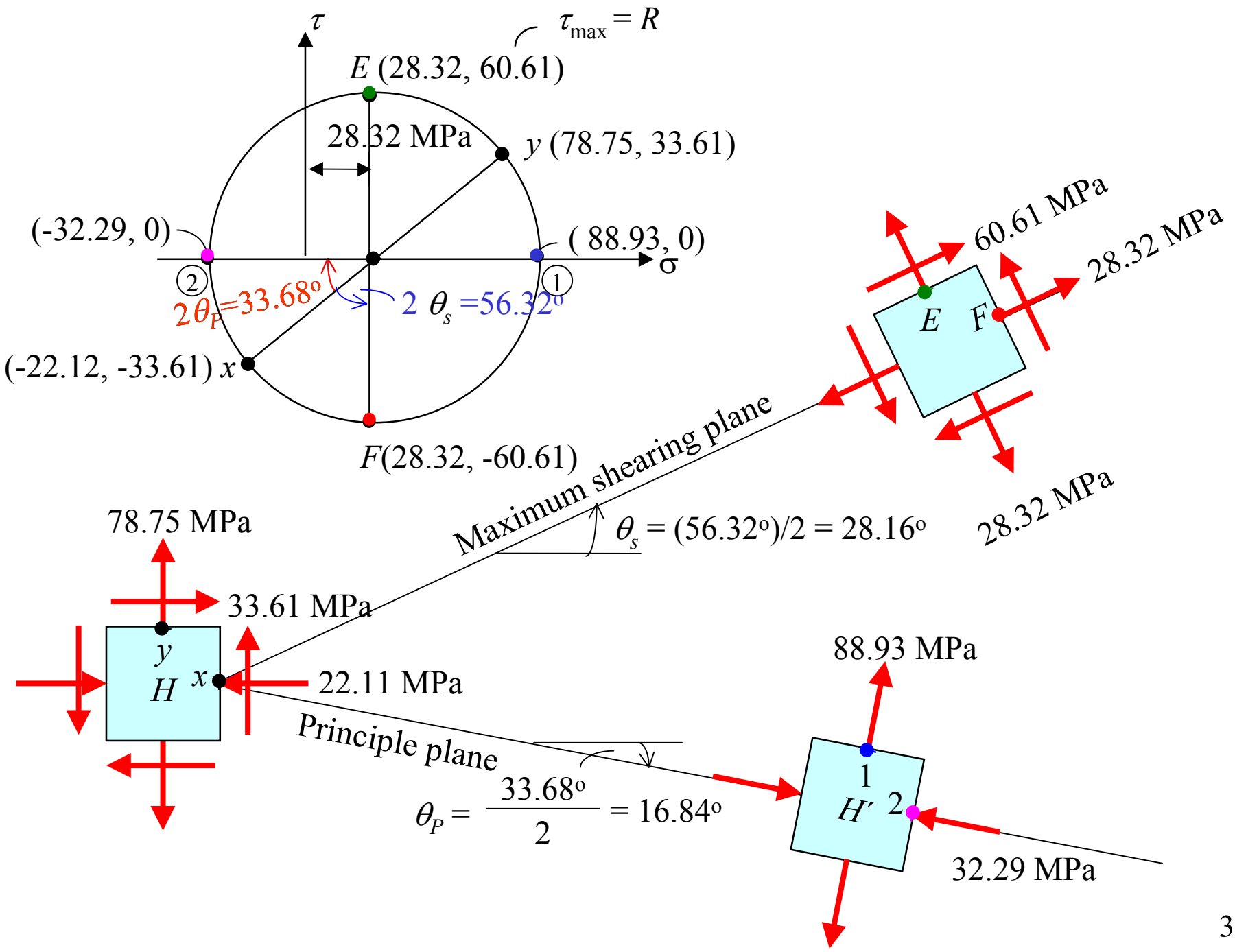


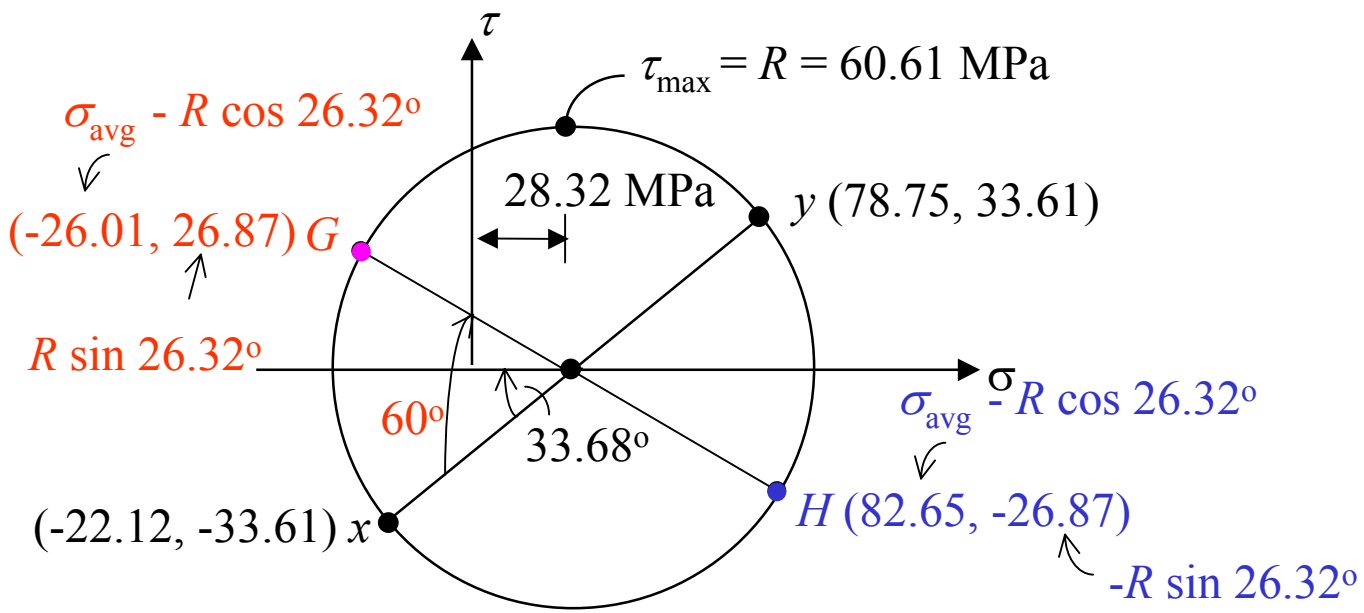
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{-22.11 + 78.75}{2} = 28.32 \text{ MPa}$$

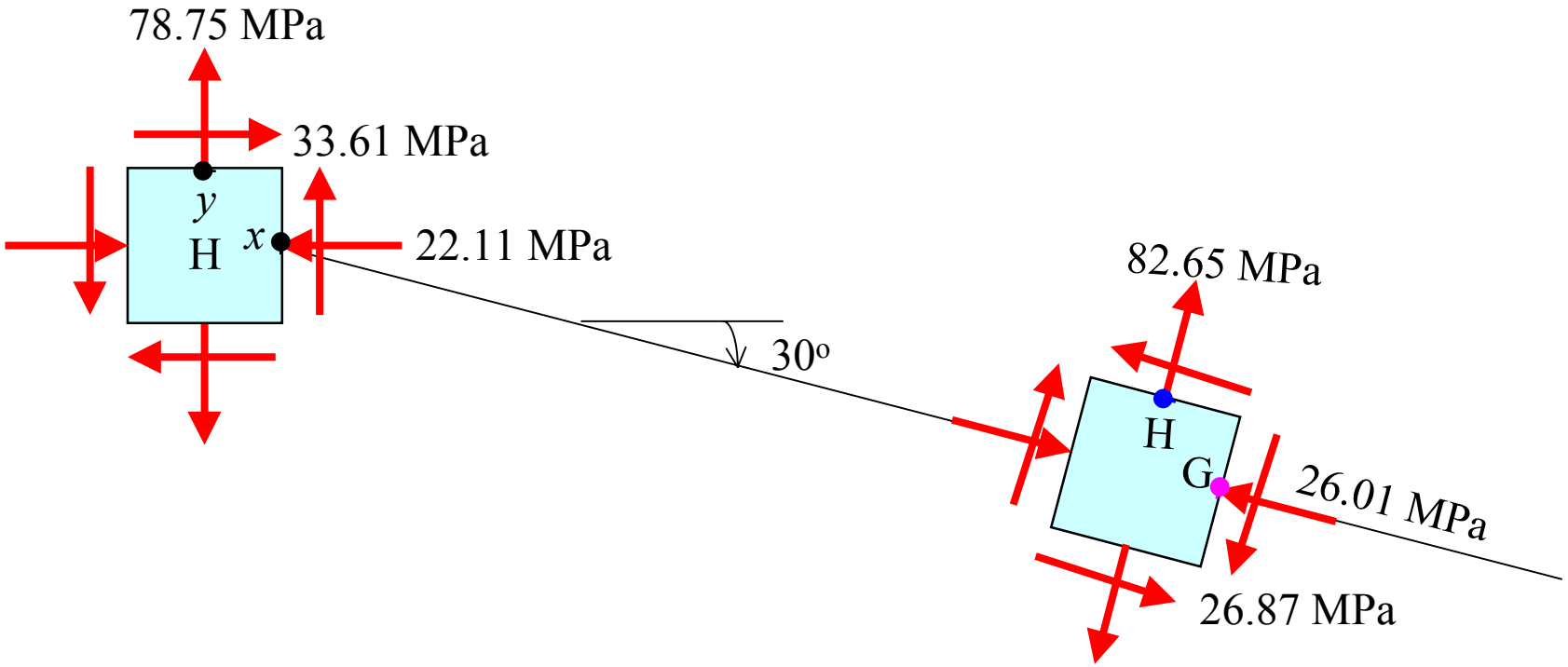
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + (\tau_{xy})^2}$$

$$= \sqrt{\left(\frac{-22.12 - 78.75}{2}\right)^2 + (33.61)^2} = \pm 60.61 \text{ MPa}$$



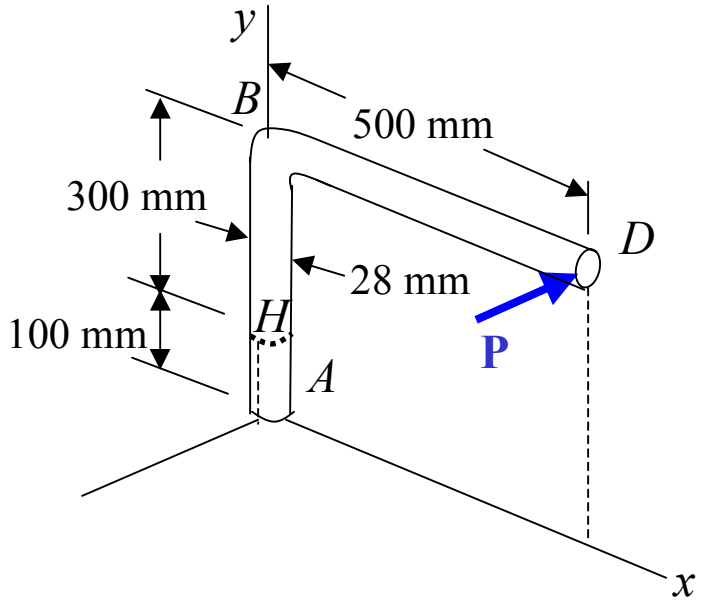

τ
τmax = R
E (28.32, 60.61)
28.32 MPa
y (78.75, 33.61)
(-32.29, 0)
(88.93, 0)
σ
2θP=33.68o
2 θs =56.32o
(-22.12, -33.61) x
F(28.32, -60.61)
60.61 MPa
28.32 MPa
E
F
28.32 MPa
Maximum shearing plane
θs = (56.32o)/2 = 28.16o
78.75 MPa
33.61 MPa
y
H
x
22.11 MPa
Principle plane
θP = 33.68o / 2 = 16.84o
88.93 MPa
1
H′
2
32.29 MPa

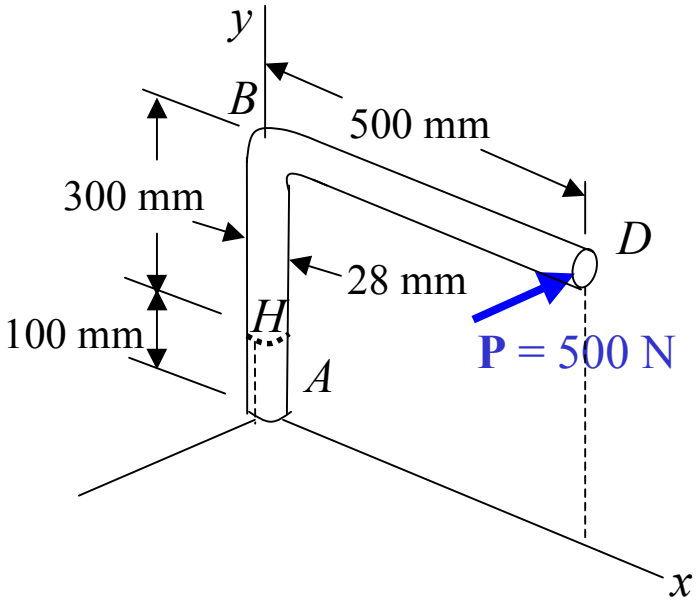

τ
τmax = R = 60.61 MPa
σavg - R cos 26.32o
(-26.01, 26.87) G
28.32 MPa
y (78.75, 33.61)
R sin 26.32o
σ
σavg - R cos 26.32o
60o
33.68o
(-22.12, -33.61) x
H (82.65, -26.87)
-R sin 26.32o

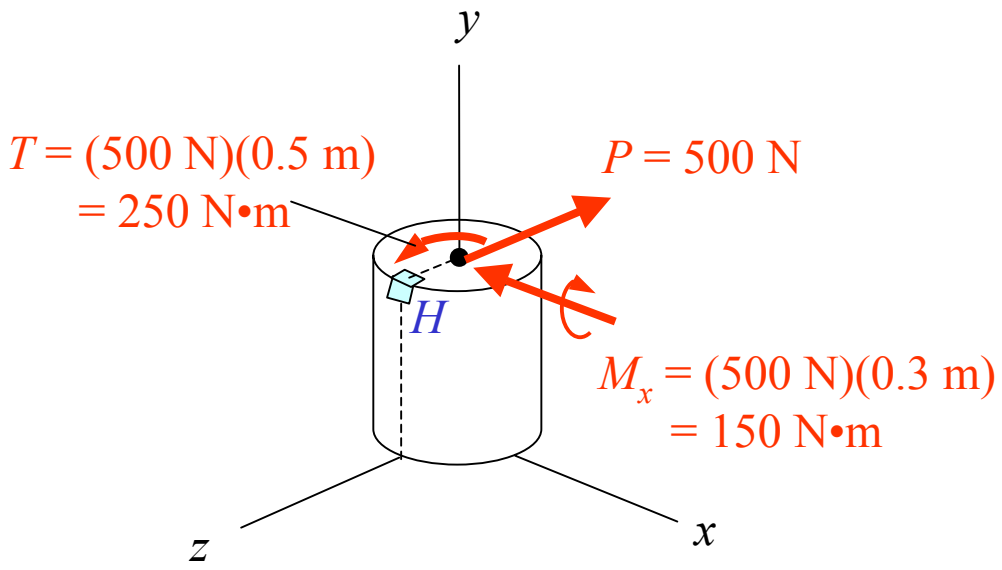

78.75 MPa
33.61 MPa
y
H
x
22.11 MPa
30o
82.65 MPa
H
G
26.01 MPa
26.87 MPa

Example 7a

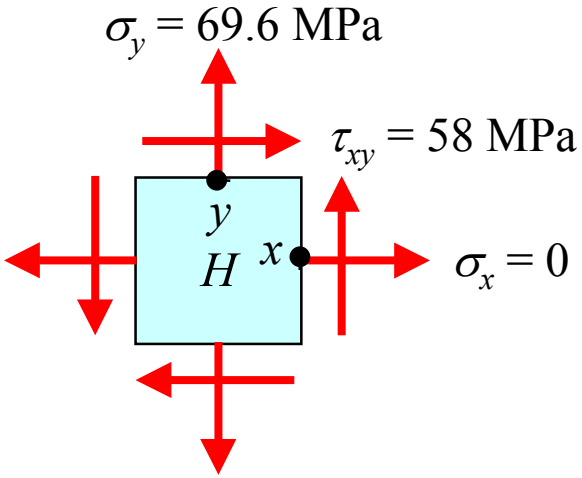
A single horizontal force *P* of magnitude 500 N is applied to end *D* of lever *ABD*. Knowing that portion *AB* of the lever has a diameter of 28 mm, determine (a) the normal and shearing stresses on an element located at point *H* and having sides parallel to the *x* and *y* axes, (b) Represent the state of stress at the point of the maximum shearing stress, and the state of stress at the point of the principle stresses (σ_1, σ_2) at point *H*, (c) Represent the state of stress at point *H* oriented 15^o outer clockwise from the horizontal axis







(a) the normal and shearing stresses on an element located at point *H*

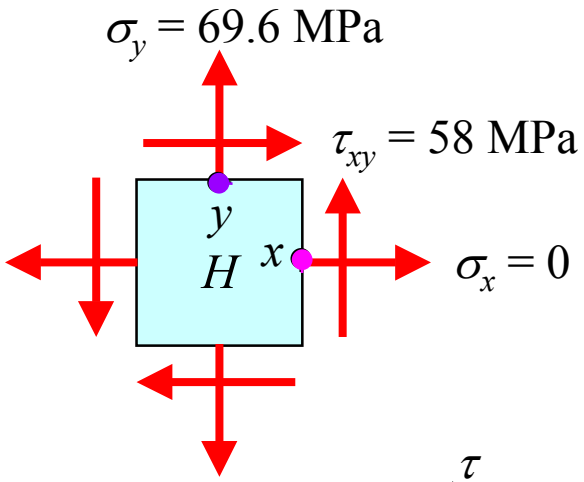


$$\sigma_x = 0$$

$$\sigma_y = +\frac{Mc}{I} = \frac{(150\ \text{N}\bullet\text{m})(0.014\ \text{m})}{(\frac{\pi}{4})(0.014\ \text{m})^4} = 69.6\ \text{MPa}$$

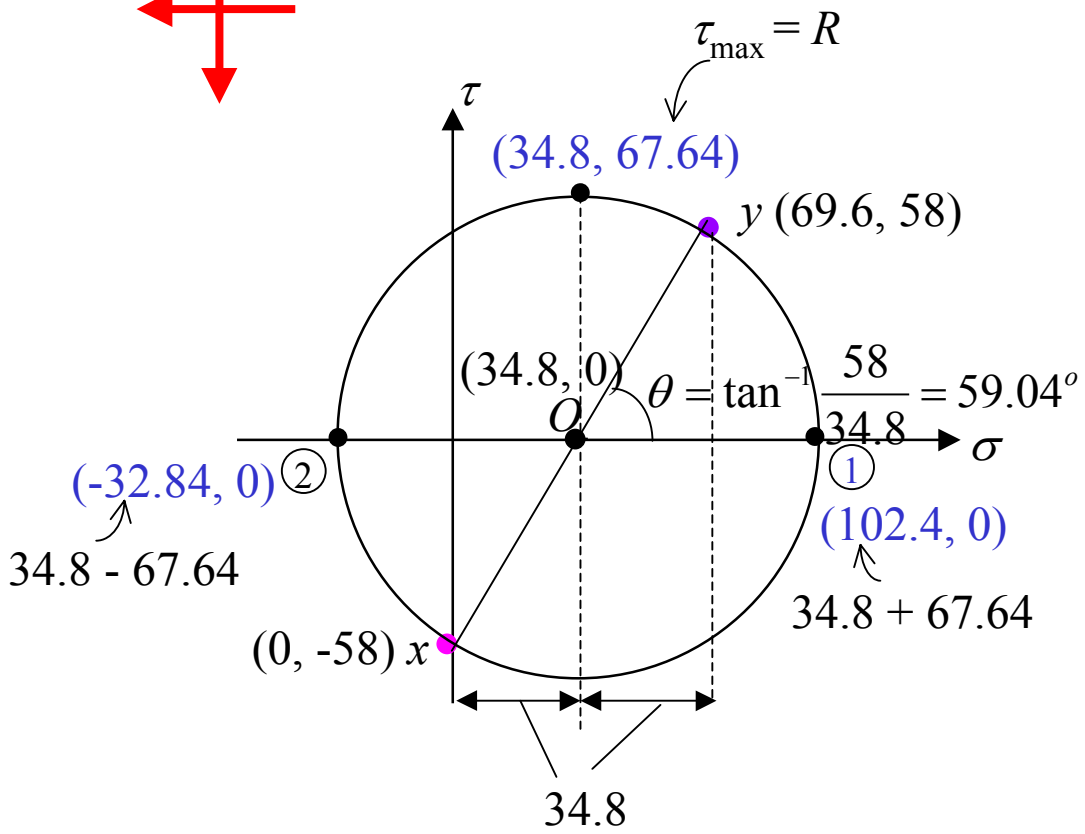
$$\tau_{xy} = +\frac{Tc}{J} = \frac{(250\ \text{N}\bullet\text{m})(0.014\ \text{m})}{(\frac{\pi}{2})(0.014\ \text{m})^4} = 58.0\ \text{MPa}$$

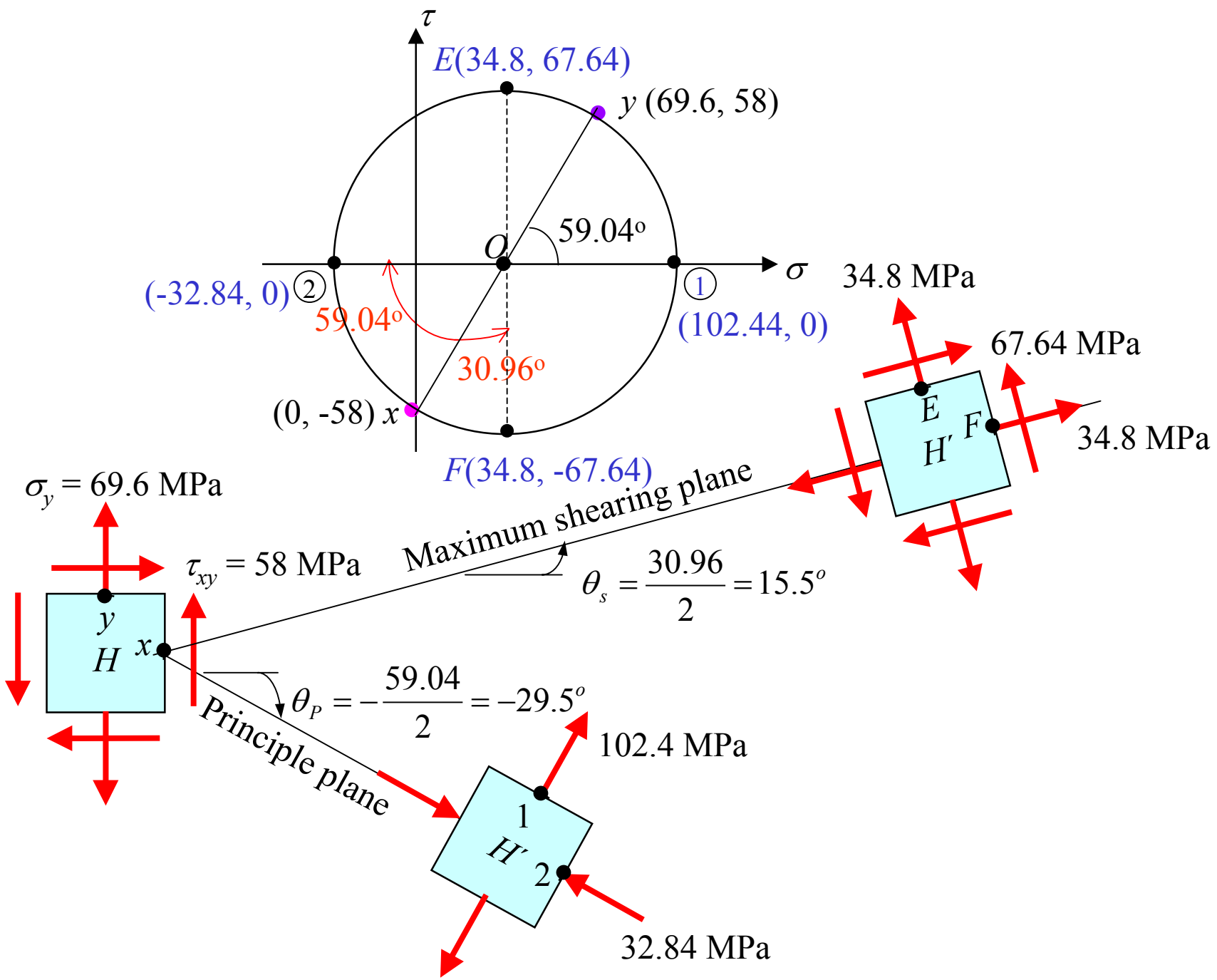
(b)



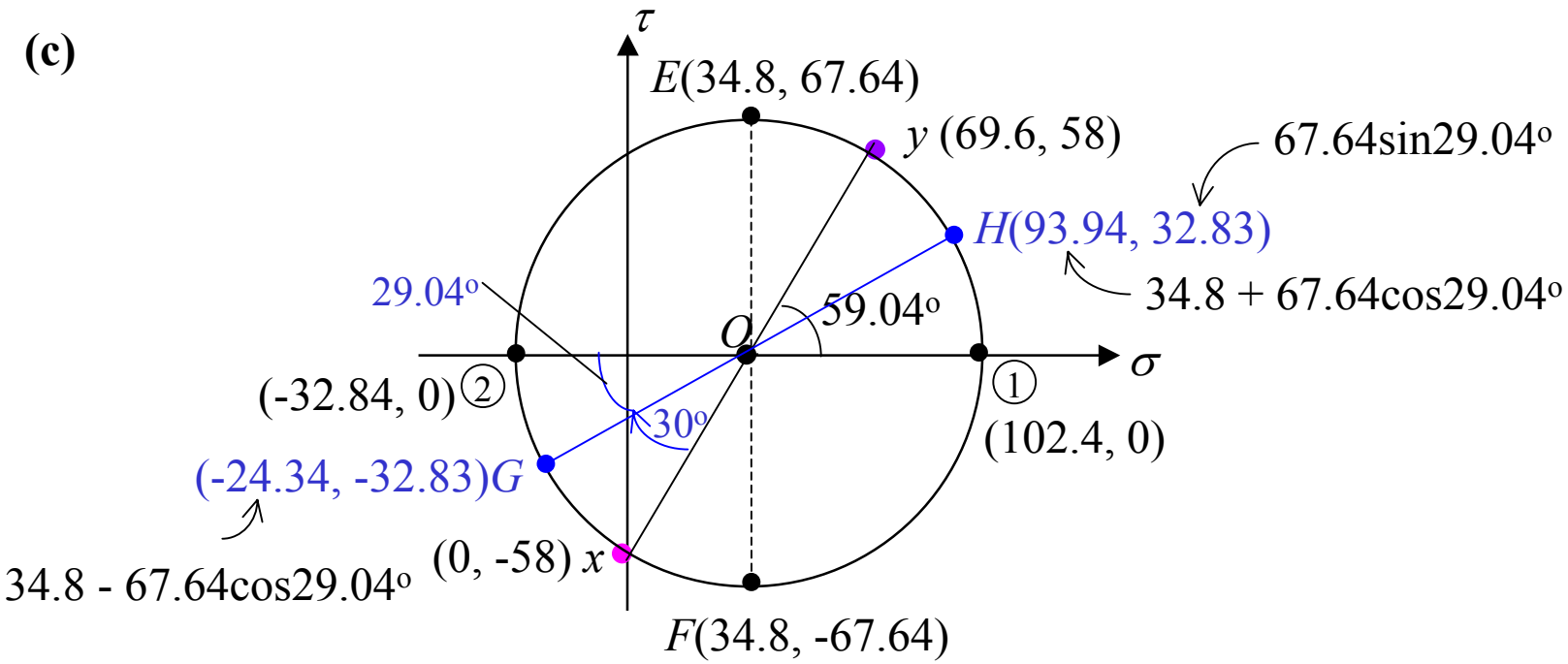
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{0 + 69.6}{2} = 34.8 \text{ MPa}$$

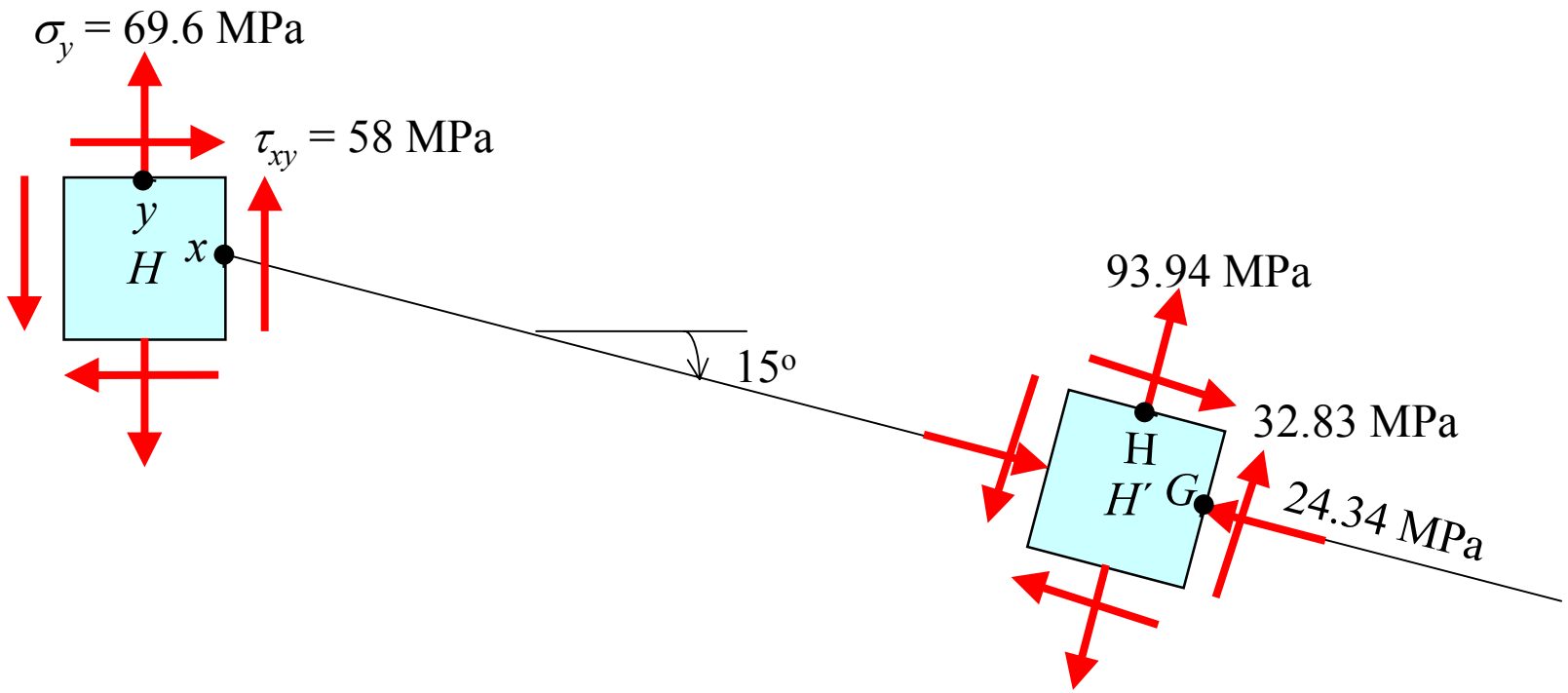
$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(34.8)^2 + 58^2} = \pm 67.64 \text{ MPa}$$




τ
E(34.8, 67.64)
y (69.6, 58)
59.04°
O
σ
(-32.84, 0) ②
59.04°
30.96°
①
(102.44, 0)
(0, -58) x
F(34.8, -67.64)
34.8 MPa
67.64 MPa
E
H′
F
34.8 MPa
σy = 69.6 MPa
τxy = 58 MPa
Maximum shearing plane
θs = 30.96/2 = 15.5°
y
H
x
θP = −59.04/2 = −29.5°
Principle plane
102.4 MPa
1
H′
2
32.84 MPa

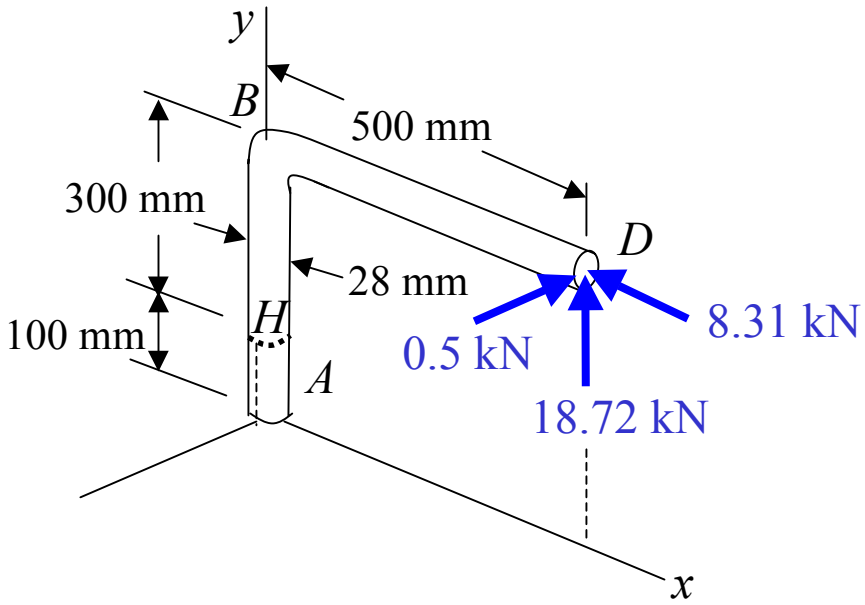
(c)

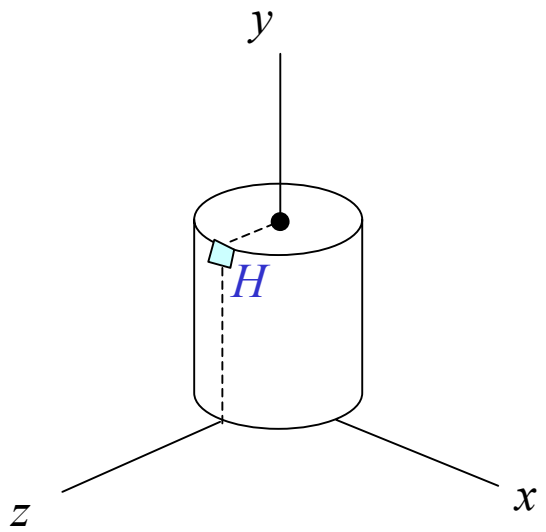

τ
E(34.8, 67.64)
y (69.6, 58)
67.64sin29.04°
H(93.94, 32.83)
34.8 + 67.64cos29.04°
29.04°
59.04°
O
σ
(-32.84, 0) ②
30°
① (102.4, 0)
(-24.34, -32.83)G
34.8 - 67.64cos29.04°
(0, -58) x
F(34.8, -67.64)

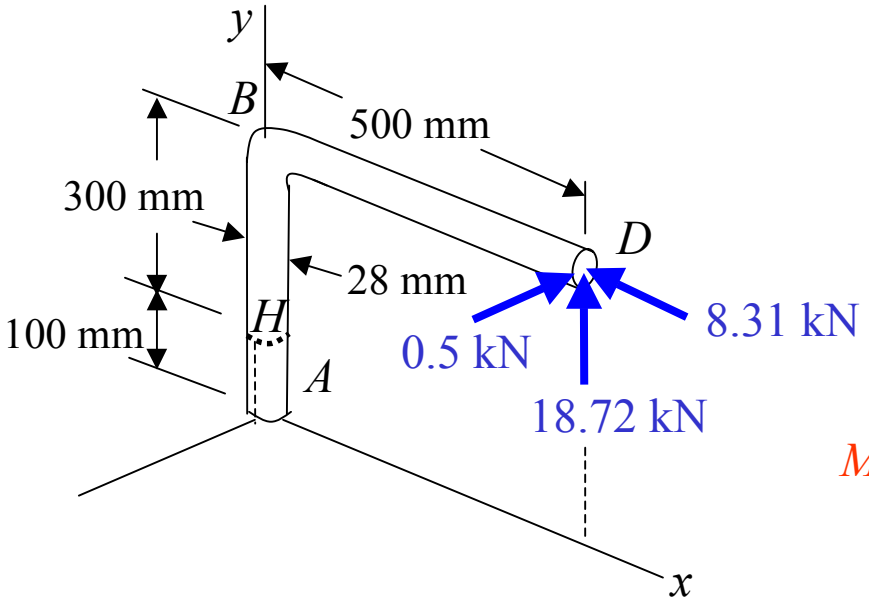

σy = 69.6 MPa
τxy = 58 MPa
y
H
x
15°
93.94 MPa
32.83 MPa
H
H′
G
24.34 MPa

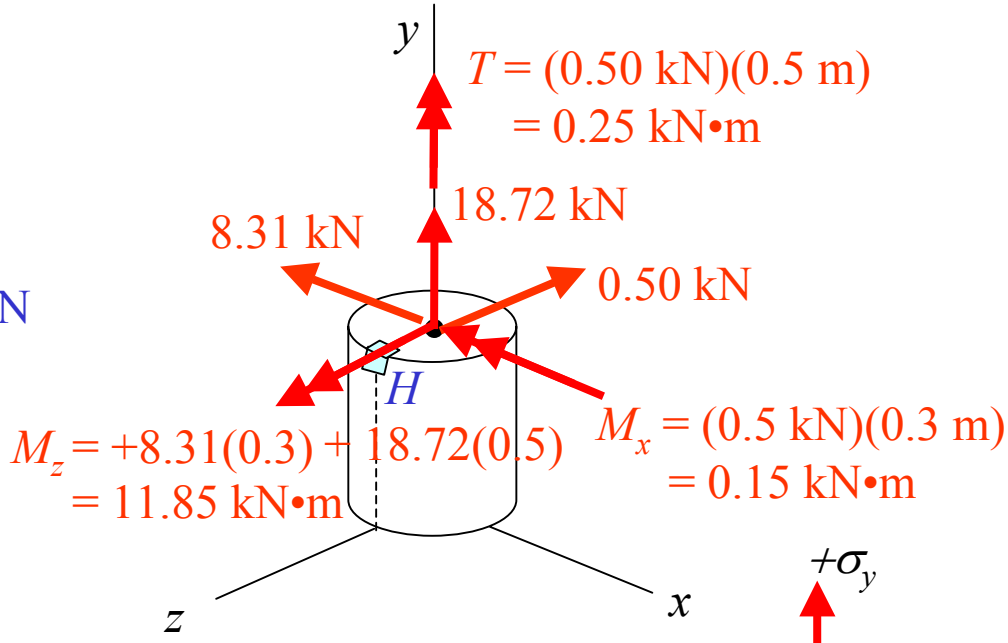
Example 7b

From the figure below, knowing that portion *AB* of the lever has a diameter of 28 mm, determine (a) the normal and shearing stresses on an element located at point *H* and having sides parallel to the *x* and *y* axes, (b) Represent the state of stress at the point of the maximum shearing stress, and the state of stress at the point of the principle stresses (σ_1, σ_2) at point *H*, (c) Represent the state of stress at point *H* oriented 15° outer clockwise from the horizontal axis

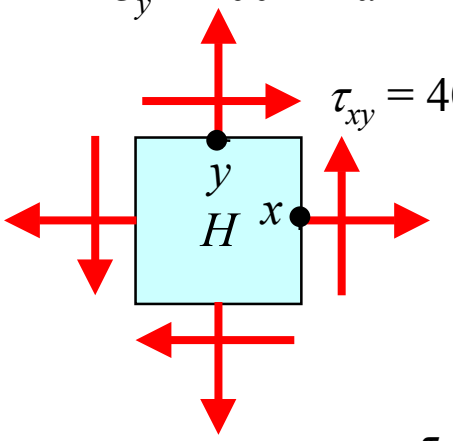








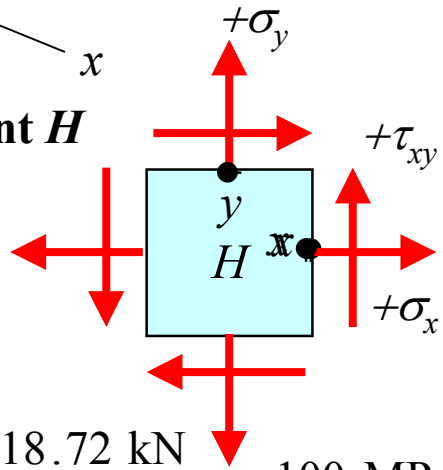
(a) the normal and shearing stresses on an element located at point *H*



$$\sigma_x = 0$$

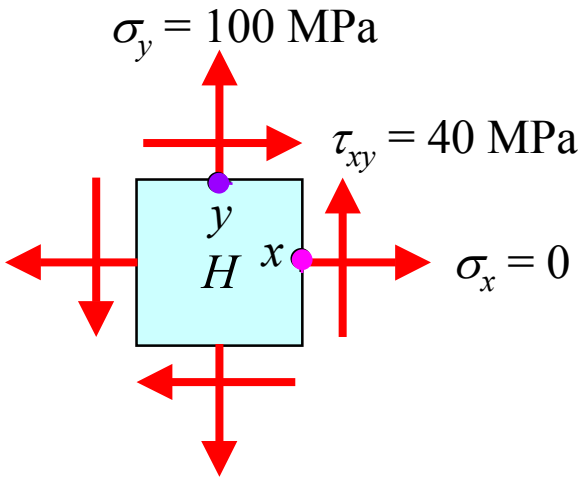
$$\sigma_y = \frac{M_x c}{I} + \frac{P}{A}$$

$$= \frac{(0.15\ \text{kN}\bullet\text{m})(0.014\ \text{m})}{(\frac{\pi}{4})(0.014\ \text{m})^4} + \frac{18.72\ \text{kN}}{\pi(0.014\ \text{m})^2} = 100\ \text{MPa}$$



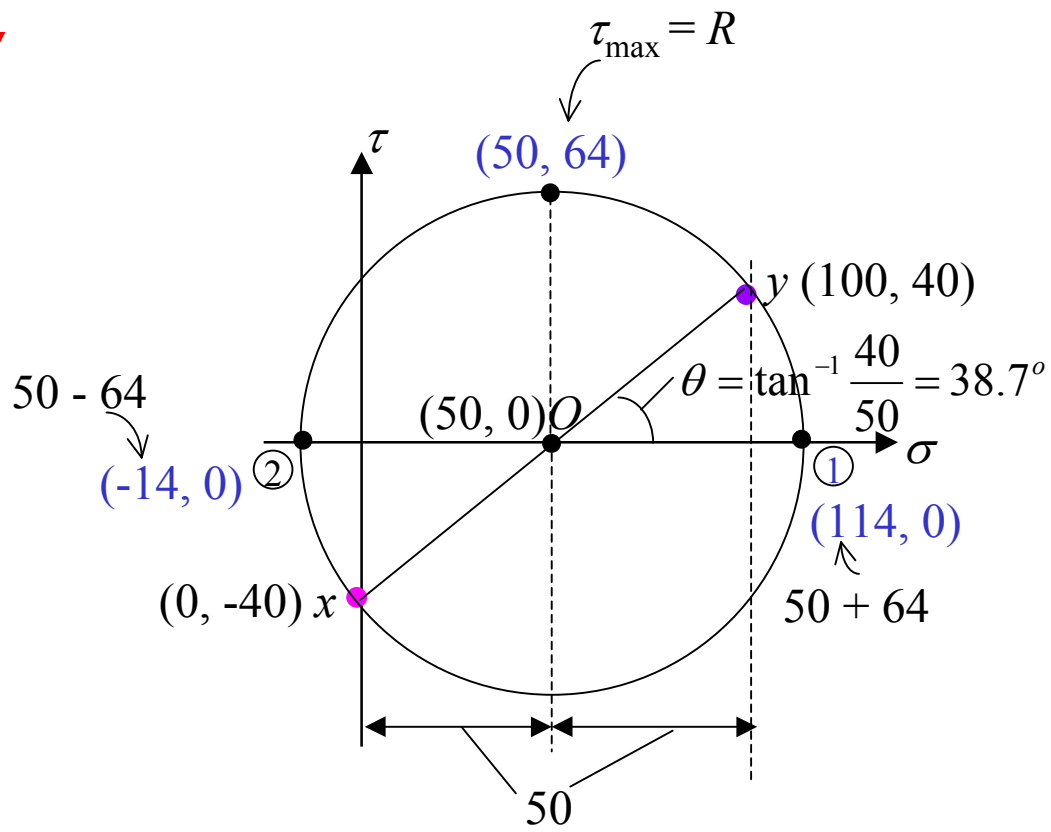
$$\tau_{xy} = \frac{Tc}{J} + \frac{4V}{3A} = \frac{(0.250\ \text{kN}\bullet\text{m})(0.014\ \text{m})}{(\frac{\pi}{2})(0.014\ \text{m})^4} - \frac{4(8.31\ \text{kN})}{3(\pi(0.014^2))} = 40\ \text{MPa}$$

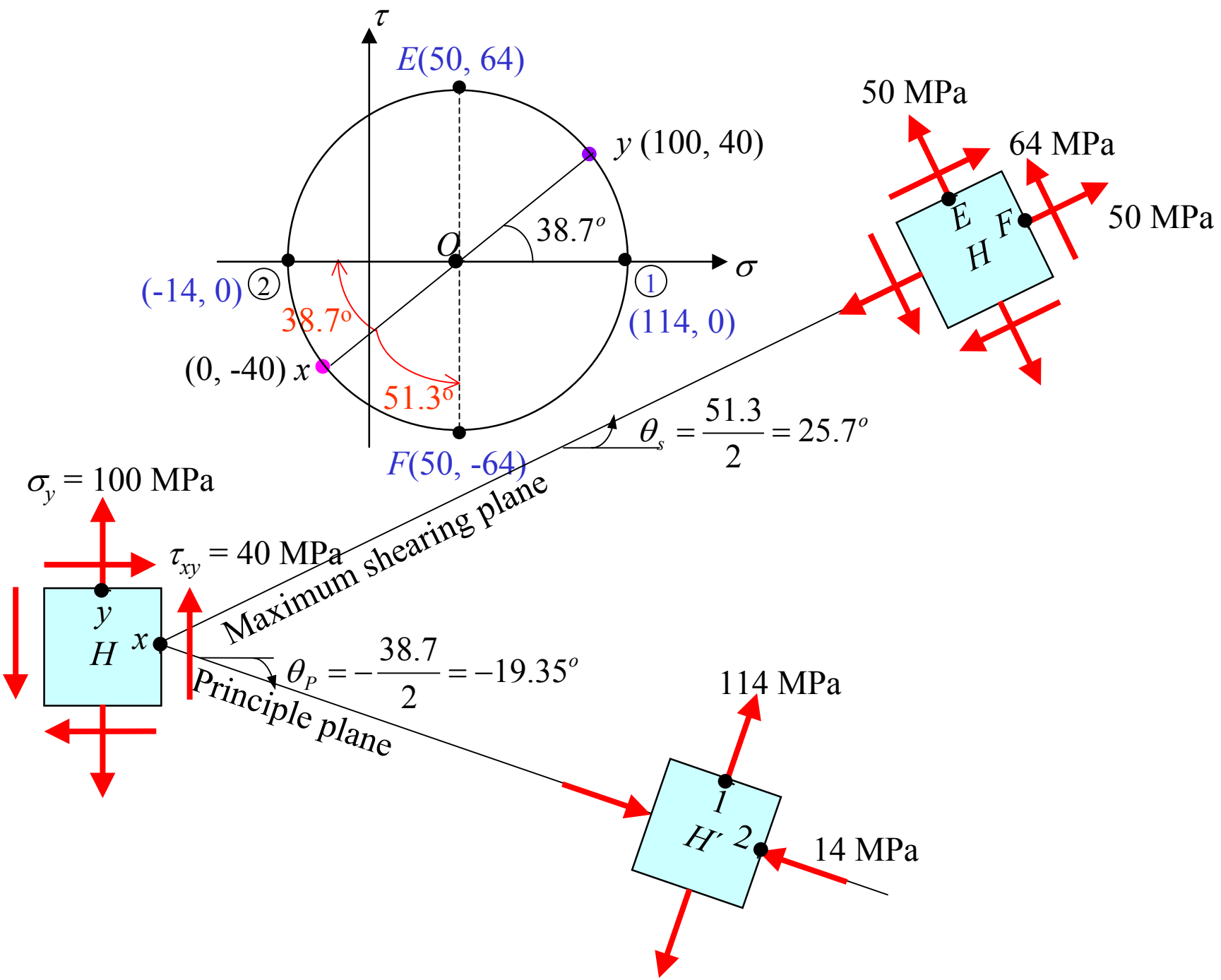
(b)

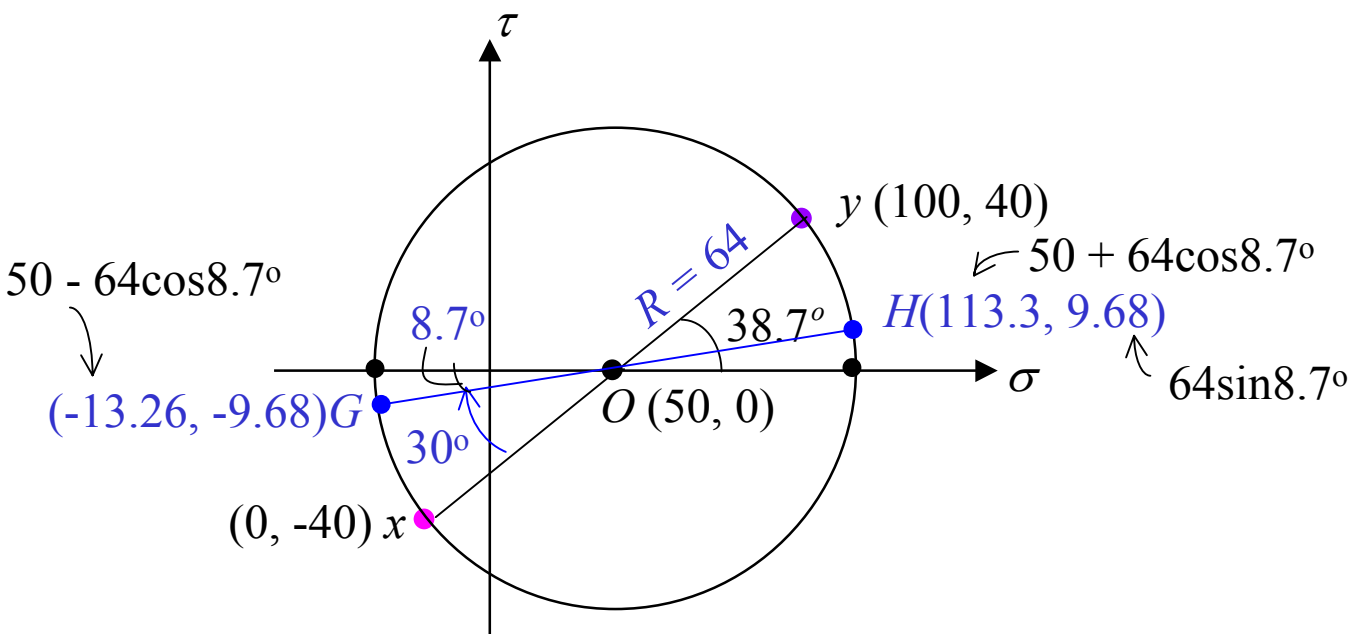


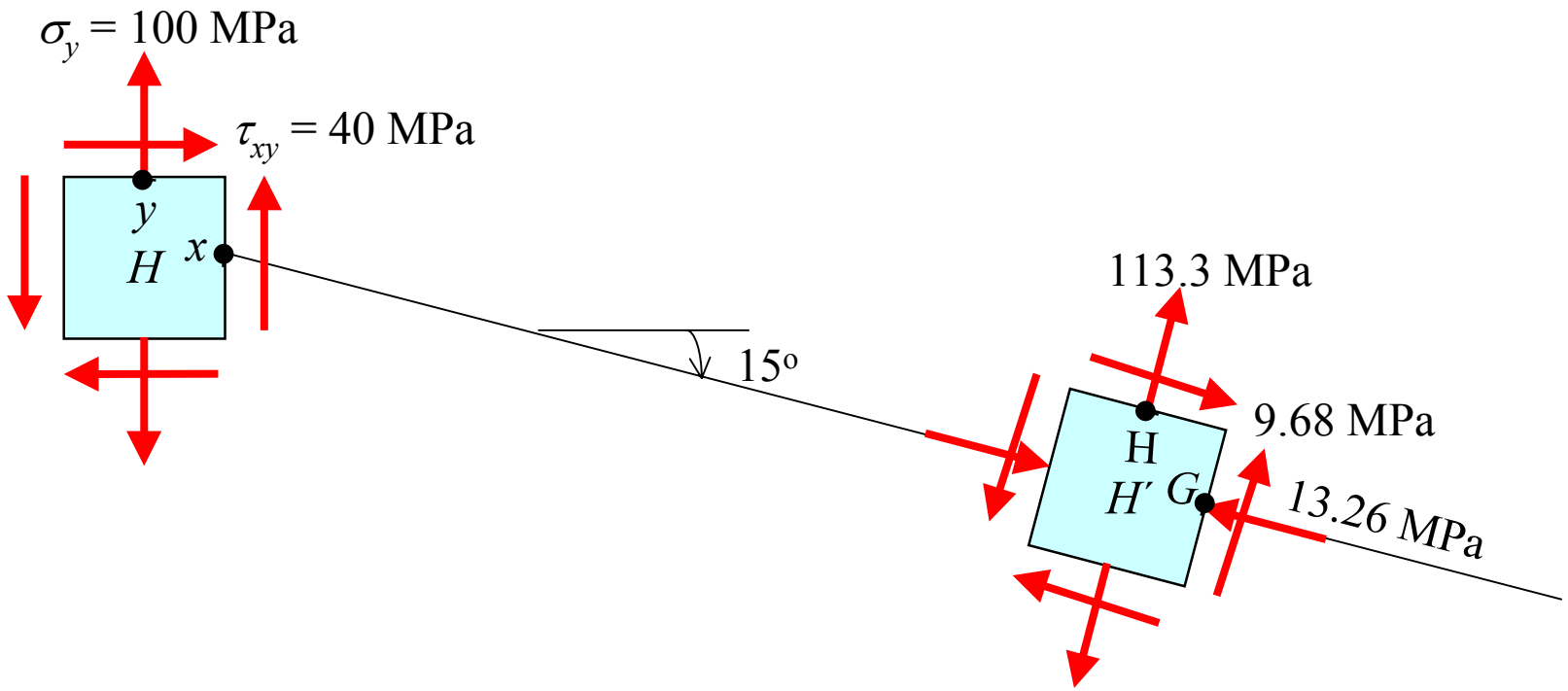
$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} = \frac{0 + 100}{2} = 50 \text{ MPa}$$

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{(50)^2 + 40^2} = \pm 64 \text{ MPa}$$




τ
E(50, 64)
y (100, 40)
38.7°
O
σ
(-14, 0) ②
①
(114, 0)
38.7°
(0, -40) x
51.3°
F(50, -64)
50 MPa
64 MPa
50 MPa
E
F
H
$\theta_s = \frac{51.3}{2} = 25.7^o$
σ_y = 100 MPa
τ_{xy} = 40 MPa
y
H
x
Maximum shearing plane
$\theta_P = -\frac{38.7}{2} = -19.35^o$
Principle plane
114 MPa
1
H′
2
14 MPa


τ
y (100, 40)
50 - 64cos8.7°
50 + 64cos8.7°
R = 64
8.7°
38.7°
H(113.3, 9.68)
σ
64sin8.7°
(-13.26, -9.68)G
O (50, 0)
30°
(0, -40) x


σy = 100 MPa
τxy = 40 MPa
y
H
x
15°
113.3 MPa
9.68 MPa
13.26 MPa
H
H′
G